REFERENCES

1. B a j p a i, S.K. – *A note on a class of meromorphic univalent functions*, Rev. Roumanie Math. Pures Appl. 22(1977), 295 - 297.
2. G a n i g i, M.D. and U r a l e g a d d i, B.A. – *New criteria for meromorphic univalent functions*, Bull. Math. de la Soc. Sci. Math. de la R.S. Roumanie, 33(81), nr.1, (1989), 9-13.
3. G o e l, R.M. and S o h i, N.S. – *Subclasses of univalent functions*, Tamkang J. Math. 11(1980), 77-81.
4. J a c k, I.S. – *Functions starlike and convex of order α*, J.London Math. Soc. (2) 3 (1971), 469 - 474.
5. R u s c h e w e y h, S. – *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49 (1975), 109 - 115.

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Department of Mathematics
Karnatak Arts College
DHARWAD - 580 001
Karnataka State
INDIA

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ON SUFFICIENT CONDITIONS FOR UNIVALENCE

BY

VIRGIL PESCAR

The aim of this paper is to obtain sufficient conditions for univalence of regular functions in the unit disc $U = \{z : |z| < 1\}$.

In the following, we shall use the result due to P o m m e r e n k e.

Theorem A.[1]. *Let r_0 be a real number, $r_0 \in (0,1], U_{r_0} = \{z : |z| < r_0\}$ and let $f(z,t) = a_1(t)z + ..., a_1(t) \neq 0$, be regular in U_{r_0}, for all $t \geq 0$ and locally absolutely continuous in $I = [0,\infty)$, locally uniformly with respect to U_{r_0}. Supposing that for almost all $t \in I, f(z,t)$ satisfies the equation*

$$z\frac{\partial f(z,t)}{\partial z} = p(z,t)\frac{\partial f(z,t)}{\partial t},\ z \in U_{r_0'} \tag{1}$$

where $p(z,t)$ is regular in U_{r_0} and $Re\ p(z,t) > 0$, for all $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and if $\{f(z,t)/a_1(t)\}$ forms a normal family in U_{r_0}, for all $t \in I, f(z,t)$ is a regular and univalent extension to the whole disc U.

Theorem 1. *Let $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be a regular function in U, a and b real positive numbers, c a complex number, $c \notin (-\infty, -1]$ and*

$$|c + 1 - \frac{a+b}{2a}| < \frac{a+b}{2a} \tag{2}$$

If

$$|ce^{-(a+b)t} + (1 - e^{-(a+b)t})e^{-at}\frac{zf''(e^{-at}z)}{f'(e^{-at}z)} + 1 - \frac{a+b}{2a}| < \frac{a+b}{2a} \tag{3}$$

for all $z \in U$ and $t \geq 0$, then the function $f(z)$ is univalent in U.

Proof. Because the function $f(z)$ is regular in U, it results that the function $L(z,t)$ defined by

$$L(z,t) = f(e^{-at}z) + \frac{1}{1+c}(e^{bt} - e^{-at})zf'(e^{-at}z), \tag{4}$$

is regular in U, for all $t \geq 0$ and, hence, $L(z,t) = a_1(t)z + ...$, where

$$a_1(t) = \frac{c}{1+c}e^{-at} + \frac{1}{1+c}e^{bt}. \tag{5}$$

Let's prove that $a_1(t) \neq 0$ for all $t \geq 0$. We observe that if $a_1(t) = 0$ then from (5) it results that $c = -e^{-(a+b)t} \in (-\infty, -1]$. Because from hypothesis $c \notin (-\infty, -1]$, it results that $a_1(t) \neq 0$, for all $t \geq 0$.

Since $f(ze^{-at})$ is regular in U, we have:

$$\begin{aligned}\frac{\partial L(z,t)}{\partial t} = \frac{1}{1+c}[(-cae^{-at} + be^{bt})zf'(e^{-at}z) -\\ - ae^{-at}(e^{bt} - e^{-at})z^2 f''(e^{-at}z)].\end{aligned} \tag{6}$$

Because the function $f(z), f'(z), f''(z)$ are regular in U, it results that for all $r_0 \in (0,1)$, there exist the numbers A, B, C (which depend upon r_0) such that

$$|f(z)| \leq A,\ |f'(z)| \leq B,\ |f''(z)| \leq C, \tag{7}$$

for all $z \in U_{r_0} = \{z : |z| \leq r_0\}$.

Let $T > 0$ be a fixed number. From (6) and (7) it results that:

$$\left|\frac{\partial L(z,t)}{\partial t}\right| \leq \frac{1}{|1+c|}(|c|a + be^{bT})B + a(e^{bT} + 1)C. \tag{8}$$

for all $z \in U_{r_0}$ and $t \in [0,T]$.

It results that a constant $M > 0$ exists such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| \leq M, \tag{9}$$

for all $z \in U_{r_0}$ and $t \geq 0$.

From (9) it results that the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to U.

Let $p(z,t)$ be the function defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t} \tag{10}$$

Then, in order to prove that the function $p(z,t)$ is regular and with a positive real part in U, it is sufficient to prove that the function:

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1} \tag{11}$$

is regular and

$$|w(z,t)| < 1, \tag{12}$$

for all $z \in U$ and $t \in [0,\infty)$. A simple calculation yields:

$$\begin{aligned} w(z,t) = \{(1+a)\{ce^{-(a+b)t}[1 - e^{-(a+b)t}]\cdot\\ \cdot e^{-at}zf''(e^{-at}z)/f'(e^{-at}z)\} + (1-b)\} :\\ : \{(1-a)\{ce^{-(a+b)t}[1 - e^{-(a+b)t}]\cdot\\ \cdot e^{-at}zf''(e^{-at}z)/f'(e^{-at}z)\} + (1+b)\}.\end{aligned} \tag{13}$$

If $V(z,t)$ is the function defined by

$$V(z,t) = ce^{-(a+b)t} + (1 - e^{-(a+b)t})e^{-at}zf''(e^{-at}z)/f'(e^{-at}z), \tag{14}$$

and

$$X = \text{Re } V(z,t),\ Y = \text{Im } V(z,t), \tag{15}$$

then substituting (14) and (15) in (13) we obtain

$$w(z,t) = \frac{(1+a)(X+iY) + (1-b)}{(1-a)(X+iY) + (1+b)}. \tag{16}$$

The inequality (12) is equivalent to the inequality

$$\left|X + iY + 1 - \frac{a+b}{2a}\right| < \frac{a+b}{2a} \tag{17}$$

we conclude that the inequality (17) has the form:

$$\left|ce^{-(a+b)t} + (1 - e^{-(a+b)t})e^{-at}\frac{zf''(e^{-at}z)}{f'(e^{-at}z)} + 1 - \frac{a+b}{2a}\right| < \frac{a+b}{2a} \tag{18}$$

which is identical to the inequality (3).

For $t = 0$, the inequality (18) has the form:

$$|c + 1 - \frac{a+b}{2a}| < \frac{a+b}{2a}$$

and is identical with the inequality (2).

Because from hypothesis the inequality (3) holds true for all $z \in U$ and $t \geq 0$, we conclude that $|w(z,t)| < 1$, for all $z \in U$ and $t \geq 0$.

If the function $w(z,t)$ has a singular point $z_0 \in U$, then z_0 is a pole for the function $w(z,t)$ and, hence, $\lim_{z \to z_0} |w(z,t)| = \infty$, which is in contradiction with the inequality $|w(z,t)| < 1$. It results that the function $w(z,t)$ is regular in U for all $t \geq 0$. Because $a > 0, b > 0$ and $c \notin (-\infty, -1]$ we have:

$$\lim_{t \to \infty} |a_1(t)| = \infty \tag{19}$$

It is easy to prove that, if $r_0 \in (0,1)$ then $\{L(z,t)/a_1 t\}$ is a normal family in U_{r_0}.

From Theorem A it results that $L(z,t)$ is univalent in U for all $t \geq 0$.

For $t = 0$ we obtain: $L(z,0) \equiv f(z)$ and, hence, f(z) is univalent in U.

Theorem 2. *Let m be a real positive number, c a complex number $c \notin (-\infty, -1]$. If f(z) is regular in U, $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ and*

$$|c + 1 - \frac{m+1}{2}| < \frac{m+1}{2} \tag{20}$$

If

$$|c|z|^{m+1} + (1 - |z|^{m+1}) \frac{zf''(z)}{f'(z)} + 1 - \frac{m+1}{2}| \leq \frac{m+1}{2} \tag{21}$$

for all $z \in U$, then the function f(z) is univalent in U.

Proof. Noting $m = \frac{b}{a}, a > 0, b > 0$ we observe that the inequality (20) is identical to inequality (2). Let's consider the function $Q(z,t)$ defined by

$$Q(z,t) = ce^{-at(m+1)} + (1 - e^{-at(m+1)}) e^{-at} \frac{zf''(e^{-at}z)}{f'(e^{-at}z)} + 1 - \frac{m+1}{2}. \tag{22}$$

From (21), for $t = 0$ we obtain:

$$|Q(z,0)| < \frac{m+1}{2}, \tag{23}$$

for all $z \in U$.

For $t > 0$, we observe that:

$$|e^{-at}z| = e^{-at}|z| < 1, \; if \; |z| \leq 1, \tag{24}$$

and hence the function $Q(z,t)$ is regular in $\bar{U} = \{z : |z| \leq 1\}$. By applying the maximum principle, it results that

$$|Q(z,t)| < \max_{|z|=1} |Q(z,t)| = |Q(e^{i\theta}, t)|, \tag{25}$$

for all $z \in U$ and $t_{i\theta} > 0$, where θ is a real number.

If $\xi = e^{-at}e^{i\theta}$, we have $|\xi| = e^{-at}$, where $e^{-t} = |\xi|^{1/a}$. Because $\xi \in U$, from (21), for $z = \xi$, we obtain for all $\xi \in U$,

$$|c|\xi|^{m+1} + (1 - |\xi|^{m+1}) \frac{zf''(\xi)}{f'(\xi)} + 1 - \frac{m+1}{2}| \leq \frac{m+1}{2}. \tag{26}$$

$$|Q(e^{i\theta}, t)| \leq \frac{m+1}{2}. \tag{27}$$

From (25), (26) and (27) we obtain:

$$|Q(z,t)| < \frac{m+1}{2}. \tag{28}$$

for all $z \in U, t > 0$.

The conditions from the hypothesis of the Theorem 1 are satisfied and, hence, $f(z)$ is univalent in U.

REFERENCES

1. P o m m e r e n k e, C. – *Uber die Subordination analytischer Funktionen*, J. reine angew. Math. 218, (1965), 159-173.
2. P o m m e r e n k e, C. – *Univalent Functions*, Venderhoeck and Ruprecht, Gottingen, 1975.

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Department of Mathematics
University of Braşov
2200 Braşov
ROMANIA