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# ON BINARY TRIPLE ALGEBRAS

BY

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The *binary triple algebras* have been emphasized by M.A. A k i v i s [1] related with the local Lie loops, who named them *W-algebras*. These algebras have been later treated by K.H. H o f m a n n and K. S t r a m b a c h [5,6] as tangent algebras of local Lie loops; they named them *Akivis-algebras*. Both these names are related to the theory of local Lie-loops. But such algebras do not give a local characterization of local Lie loops. Indeed, it was proved in [2] that the tangent space at the identity element of a local Lie loop may be organized as a homogeneous system of K. Y a m a g u t i (see [9]), this construction being compatible with those given by the authors above mentioned.

On the other hand, these algebras arise in the study of nonassociativity of binary algebras, simultaneously being generalizations of Lie triple algebras [8]. That is why it seems to be appropriate to name such algebras *binary triple algebras (briefly BT - algebras)*. Moreover, as it has been proved in [3], with any binary algebra a homogeneous system of K. Yamaguti may be associated. For some binary algebras (for ex. see [4]), only the binary and ternary operations of the corresponding homogeneous systems determine the other ones. In this case, the two operations endow the underlying space of the algebra as a binary triple algebra, what motivates the interest in the study of such an algebraic structure.

The present paper deals with the semi-simplicity of such algebras. We introduce the concept of solvability of $BT$-algebras, too. The existence of the solvable radical of $BT$-algebras is shown and some results about semi-simple $BT$-algebras are given, under the solvability just introduced.

**I.Definitions. Examples.** Throughout this paper $K$ will be a commutative ring with an unit element.

**Definition 1.** *A binary triple algebra (shortly BT-algebra) over $K$ is a $K$-module with a bilinear operation $A \times A \to A$, $(x,y) \to [x,y]$ and a trilinear operation $A \times A \times A \to A$, $(x,y,z) \to < x,y,z >$, satisfying the following axioms:*

*A.1.* $[x,y] = -[y,x], \quad \forall x,y \in A,$

*A.2.* $< x,y,z > + < y,z,x > + < z,x,y > - < x,z,y > - < y,x,z > - < z,y,x > = [x,[y,z]] + [y,[z,x]] + [z,[x,y]], \quad \forall x,y,z \in A.$*

*Such an algebra will be denoted by $(A;[,];<,>)$ or, shorthly, by $A$.*

**Examples.**

*1. The tangent space at the identity element of a local Lie loop may be organized as BT-algebra (see [1],[2],[5],[6]).*

*2. Let $A(\cdot)$ be a nonassociative binary algebra over $K$. The operations*

$$[x,y] = x \cdot y - y \cdot x \qquad \forall x,y \in A, \tag{1}$$

$$< x,y,z > = x \cdot (y \cdot x) - (x \cdot y) \cdot z \quad \forall x,y,z \in A, \tag{2}$$

*endow $A$ with a BT-algebra structure.*

*3. Any Lie triple algebra (see [8]) becomes a BT-algebra by an obvious modification of its ternary operation.*

*4. Let $M$ be an $n$-dimensional $C^\infty$-manifold and $\nabla$ be a linear connection on it having the property $\nabla T = 0$ ($T$ being the torsion tensor for $\nabla$). Then $F(M)$-module $\chi(M)$ of all vector fields on $M$ becomes a BT-algebra with operations:*

$$[X,Y] = T(X,Y) \qquad < X,Y,Z > = \frac{1}{2} R(X,Y)Z \quad \forall X,Y,Z \in \chi(M)$$

*(here $F(M)$ is the algebra of all real $C^\infty$-maps on $M$ and $R$ is the curvature tensor for $\nabla$).*

**Definition 2.** *Any BT-algebra $A$ for which $[A,A] = \{0\}$, $< A,A,A > = \{0\}$ is called an abelian BT-algebra.**

*Obviously, any $K$-module may be viewed as an abelian BT-algebra.*

**II. Ideals.** *Given a BT-algebra $A$, a BT-subalgebra is a $K$-submodule $B$ of $A$ closed under the two multiplications of $A$. An ideal of $A$ is a $K$-submodule $B$ of $A$ such that $[A,B] \subset B$, $< B,A,A > \subset B$, $< A,B,A > \subset B$, $< A,A,B > \subset B$.*

*A homomorphism from BT-algebra $(A;[,];<,>)$ to BT-algebra $(A';[,]';<,>')$ is a $K$-linear map $f : A \to A'$ such that $f([x_1,x_2]) = [f(x_1),f(x_2)]'$, $f(< x_1,x_2,x_3 >) = < f(x_1),f(x_2),f(x_3) >'$, $\forall x_1,x_2,x_3 \in A$. The definitions of isomorphisms and automorphism are ovbious. It is easily checked that the kernel of a homomorphism is an ideal.*



*If $B$ and $C$ are ideals of the BT-algebra $A$ such that $A = B \oplus C$ is the direct sum of the $K$-submodules $B$ and $C$, then $[B,C] = \{0\}$ and $< x_1,x_2,x_3 > = 0$ as soon as $x_i \in B$ and $x_j \in C$ for some $i \neq j, 1 \leq i,j \leq 3$. Thus, for any $b_1,b_2,b_3 \in B$ and any $c_1,c_2,c_3 \in C$, we have:*

$$[b_1 + c_1, b_2 + c_2] = [b_1,b_2] + [c_1,c_2],$$

$$< b_1 + c_1, b_2 + c_2, b_3 + c_3 > = < b_1,b_2,b_3 > + < c_1,c_2,c_3 >,$$

*i.e. the multiplications can be carried out componentwise. In this situation we say that $A$ is the (inner) direct sum of the ideals $B$ and $C$. If, on the other hand, $A_1$ and $A_2$ are BT-algebras over $K$, then we can make the cartesian product $A_1 \times A_2$ a BT-algebra by defining operations componentwise. We shall denote this BT-algebra by $A_1 \otimes A_2$ and call it the (exterior) direct sum of $A_1$ and $A_2$. There is no essential differcnce between inner and exterior sums. Indeed, if $A$ is the inner direct sum of the ideals $B$ and $C$, then $A$ is isomorphic to the exterior direct sum of the algebras $B$ and $C$. Conversely, $(A_1,0)$ and $(0,A_2)$ are ideals of the exterior direct sum $A_1 \otimes A_2$ and $A_1 \otimes A_2$ is the inner direct sum of these ideals. That is why we write $A_1 \oplus A_2$ both for inner and exterior sum.*

*Any BT-algebra $A$ has at least two ideals: $A$ and $\{0\}$. The $K$-submodule $\{A\}$ of all finite linear combinations of products $[x,y]$ or $< z,u,v >$ for any $x,y,z,u,v \in A$ is, obviously, an ideal of $A$.*

**Definition 3.** *The BT-algebra $A$ is called simple if $A = \{A\} \neq 0$ and its only ideals are the trivial ideals. A BT-algebra $A$ is called semi-simple if it is direct sum of finitely many ideals $A_1, \ldots, A_r$, each of which being simple as a BT-algebra.*

*In the following, we state some theorems about BT-algebras. We will omit the proofs, since these can be carried over almost word-by-word from the proofs of analogous statements for binary algebras.*

**Theorem 1.** *Let $A$ be a BT-algebra and $B$ be an ideal of $A$. Then the quotient space $A/B$ becomes a BT-algebra by defining*

$$[x_1 + B, x_2 + B] = [x_1,x_2] + B,$$

$$< x_1 + B, x_2 + B, x_3 + B > = < x_1,x_2,x_3 > + B.$$

*The canonical projection*

$$\pi : A \to A/B, \quad x \to x + B$$

*is a homomorphism with the kernel $B$.**

**Theorem 2.** *Let $A$ and $A'$ be $BT$-algebras, $B$ be an ideal of $A$, $B'$ be an ideal of $A'$ and $f : A \to A'$ be a homomorphism.*

*a) There exists a homomorphism $\bar{f} : A/B \to A'$ with $f = \bar{f} \circ \pi$ if and only if $B \subset \ker f$.*

*b) The image of $f$, $\mathrm{Im}\, f$, is a $BT$-subalgebra of $A'$ and $A/\ker f \simeq \mathrm{Im}\, f$.*

*c) $f^{-1}(B')$ is an ideal of $A$.**

**Theorem 3.** *Let $A$ be a $BT$-algebra, and $B, C$ be ideals of $A$.*

*a) $B + C$ and $B \cap C$ are ideals of $A$ and $(B+C)/C \simeq B/(B \cap C)$.*

*b) If $C \subset B$, then $B/C$ is an ideal of $A/C$ and $(A/C)/(B/C) \simeq A/B$.*

**Remark.** Then statement a) of Theorem 3 may be improved by requiring $B$ be a $BT$-subalgebra, only.

**Theorem 4. (Chinese Remainder Theorem.)** *Let $A$ be a $BT$-subalgebra and $B, C$ be two ideals of $A$ such that $A = B + C$. Then*

$$A/(B \cap C) \simeq (A/B) \oplus (A/C).*$$

**Corollary.** *Let $A$ be a $BT$-subalgebra and $B_1, \ldots, B_r$ be ideals of $A$ such that $B_i + \bigcap_{j \neq i} B_j = A$ $(1 \leq i \leq r)$. Then*

$$A/(\bigcap_{1 \leq i \leq r} B_i) \simeq (A/B_1) \oplus \ldots \oplus (A/B_r).*$$

**III.Semi-simple binary triple algebras.** In the following we shall do, as much as possible, the structure theory for general $BT$-algebras of finite dimension.

**Lemma 1.** *Let $A$ be a semi-simple $BT$-algebra. Then, for every nontrivial ideal $B$ of $A$ there is an ideal $C$ of $A$ such that $A = B \oplus C$.**

**Proof.** If $A$ is simple, then it is nothing to prove. Otherwise, let $A = B_1 \oplus \ldots \oplus B_r$ with ideals $B_1, \ldots, B_r$ which are simple $BT$-algebras. Let $B$ be a nontrivial ideal of $A$ and $c \in B$, $c \neq 0$. Then $c = b_1 + \ldots + b_r$ with uniquely determined $b_i \in B_i$, $1 \leq i \leq r$. Assume $b_j \neq 0$. Then we assert $B_j \subset B$. To see this, note that

$$[c, b'_j] = [b_j, b'_j] \in B \cap B_j, \qquad \forall b'_j \in B_j,$$
$$< c, b'_{j_2}, b'_{j_3} > = < b_j, b'_{j_2}, b'_{j_3} > \in B \cap B_j, \quad \forall b'_{j_2}, b'_{j_3} \in B_j.$$

Also $< b'_{j_2}, c, b'_{j_3} > \in B \cap B_j$ and $< b'_{j_2}, b'_{j_3}, c > \in B \cap B_j$ for any $b'_{j_2}, b'_{j_3} \in B_j$. If $B_j \cap B = \{0\}$ then $Kb_j$ is a nonzero ideal of $B_j$, hence $B_j = Kb_j$ and $\{B_j\} = \{0\}$, that is a contradiction. Thus $B_j \cap B \neq \{0\}$. But $B_j \cap B$ is an ideal of $B_j$ hence $B_j \cap B = B_j$ or $B_j \subset B$. This argument shows that $B$ is direct sum of those $B_j$ with $B_j \cap B \neq \{0\}$. Let $C$ be the direct sum of remaining ones. Then clearly $B \oplus C = A$.*



**Lemma 2.** *If $A = \{A\}$ and for every nontrivial ideal $B$ of $A$ there is an ideal $C$ of $A$ such that $A = B \oplus C$, then $A$ is semi-simple.*

**Proof.** The essential remark in the proof of this assertion is that an ideal of an ideal of such $A$ is an ideal of $A$. Indeed, if $B$ is an ideal of $A$ and $C$ is an ideal of $B$, then there exists an ideal $D$ of $A$ such that $A = B \oplus D$. Because $[B, D] = \{0\}$, $< B, D, D > = < D, B, D > = < D, D, B > = < B, B, D > = < B, D, B > = < D, B, B > = \{0\}$ we get $[c, a] = [c, b + d] = [c, b] + [c, d] = [c, b] \in C$ for any $a = b + d \in A = B \oplus D$, $< c, a_2, a_3 > = < c, b_2 + d_2, b_3 + d_3 > = < c, b_2, b_3 > \in C$ for any $a_2 = b_2 + d_2$, $a_3 = b_3 + d_3 \in A$. Similarly, we have $< a_1, c, a_2 >, < a_1, a_2, c > \in C$, $\forall a_1, a_2 \in A$. Thus, $C$ is an ideal of $A$. The remainder of the proof is standard.*

**Definition 1.** *An ideal $B$ of $A$ is called maximal if $B \neq A$ and every ideal $C$ of $A$ satisfying $B \subset C \subset A$ satisfies $C = B$ or $C = A$.**

The existence of maximal ideals of $A$ is guaranted by the finite dimension of $A$. It is easily checked that an ideal $C$ of $A$ is maximal if and only if $A/C$ has only trivial ideals (and, hence is either simple or one-dimensional with trivial multiplications). Obviously, every ideal of $A$ different from $A$ is contained in some maximal ideal.

Let $(B_i)_{i \in I}$ be the collection of maximal ideals of $A$.

**Lemma 3.** *If $A$ is semi-simple, then, $\bigcap_{i \in I} B_i = \{0\}$.*

The proof is standard and will be omitted.

**Corollary.** *Let $C$ be an ideal of $A$. If $A/C$ is semisimple, then $\bigcap_{i \in I} B_i \subset C$.**

**Proof.** It is easy to verify that there is a 1-1-corespondence between the maximal ideals of $A/C$ and these maximal ideals of $A$ containing $C$. Then the assertion follows from Lemma 3.*

The main result is:

**Theorem 5.** *Let $A$ be $BT$-algebra, $(B_i)_{i \in I}$ its collection of (all) maximal ideals and $S = \bigcap_{i \in I} B_i$. Then $A/S = N \oplus W$, where $N$ and $W$ are ideals of $A/S$, $N$ has trivial multiplications and $W$ is semi-simple. Moreover, $N = \{x \in A/S;\ [x, y] = 0,\ < x, y_1, y_2 > = < y_1, x, y_2 > = < y_1, y_2, x > = 0,\ \forall y_1, y_2, y \in A/S\}$ and $W = \{A/S\}$. In particular, $N$ and $W$ are uniquely determined.*

**Proof.** We first note that there are finitely many maximal ideals $B_{i_1},\ldots,B_{i_r}$ such that $S = B_{i_1}\cap\ldots\cap B_{i_r}$. This, again, follows from the finite dimension of $A$. Relabelling, we write $S = B_1\cap\ldots\cap B_r$. We may also assume that the set $B_1,\ldots,B_r$ is minimal in the sense that $S \neq \bigcap_{j\neq k} B_j$ for all $k, 1\leq k\leq r$. This implies $B_k + \bigcap_{j\neq k} B_j = A$, because $B_k + \bigcap_{j\neq k} B_j$ is an ideal of $A$ which contains $B_k$ and it is equal to $B_k$ if it is no equal to $A$. But then $\bigcap_{j\neq k} B_j \subset B_k$ and $S = \bigcap B_j = \bigcap_{j\neq k} B_j$ which is a contradiction to the minimality of $B_1,\ldots,B_r$. Therefore we can apply the Corollary to Theorem 4 to obtain $A/S = (A/B_1)\oplus\ldots\oplus(A/B_r)$, and each $A/B_k$ is either simple or one-dimensional with trivial multiplication. Arrange these terms appropriately to finish the proof.*

**Definition 5.** *Let $\pi : A \to A/S$ the canonical projection. Then $R = \pi^{-1}(N)$ is called the radical of $A$.**

**Theorem 6.**

a) *$A/R$ is semi-simple or $A = R$.*

b) *If $U$ is an ideal of $A$ and $A/U$ is semi-simple, then $R\subset U$.*

**Proof.** a) Obviously, $S\subset R$ and $R/S \simeq N$. Using the standard isomorphism theorem we find: $A/R \simeq (A/S)/(R/S) \simeq (N\oplus W)/N \simeq$ $\simeq W$ =semi-simple.

b) We already know $S\subset U$ (Corollary to Lemma 3); hence $A/U \simeq (A/S)/(U/S) = \bar{A}/\bar{U}$, where $\bar{A} = A/S$ and $\bar{U} = U/S$. Now $R\subset U$ if and only if $N\subset \bar{U}$ and this is equivalent to $(N+\bar{U})/\bar{U} = \{0\}$. Let us prove $(N+\bar{U})/\bar{U} = \{0\}$. Indeed, $(N+\bar{U})/\bar{U}$ is an ideal of $\bar{A}/\bar{U}$, hence there is an ideal $T$ of $\bar{A}/\bar{U}$ such that $\bar{A}/\bar{U} = ((N+\bar{U})/\bar{U})\oplus T$. Now $\{N+\bar{U}\}\subset \bar{U}$ hence $\{(N+\bar{U})/\bar{U}\} = \{0\}$, and therefore $\bar{A}/\bar{U} = \{\bar{A}/\bar{U}\}\subset T$ showing $\bar{A}/\bar{U} = T$ and $(N+\bar{U})/\bar{U} = \{0\}$.*

**IV. Solvability of binary triple algebras.** Let $A$ be $BT$-algebra and $B$ be an ideal of it. Then we define $BT$-algebras $B^{(i)}$ $(i = 0,1,2,\ldots)$ inductively as follows:

$$B^{(0)} = B,\quad B^{(1)} = \{B\},\quad B^{(i+1)} = \{B^{(i)}\}.$$

**Theorem 7.** *For each ideal $B$ of $A$, $B^{(i)}$'s form a chain $B = B^{(0)} \supset B^{(1)} \supset\ldots\supset B^{(i)} \supset B^{(i+1)} \supset\ldots$ of $BT$-subalgebras of $A$ such that each $B^{(i+1)}$ is an ideal of $B^{(i)}$ and the quotient $BT$-algebras $B^{(i)}/B^{(i+1)}$ $(i = 0,1,2,\ldots)$ are abelian.*

**Proof.** The relations $B^{(i+1)}\subset B^{(i)}$ $(i = 0,1,2,\ldots)$ are clear from their constructions. Since $[B^{(i)},B^{(i+1)}]\subset B^{(i+1)}$, $< B^{(i+1)},B^{(i)},B^{(i)} >\subset B^{(i+1)}$, $< B^{(i)},B^{(i+1)},B^{(i)} >\subset B^{(i+1)}$, $< B^{(i)},B^{(i)},B^{(i+1)} >\subset B^{(i+1)}$ hold, we see that $B^{(i+1)}$ is an ideal of $B^{(i)}$ and each $B^{(i)}$ is a $BT$-subalgebras of $A$. It is easy to verify that the quotient $BT$-algebra $B^{(i)}/B^{(i+1)}$ is abelian.*



**Lemma 4.** *If $B$ and $C$ are ideals of $A$, then $B\cap C$ and $B+C$ are ideals of $A$ satisfying the following inclusion relations:*

$$a)\ (B\cap C)^{(i)}\subset B^{(i)}\cap C^{(i)},$$

$$b)\ (B+C)^{(i)}\subset B^{(i)}+C^{(i)}+B\cap C.$$

**Proof.** The inclusions may be checked up by induction on $i$.*

**Definition 6.** *An ideal $B$ of $A$ is solvable if $B^{(i)} = \{0\}$ for some integer $i$.**

**Theorem 8.** *If two ideals $B$ and $C$ of a $BT$-algebra $A$ are solvable so is $B+C$.**

**Proof.** If $B^{(i)} = C^{(i)} = 0$ for a large $i$, then the relation b) in Lemma 4 implies $(B+C)^{(i)}\subset B\cap C$ and a) in the same lemma implies $(B+C)^{(2i)}\subset B^{(i)}\cap C^{(i)} = \{0\}$ which proves the theorem.*

This theorem allows us to give the following definition.

**Definition 7.** *The maximal solvable ideal of the $BT$-algebra $A$ is called the solvable radical of $A$.**

**Theorem 9.** *Let $R$ be the solvable radical of a $BT$-algebra $A$. The quotient $BT$-algebra $A/R$ is s-semisimple, i.e. its solvable radical is $\{0\}$.*

**Proof.** Let $p : A\to A/R$ the natural homomorphism of $BT$-algebras. if $\bar{B}$ is an ideal of $A/R$ then $B = p^{-1}(\bar{B})$ is an ideal of $A$ and $p^{-1}(\bar{B}^{(i)}) =$ $= B^{(i)} + R$ $(i = 1,2,\ldots)$. Moreover, if $\bar{B}$ is solvable, then $\bar{B}^{(i)} = \{0\}$ for some integer $i$ and $B^{(i)}\subset R$. Since $R$ is sovable, $B^{(j)} = \{0\}$ for some $j$, that is $B$ is contained in $R$. Hence $\bar{B} = p(B) = \{0\}$ so the radical of $A/R$ is zero.*

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# A UNIFIED THEORY OF PERFECT AND RELATED FUNCTIONS

BY

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**1.Introduction.**Perfect functions and certain allied types of functions refered to the literature have been studied thoroughly by several authors. These types of functions have similarity in many aspects. Certain characterizations are quite analogous. The analogous features existing in their definitions and the types of results obtained in the process of their study point to the necessity of promoting a unified theory. The purpose of this paper is thus to present a unified theory by agglomeration of the individual findings.In the next section, we display the development of the unified theory in a general perspective. In Section 3, we bring about certain concepts viz. those of $\beta$ -sets, $y$ -sets and $\beta$ - continuity to find other characterizations of $\beta$ -perfect functions from an altogether different perspective. In the concluding section, we intend to show up the individual behaviours in particularized settings where the underlying spaces under discussion are topological spaces.

**2.$\beta$ - perfect functions in general.**

**Definition 2.1.** *Let $X$ be a non - empty set and let $\mathcal{B}$ be a family of subsets of $X$ such that each point $x$ of $X$ belongs to some member of $\mathcal{B}$, and that $\mathcal{B}$ is closed under finite intersection. The set of all members of $\mathcal{B}$, each containing a given point $x$ of $X$ denoted by $\mathcal{B}x$. In addition, let us consider an operator $\alpha$ on the power set $\mathcal{P}(X)$ of $X$ such that*

$$(*) \qquad A \subset B(\subset X) \Rightarrow \alpha(A) \subset \alpha(B)$$

*Henceforth by a space $X$ (or $Y$) or simply by $X$ (or $Y$) we mean a non-empty set $X$ (respectively $Y$) endowed with an operator $\alpha$ and associated*