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CERTAIN CLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS

BY

M.D. GANIGI

1.Introduction. Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n$$

regular in $D=\{z:0<|z|<1\}$. The Hadamard product or convolution of f and g will be denoted by $f*g$. Let

$$D^nf(z)=\frac{1}{z(1-z)^{n+1}}*f(z),\ n=0,1,2,\ldots\ldots=$$
$$=\frac{1}{z}\left(\frac{z^{n+1}f(z)}{n!}\right)^{(n)}=$$
$$=\frac{1}{z}+(n+1)a_0+\frac{(n+1)(n+2)}{2!}a_1z+\ldots..$$

Following R u s c h e w e y h [5], G a n i g i and U r a l e g a d d i [2] introduced the class M_n of functions in Σ that satisfy the condition

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^nf(z)}-2\right)<-\frac{n}{n+1},\ |z|<1$$

and $n\in N_0=N\cup\{0\}$, the set of non - negative integers and obtain a new criteria for univalence of functions in Σ, through basic inclusion relationship $M_{n+1}\subset M_n$, $n\in N_0$. In this paper along with other things it is shown that a function $f\in\Sigma$ which satisfies one of the conditions

$$(1)\qquad \operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^nf(z)}-2\right)<-\frac{n+\alpha}{n+1},\qquad |z|<1,\ 0\le\alpha<1$$

and $n \in N_0$ is univalent in D. More precisely it is shown that for the classes $M_n(\alpha)$ of functions in Σ satisfying (1)

$$M_{n+1}(\alpha) \subset M_n(\alpha),\ n \in N_0,\ 0 \le \alpha < 1, \tag{2}$$

holds. Since $M_0(\alpha)$ equals $\Sigma^*(\alpha)$ the class of meromorphically starlike functions of order α, the univalence of functions in $M_n(\alpha)$ follows from (2). For $\alpha = 0$, the results obtained here reduce to those obtained by G a n i g i and U r a l e g a d d i [2].

2.The classes $M_n(\alpha)$.We need the following Lemma due to I.S. J a c k [4].

Lemma. *Let w be non-constant and regular in $|z| < 1$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$, where k is a real number greater than 1.*

Theorem 1. $M_{n+1}(\alpha) \subset M_n(\alpha),\ n \in N_0,\ 0 \le \alpha < 1$.

Proof. Let $f \in M_{n+1}(\alpha)$. Then

$$\operatorname{Re}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 2\right) < -\frac{n+1+\alpha}{n+2} \tag{3}$$

It is sufficient to prove that (3) implies the inequality

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^{n}f(z)} - 2\right) < -\frac{n+\alpha}{n+1}.$$

Define $w(z)$ in $E = \{z : |z| < 1\}$ by

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} - 2 = -\left[\frac{n}{n+1} + \frac{1}{n+1}\frac{1+(2\alpha-1)w(z)}{1+w(z)}\right] \tag{4}$$

clearly $w(z)$ is regular in E and $w(0) = 0$. Equation (4) may be written as

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} = \frac{n+1+(n+3-2\alpha)w(z)}{(n+1)(1+w(z))}. \tag{5}$$

Differentiating (5) logarithmically and using the identity

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - (n+2)D^n f(z) \tag{6}$$

we obtain

$$\begin{aligned}
&\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 2 + \frac{n+1+\alpha}{n+2} = \\
&= \frac{n+1+(n+3-2\alpha)w(z)}{(n+2)(1+w(z))} - 1 + \frac{\alpha}{n+2} + \\
&+ \frac{2(1-\alpha)zw'(z)}{(n+2)(n+1+(n+3-2\alpha)w(z))(1+w(z))} = \\
&= \frac{1}{n+2}\left[\alpha - \frac{1+(2\alpha-1)w(z)}{1+w(z)} + \right.\\
&\left. + \frac{2(1-\alpha)zw'(z)}{(n+1+(n+3-2\alpha)w(z))(1+w(z))}\right].
\end{aligned} \tag{7}$$

We claim that $|w(z)| < 1$. For otherwise by the Lemma there exists z_0, $|z_0| < 1$ such that

$$zw'(z_0) = kw(z_0) \tag{8}$$

where $|w(z_0)| = 1$, $k \ge 1$.

From (7) and (8) we get

$$\begin{aligned}
&\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 2 + \frac{n+1+\alpha}{n+2} = \\
&= \frac{1}{n+2}\left[\alpha - \frac{1+(2\alpha-1)w(z_0)}{1+w(z_0)} + \right.\\
&\left. + \frac{2k(1-\alpha)w(z_0)}{(n+1+(n+3-2\alpha)w(z_0))(1+w(z_0))}\right].
\end{aligned}$$

Thus

$$\operatorname{Re}\left[\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 2 + \frac{n+1+\alpha}{n+2}\right] \ge \frac{1-\alpha}{2(n+2)(n+2-\alpha)} \ge 0$$

which contradicts (3) and from (4) it follows that $f \in M_n(\alpha)$.

Theorem 2. *Let $f \in \Sigma$ and for a given $n \in N_0$, $c > 0$ satisfy the condition*

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^{n}f(z)} - 2\right) < \frac{(1-\alpha) - 2(n+\alpha)(1-\alpha+c)}{2(n+1)(1-\alpha+c)},\ z \in E. \tag{9}$$

Then $F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt$ *belongs to* $M_n(\alpha)$.

Proof. Using the identities

$$z(D^n F(z))' = cD^n f(z) - (c+1)D^n F(z) \tag{10}$$

and

$$z(D^n F(z))' = (n+1)D^{n+1}F(z) - (n+2)D^n F(z), \tag{11}$$

the condition (9) may be written as

$$\begin{aligned}\mathrm{Re}\left(\frac{(n+2)D^{n+2}F(z)/D^{n+1}F(z) - (n+2-c)}{n+1-(n+1-c)D^nF(z)/D^{n+1}F(z)} - 2\right) < \\ < \frac{(1-\alpha) - 2(n+\alpha)(1-\alpha+c)}{2(n+1)(1-\alpha+c)}.\end{aligned} \tag{12}$$

We have to prove that (12) implies the inequality

$$\mathrm{Re}\left(\frac{D^{n+1}F(z)}{D^nF(z)} - 2\right) < -\frac{n+\alpha}{n+1}$$

Define a regular function $w(z)$ in E by

$$\mathrm{Re}\frac{D^{n+1}F(z)}{D^nF(z)} - 2 = -\left[\frac{n}{n+1} + \frac{1+(2\alpha-1)w(z)}{(n+1)(1+w(z))}\right] \tag{13}$$

Clearly $w(0) = 0$. The equation (13) may be written as

$$\mathrm{Re}\frac{D^{n+1}F(z)}{D^nF(z)} = \frac{n+1+(n+3-2\alpha)w(z)}{(n+1)(1+w(z))} \tag{14}$$

Differentiating (14) logarithmically and simplifying we get

$$\begin{aligned}&\frac{(n+2)D^{n+2}F(z)/D^{n+1}F(z) - (n+2-c)}{n+1-(n+1-c)D^nF(z)/D^{n+1}F(z)} - 2 = \\ &= -\left[\frac{n}{n+1} + \frac{1+(2\alpha-1)w(z)}{(n+1)(1+w(z))}\right] + \\ &+ \frac{2(1-\alpha)zw'(z)}{(n+1)(1+w(z))(c+(c+2-2\alpha)w(z))}.\end{aligned} \tag{15}$$

The remaining part of the proof is similar to that of Theorem 1. Putting, $n = 0, \alpha = 0$ and $c = 1$ in the statement of Theorem 2 we obtain the following result of Goel and Sohi [3].

Corollary. *If* $f \in \Sigma$ *and satisfy* $\mathrm{Re}\,\frac{zf'(z)}{f(z)} < \frac{1}{4}$ *then* $F(z) = \frac{1}{z^2}\int_0^z tf(t)dt$ *belongs to* Σ^* *if* $F(z) \neq 0$ *for all* z *such that* $0 < |z| < 1$.

The above Corollary is the extension of the following result of Bajpai [1]. If $f \in \Sigma^*$ then so is $F(z) = \frac{1}{z^2}\int_0^z tf(t)dt$.

Theorem 3. *If* $f \in M_n(\alpha)$ *then* $F(z) = \frac{n+1}{z^{n+2}}\int_0^z t^{n+1}f(t)dt$ *belongs to* $M_{n+1}(\alpha)$.

Proof. For $F(z) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)dt$, we have

$$\begin{aligned}cD^nf(z) &= (n+1)D^{n+1}F(z) - (n+1-c)D^nF(z)\\ cD^{n+1}f(z) &= (n+2)D^{n+2}F(z) - (n+2-c)D^{n+1}F(z)\end{aligned}$$

Taking $c = n+1$ in the above relation we have

$$\frac{(n+2)D^{n+2}F(z) - D^{n+1}F(z)}{(n+1)D^{n+1}F(z)} = \frac{D^{n+1}f(z)}{D^nf(z)}$$

which reduces to

$$\frac{n+2}{n+1}\frac{D^{n+2}F(z)}{D^{n+1}F(z)} - \frac{1}{n+1} = \frac{D^{n+1}f(z)}{D^nf(z)}$$

Thus

$$\begin{aligned}&\mathrm{Re}\left(\frac{n+2}{n+1}\frac{D^{n+2}F(z)}{D^{n+1}F(z)} - \frac{1}{n+1} - 2\right) = \\ &= \mathrm{Re}\left(\frac{D^{n+1}f(z)}{D^nf(z)} - 2\right) < -\frac{n+\alpha}{n+2}\end{aligned}$$

From which it follows that

$$\mathrm{Re}\left(\frac{D^{n+2}F(z)}{D^{n+1}F(z)} - 2\right) < -\frac{n+1+\alpha}{n+2}$$

This completes the proof of Theorem 3.

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Department of Mathematics
Karnatak Arts College
DHARWAD - 580 001
Karnataka State
INDIA

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ON SUFFICIENT CONDITIONS FOR UNIVALENCE

BY

VIRGIL PESCAR

The aim of this paper is to obtain sufficient conditions for univalence of regular functions in the unit disc $U = \{z : |z| < 1\}$.

In the following, we shall use the result due to P o m m e r e n k e.

Theorem A.[1]. *Let r_0 be a real number, $r_0 \in (0,1], U_{r_0} = \{z : |z| < r_0\}$ and let $f(z,t) = a_1(t)z + \ldots, a_1(t) \neq 0$, be regular in U_{r_0}, for all $t \geq 0$ and locally absolutely continuous in $I = [0,\infty)$, locally uniformly with respect to U_{r_0}. Supposing that for almost all $t \in I$, $f(z,t)$ satisfies the equation*

$$z\frac{\partial f(z,t)}{\partial z} = p(z,t)\frac{\partial f(z,t)}{\partial t}, \quad z \in U_{r'_0} \tag{1}$$

where $p(z,t)$ is regular in U_{r_0} and $Re\ p(z,t) > 0$, for all $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and if $\{f(z,t)/a_1(t)\}$ forms a normal family in U_{r_0}, for all $t \in I$, $f(z,t)$ is a regular and univalent extension to the whole disc U.

Theorem 1. *Let $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be a regular function in U, a and b real positive numbers, c a complex number, $c \notin (-\infty,-1]$ and*

$$|c + 1 - \frac{a+b}{2a}| < \frac{a+b}{2a} \tag{2}$$

If

$$|ce^{-(a+b)t} + (1 - e^{-(a+b)t})e^{-at}\frac{zf''(e^{-at}z)}{f'(e^{-at}z)} + 1 - \frac{a+b}{2a}| < \frac{a+b}{2a} \tag{3}$$

for all $z \in U$ and $t \geq 0$, then the function $f(z)$ is univalent in U.