4. E n e s c u, I. – *Asupra unei clase de algebre neasociative*, Bul.I.P.Iaşi, t.XXIII, 1977, Sect.I, p.39-42.
5. H o f m a n n, K.H., S t r a m b a c h, K. – *Lie's fundamental theorems for local analytical loops*, Pacif.J.Math., 123 (1986), p.301-327.
6. H o f m a n n, K.H., S t r a m b a c h, K. – *The Akivis algebra of a homogeneous loop*, Mathematika, 33 (1986), p.87-95.
7. K i k k a w a, M. – *Remarks on solvability of Lie Triple Algebras*, Mem.of Fac.Sci. Shimane Univ., 13 (1979), p.17-22.
8. Y a m a g u t i, K. – *Lie the triple systems and its generalizations*, J.Sci. Hiroshima Univ., Ser.A, v.21 (1957/1958), p.155-160.
9. Y a m a g u t i, K. – *A Note on a Theorem of N.Jacobson*, J.Sci. Hiroshima Univ., Ser.A, v.22 (1958), p.187-190.

Received:18.XI.1992

Departament of Mathematics
Technical University "Gh.Asachi"
Blvd.Copou, 22
6600 Iaşi
ROMANIA

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXIX, s.I.a, Matematica, 1993, f.1.

A UNIFIED THEORY OF PERFECT AND RELATED FUNCTIONS

BY

M.N. MUKHERJEE and S. RAYCHAUDHURI

1.Introduction. Perfect functions and certain allied types of functions refered to the literature have been studied thoroughly by several authors. These types of functions have similarity in many aspects. Certain characterizations are quite analogous. The analogous features existing in their definitions and the types of results obtained in the process of their study point to the necessity of promoting a unified theory. The purpose of this paper is thus to present a unified theory by agglomeration of the individual findings.In the next section, we display the development of the unified theory in a general perspective. In Section 3, we bring about certain concepts viz. those of β -sets, y -sets and β - continuity to find other characterizations of β -perfect functions from an altogether different perspective. In the concluding section, we intend to show up the individual behaviours in particularized settings where the underlying spaces under discussion are topological spaces.

2.β - perfect functions in general.

Definition 2.1. *Let X be a non - empty set and let $\mathcal{B}$ be a family of subsets of X such that each point x of X belongs to some member of $\mathcal{B}$, and that $\mathcal{B}$ is closed under finite intersection. The set of all members of $\mathcal{B}$, each containing a given point x of X denoted by $\mathcal{B}x$. In addition, let us consider an operator α on the power set $\mathcal{P}(X)$ of X such that*

$$(*) \qquad A \subset B(\subset X) \Rightarrow \alpha(A) \subset \alpha(B)$$

Henceforth by a space X (or Y) or simply by X (or Y) we mean a non-empty set X (respectively Y) endowed with an operator α and associated *with a collection $\mathcal{B}$ of subsets of X (or Y), where α and $\mathcal{B}$ are defined as above.*

Definition 2.2. *A point x of a space X is called a β -adherent point of a set $A \subset X$ if for each $U \in \mathcal{B}_x, \alpha(U) \cap A \neq \emptyset$. The set of all β -adherent points of A is denoted by $[A]_\beta$.*

Remark 2.3. It is clear from the above definition that the operator β is increasing, i.e., whenever $A \subset B(\subset X), [A]_\beta \subset [B]_\beta$. When we consider a function f from a space X to a space Y, we use the same notations α, β, etc. for the space Y as well, with the hope that the context will leave no scope for confusion.

Definition 2.4. *A point $x \in X$ is called a β -adherent point of a filterbase $\mathcal{F}$ on X, written as $x \in \beta$ -ad $\mathcal{F}$, if $x \in \cap\{[F]_\beta : F \in \mathcal{F}\}$.*

Definition 2.5. *A filterbase $\mathcal{F}$ on X is said to β -converge to a point x of X, written as $\mathcal{F} \to^\beta x$, if for each $U \in \mathcal{B}_x$, there exists an $F \in \mathcal{F}$ such that $F \subset \alpha(U)$.*

Definition 2.6. *A filterbase $\mathcal{F}$ on X is said to be β -directed to a subset A of X, denoted as $\mathcal{F} \to^{\beta-d} A$, if for each filterbase $\mathcal{G}$ finer that $\mathcal{F}, A \cap \beta$ -ad $\mathcal{G} \neq \emptyset$.*

Proposition 2.7. *For a filterbase $\mathcal{F}$ on X, if $\mathcal{F}$ β - converges to $x \in X$, then $x \in \beta$ -ad $\mathcal{F}$.*

Proof. Let $\mathcal{F} \to^\beta x$ and if possible, $x \notin \beta$ -ad $\mathcal{F}$. Then there exists $F \in \mathcal{F}$ such that $x \notin [F]_\beta$. So there exists $U \in \mathcal{B}_x$ such that $\alpha(U) \cap F = \emptyset$. Since $\mathcal{F} \to^\beta x$, there exists $F_1 \in \mathcal{F}$ such that $F_1 \subseteq \alpha(U)$. Let $F_2 \in \mathcal{F}$ such that $F_2 \subseteq F_1 \cap F$. Then $F_2 \subset F_1 \subset \alpha(U)$; but by (1), $F_2 \cap \alpha(U) = \emptyset$. Hence $F_2 = \emptyset$ which is a contradiction.

Proposition 2.8. *If $x \in [A]_\beta$, where $x \in X$ and $A \subset X$, then there exists a filterbase $\mathcal{F}$ on A such that $\mathcal{F} \to^\beta x$.*

Proof. *For each $U \in \mathcal{B}_x, \alpha(U) \cap A \neq \emptyset$. Let $\mathcal{F} = \{\alpha(U) \cap A : U \in \mathcal{B}_x\}$. For members $F_1 = \alpha(U) \cap A$ and $F_2 = \alpha(V) \cap A$ of $\mathcal{F}$ (where $U, V \in \mathcal{B}_x$), $F_3 = \alpha(U \cap V) \cap A \in \mathcal{F}$ (since $U \cap V \in \mathcal{B}_x$) such that $F_3 \subset F_1 \cap F_2$, using (*). Hence $\mathcal{F}$ is a filterbase on A and clearly $\mathcal{F} \to^\beta x$.*

Proposition 2.9. *If $x \in \beta$ -ad $\mathcal{F}$, for some filterbase $\mathcal{F}$ on X (where $x \in X$), then there exists a filterbase $\mathcal{G}$ on X finer than $\mathcal{F}$ such that $\mathcal{G} \to^\beta x$.*

Proof. If $x \in \beta$ -ad $\mathcal{F}$, then $\alpha(U) \cap F \neq \emptyset$, for all $F \in \mathcal{F}$ and for all $U \in \mathcal{B}_x$. Put $\mathcal{G} = \{\alpha(U) \cap F : U \in \mathcal{B}_x, F \in \mathcal{F}\}$. It is easy to see that $\mathcal{G}$ is a filterbase finer than $\mathcal{F}$ such that $\mathcal{G} \to^\beta x$.

Proposition 2.10. *For a filterbase $\mathcal{F}$ on X and $x \in X$, $\mathcal{F} \to^\beta x$ iff $\mathcal{F} \to^{\beta-d} x$.*

Proof. If $\mathcal{F} \to^\beta x$, then for every filterbase $\mathcal{G}$ on X finer than $\mathcal{F}$, it is clear that $\mathcal{G} \to^\beta x$ and hence by Proposition 2.7., $x \in \beta$ -ad $\mathcal{G}$. This proves that $\mathcal{F} \to^{\beta-d} x$. Conversely , if $\mathcal{F}$ does not β -converge to x, then there exists $U \in \mathcal{B}_x$ such that $F \not\subset \alpha(U)$, for all $F \in \mathcal{F}$. Consider $\mathcal{G} = \{[X - \alpha(U)] \cap F : F \in \mathcal{F}\}$. Then $\mathcal{G}$ is a filterbase on X finer than $\mathcal{F}$ and clearly $x \notin \beta$ -ad $\mathcal{G}$. Hence $\mathcal{F}$ is not β -directed to x.

Definition 2.11. *Let X and Y be two spaces. A function $f : X \to Y$ is called β - perfect iff for each filterbase $\mathcal{F}$ on $f(X), \mathcal{F} \to^{\beta-d} B(\subset f(X))$ implies $f^{-1}(\mathcal{F}) \to^{\beta-d} f^{-1}(B)$.*

Theorem 2.12. *For a function $f : X \to Y$, the following are equivalent:*

a) f is β -perfect.

b) For each filterbase $\mathcal{F}$ on $f(X)$ such that $\mathcal{F} \to^\beta y(\in f(X))$, $f^{-1}(\mathcal{F}) \to^{\beta-d} f^{-1}(y)$.

c) For any filterbase $\mathcal{F}$ on X, β ad $f(\mathcal{F}) \subset f(\beta - ad\mathcal{F})$.

Proof. a) $\Rightarrow$ b) : Follows from Definition 2.11 and Proposition 2.10.

b) $\Rightarrow$ c): Let $y \in \beta - ad\ f(\mathcal{F})$. Then by Proposition 2.9. there is a filterbase $\mathcal{G}$ on $f(X)$ finer than $f(\mathcal{F})$ such that $\mathcal{G} \to^\beta y$. Let $\mathcal{U} = \{f^{-1}(G) \cap F : G \in \mathcal{G}$ and $F \in \mathcal{F}\}$. Then $\mathcal{U}$ is a filterbase on X finer than $f^{-1}(\mathcal{G})$. Since $\mathcal{G} \to^\beta y$, by b) we have $f^{-1}(\mathcal{G}) \to^{\beta-d} f^{-1}(y)$. $\mathcal{U}$ being finer than $f^{-1}(\mathcal{G})$, we have $f^{-1}(y) \cap (\beta - ad\ \mathcal{U}) \neq \emptyset$. Since $\mathcal{U}$ is finer than $\mathcal{F}$, we have in view of (*), that $f^{-1}(y) \cap \beta - ad\ \mathcal{F} \neq \emptyset$. Then $y \in f(\beta - ad\ \mathcal{F})$.

c) $\Rightarrow$ a) : Let $\mathcal{F}$ be a filterbase on $f(X)$ such that $\mathcal{F} \to^{\beta-d} B$ $(\subset f(X))$. Let $\mathcal{G}$ be a filterbase on X finetr than $f^{-1}(\mathcal{F})$. Then $f(\mathcal{G})$ is a filterbase on f(X) finer than $\mathcal{F}$ and hence $B \cap \beta - ad\ f(\mathcal{G}) \neq \emptyset$. Then (c) implies that $B \cap f(\beta - ad\ \mathcal{G}) \neq \emptyset$ and hence $f^{-1}(B) \cap \beta - ad\ \mathcal{G} \neq \emptyset$. Hence $f^{-1}(\mathcal{F}) \to^{\beta-d} f^{-1}(B)$ and consequently f becomes β -perfect.

Definition 2.13. *A function $f : X \to Y$ is called β -closed if $[f(A)]_\beta \subset f([A]_\beta)$, for each $A \subset X$.*

Proposition 2.14. *A β -perfect function $f : X \to Y$ is β -closed.*

Proof. Let $A \subset X$ and $y \in [f(A)]_\beta$. By Proposition 2.8, there exists a filterbase $\mathcal{F}$ on $f(A)$ such that $\mathcal{F} \to^\beta y$. Put $\mathcal{G} = \{f^{-1}(F) \cap A : F \in \mathcal{F}\}$. Then $\mathcal{G}$ is a filterbase on A such that $\mathcal{G}$ is finer than $f^{-1}(\mathcal{F})$. Since f is β -perfect we have in view of Theorem 2.12, $f^{-1}(\mathcal{F}) \to^{\beta-d} f^{-1}(y)$. It then follows that $f^{-1}(y) \cap \beta - ad\ \mathcal{G} \neq \emptyset$. $\mathcal{G}$ being a filterbase on A and since β is increasing , we have $y \in f([A]_\beta)$. Hence f is β -closed.

3. β- and y -sets, β -continuity and β -perfect functions. We have just seen in Proposition 2.14 that a β -perfect function is β -closed. This motivates us to find conditions which along with the β -closedness of functions yield β -perfect functions. For this purpose we introduce β -sets as follows and ultimately characterize β -perfect functions in terms of this type of sets.

Definition 3.1. *A subset A of a space X is called a β -set if for every filterbase $\mathcal{F}$ on $X, A \cap \beta$ -ad $\mathcal{F} = \emptyset$ implies the existence of an $F \in \mathcal{F}$ such that $A \cap [F]_\beta = \emptyset$.*

Theorem 3.2. *If $f : X \to Y$ is β -closed such that $f^{-1}(y)$ is a β -set for each $y \in Y$, then f is β -perfect.*

Proof. Let $\mathcal{F}$ be a filterbase in $f(X)$ such that $\mathcal{F} \to^{\beta} y$, for some $y \in f(X)$. If $\mathcal{G}$ is a filterbase on X finer than the filterbase $f^{-1}(\mathcal{F})$, then $f(\mathcal{G})$ is a filterbase on Y finer than $\mathcal{F}$. As $\mathcal{F} \to^{\beta-d} y$ (by Proposition 2.10), $y \in \beta$ -ad $f(\mathcal{G})$, i.e., $y \in \cap\{[f(G)]_\beta : G \in \mathcal{G}\}$. Since f is β -colsed, $y \in \cap\{f([G]_\beta) : G \in \mathcal{G}\}$. Hence $f^{-1}(y) \cap [G]_\beta \neq \emptyset$, for all $G \in \mathcal{G}$. $f^{-1}(y)$ being a β -set we obtain $f^{-1}(y) \cap \beta$ -ad $\mathcal{G} \neq \emptyset$. Then $f^{-1}(\mathcal{F}) \to^{\beta-d} f^{-1}(y)$ and hence f is β -perfect, by Theorem 2.12.

Definition 3.3. *A function $f : X \to Y$ is called β - continuous if for each $x \in X$ and each $B \in \mathcal{B}_{f(x)}$ in Y, there exists $A \in \mathcal{B}_x$ in X such that $f(\alpha(A)) \subset \alpha(B)$.*

Proposition 3.4. *If $f : X \to Y$ is β -continuous, then $f([A]_\beta) \subset [f(A)]_\beta$, for $A \subset X$.*

Proof. Let for some $A \subset X$ and some $x \in X, f(x) \notin [f(A)]_\beta$. Then there exists $V \in \mathcal{B}_{f(x)}$ such that $\alpha(V) \cap f(A) = \emptyset$. Since f is β -continuous, there is $U \in \mathcal{B}_x$ in X such that $f(\alpha(U)) \subset \alpha(V)$. Then $f(\alpha(U)) \cap f(A) = \emptyset$ which shows that $\alpha(U) \cap A = \emptyset$. Hence $x \notin [A]_\beta$ so that $f(x) \notin f([A]_\beta)$.

Theorem 3.5. *If a β -continuous function $f : X \to Y$ is β -perfect, then f is β -colsed and $f^{-1}(y)$ is a β -set for each $y \in Y$.*

Proof. It has already been proved in Result 2.14 that a β -perfect function is β -closed. Let now $y \in Y$ and $\mathcal{F}$ be a filterbase on X such that $f^{-1}(y) \cap \beta$ -ad $\mathcal{F} = \emptyset$. Then $y \notin f(\beta - ad\mathcal{F})$ so that $y \notin \beta$ -ad $f(\mathcal{F})$, by Theorem 2.12. Thus $y \notin [f(F)]_\beta$, for some $F \in \mathcal{F}$ so that there exists $V \in \mathcal{B}_y$ with $\alpha(V) \cap f(F) = \emptyset$. Since f is β -continuous, for each $x \in f^{-1}(y)$ there is $U_x \in \mathcal{B}_x$ in X such that $f(\alpha(U_x)) \subset \alpha(V) \subset Y - f(F)$. Then $f(\alpha(U_x)) \cap f(F) = \emptyset$ so that $\alpha(U_x) \cap F = \emptyset$. Hence $x \notin [F]_\beta$, for all $x \in f^{-1}(y)$. In otherwords, $f^{-1}(y) \cap [F]_\beta = \emptyset$. Consequently, $f^{-1}(y)$ is a β -set in X.

Combining Theorem 3.2 and Theorem 3.5 we obtain:

Theorem 3.6. *A β -continuous function $f : X \to Y$ is β -perfect iff f is β -closed and $f^{-1}(y)$ is a β -set, for each $y \in Y$.*

It now follows from the above theorem and Proposition 3.4 that

Theorem 3.7. *A β -continuous function $f : X \to Y$ is β -perfect iff $f([A]_\beta) = [f(A)]_\beta$, for each $A \subset X$ and $f^{-1}(y)$ is a β -set for each $y \in Y$.*

Definition 3.8. *A subset A of X is said to be y -set if for every cover $\mathcal{U}$ of A by members of $\mathcal{B}$, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subset \cup\{\alpha(B) : B \in \mathcal{U}_0\}$.*

Theorem 3.9. *For a subset A of a space X, the following are equivalent:*

a) A is y -set.

b) Every maximal filterbase on X which meets A β - converges to some point of A.

c) Every filterbase on X which meets A, has a β -adherent point in A.

Proof. *a) $\Rightarrow$ b)* : Let A be a y -set and suppose that $\mathcal{U}$ is a maximal filterbase on X which meets A, but does not β -converge to any point of A. Then for each $x \in A$, there exists a $B_x \in \mathcal{B}_x$ such that $U \cap (X - \alpha(B_x)) \neq \emptyset$, for every $U \in \mathcal{U}$. The maximality of $\mathcal{U}$ implies that $(X - \alpha(B_x)) \in \mathcal{U}$. Then $U_x \cap \alpha(B_x) = \emptyset$ for some $U_x \in \mathcal{U}$. Since $\{B_x : x \in A\}$ is a cover of A by members of $\mathcal{B}$ and A is a y -set, we have $A \subset \cup_{i=1}^{n} \alpha(B_{x_i})$, where $\{x_1, ..., x_n\}$ is a finite subset of A. We can find $U \in \mathcal{U}$ such that $U \cap A \subset (\cap_{i=1}^{n} U_{x_i}) \cap (\cup_{i=1}^{n} \alpha(B_{x_i})) = \emptyset$ and this is a contradiction, since $\mathcal{U}$ meets A.

b) $\Rightarrow$ c) : Let $\mathcal{F}$ be a filterbase on X which meets A. Then $\mathcal{F}$ is contained in a maximal filterbase $\mathcal{F}_0$ which meets A. Since $\mathcal{F}_o$ β -converges to some $x \in A$ (by b)), for every $V \in \mathcal{B}_x$ there exists an $F_0 \in \mathcal{F}_o$ such that $F_0 \subset \alpha(V)$. Since $F \cap F_0 \neq \emptyset$ for each $F \in \mathcal{F}$, we have $\alpha(V) \cap F \neq \emptyset$. for all $F \in \mathcal{F}$ and all $V \in \mathcal{B}_x$. Hence $x (\in A)$ is a β -adherent point of $\mathcal{F}$.

c) $\Rightarrow$ a): If A is not a y-set , there is a cover $\mathcal{U}$ of A by members of $\mathcal{B}$ such that for every finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$, $A - \cup_{U \in \mathcal{U}_0} \alpha(U) \neq \emptyset$. Then

$$\mathcal{F} = \{X - \cup_{U \in \mathcal{U}_0} \alpha(U) : \mathcal{U}_0 \text{ is a finite subfamily of } \mathcal{U}\}$$

is a filterbase on X which meets A. By c), there exists $x \in A$ such that $x \in \beta - ad\mathcal{F}$. Since $\mathcal{U}$ is a cover of A, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Then $U_x \in \mathcal{B}_x$ and $X - \alpha(U_x) \in \mathcal{F}$ so that $x \notin [X - \alpha(U_x)]_\beta$ which is a contradiction.

Theorem 3.10. *If $f : X \to Y$ is β -perfect, then inverse images of y -sets are y -sets.*

Proof. Let A be a y -sets in Y and to avoid triviality we assume that $f^{-1}(A) \neq \emptyset$. Let $\mathcal{F}$ be a filterbase on X which meets $f^{-1}(A)$ and set $\mathcal{G} = \{f^{-1}(A) \cap F : F \in \mathcal{F}\}$. Then $f(\mathcal{G})$ is a filterbase on Y which meets A and $f(\beta - ad\ \mathcal{G}) \cap A \supset \beta - ad\ f(\mathcal{G}) \cap A \neq \emptyset$ (by Theorems 3.9 and 2.12). β being increasing, we have β -ad $\mathcal{F} \cap f^{-1}(A) \neq \emptyset$. Hence $f^{-1}(A)$ is a y -set by Theorem 3.9.

In certain usual situations it is found that there remains no difference between the notions of β-sets and y -sets. For example, as we shall see in the next section, such a collapse takes place when the operator α is taken to stand for the identity operator and $\mathcal{B}$ for the collection of all open sets in a topological space X. Under such a condition which we shall denote by 'Condition C' (i.e., when the concepts of β -sets and y -sets coincide), the requirement of β -continuity in Theorem 3.6 can be dispensed with, i.e., we have:

Theorem 3.11 *Under 'Condition C' a function $f : X \to Y$ is β -perfect iff f is β -closed and $f^{-1}(y)$ is a y -set for each $y \in Y$.*

Proof. Follows from Proposition 2.14, Theorem 3.2 and Theorem 3.10.

Definition 3.12. *A space X is said to be α -separated if for each pair of distinct points x, y of X, there exist $U_x \in \mathcal{B}_x$ and $U_y \in \mathcal{B}_y$ such that $\alpha(U_x) \cap \alpha(U_y) = \emptyset$.*

Theorem 3.13. *Let $f : X \to Y$ be β - continuous and Y be an α -separated space. Then f is β -perfect iff for every filterbase $\mathcal{F}$ on X, if $f(\mathcal{F}) \to^{\beta} y \in f(X)$ then β -ad $\mathcal{F} \neq \emptyset$.*

Proof. Suppose f is β -perfect , and $f(\mathcal{F}) \to^{\beta} y \in f(X)$. So $f^{-1}(f(\mathcal{F})) \to^{\beta-d} f^{-1}(y)$. $\mathcal{F}$ being finer than $f^{-1}(f(\mathcal{F}))$, β -ad $\mathcal{F} \neq \emptyset$. Conversely, suppose $\mathcal{G}$ is a filterbase on $f(X)$ such that $\mathcal{G} \to^{\beta} y \in f(X)$, and suppose $\mathcal{H}$ is a filterbase on X finer than $f^{-1}(\mathcal{G})$. Then $f(\mathcal{H})$ is finer than $\mathcal{G}$. Hence β -ad $\mathcal{H} \neq \emptyset$. Let $z \in Y - \{y\}$. Y being an α -separated space, there exist $U_z \in \mathcal{B}_z$ and $U_y \in \mathcal{B}_y$ in Y such that $\alpha(U_z) \cap \alpha(U_y) = \emptyset$. Since $f(\mathcal{H}) \to^{\beta} y$, there exists $H \in \mathcal{H}$ such that $f(H) \subset \alpha(U_y)$. For each $x \in f^{-1}(z)$, there exists $V_x \in \mathcal{B}_x$ such that $f(\alpha(V_x)) \subset \alpha(U_z)$. Hence $\alpha(V_x) \cap H = \emptyset$. It follows that $f^{-1}(z) \cap \beta$ -ad $\mathcal{H} = \emptyset$, for each $z \in Y - \{y\}$. So β -ad $\mathcal{H} \cap f^{-1}(y) \neq \emptyset$ and hence f is β -perfect.

Corollary 3.14. *If $f : X \to Y$ is β -continuous, X is y -set and Y is α -separated, then f is β -perfect.*

Proof. Since X is a y-set, then every filterbase on X has non-void β -adherence. Now the corollary follows directly from Theorem 3.13.

Theorem 3.15 *The composition of β -perfect functions is β -perfect, if 'Condition C' is satisfied.*

Proof. Let $f : X \to Y, g : Y \to Z$ be β -perfect functions. Then in view of Proposition 2.14, $g \circ f$ is clearly a β -closed function. If $z \in Z$, then $g^{-1}(z)$ is a y-set. As f is β -perfect, by Theorem 3.10 it follows that $f^{-1}(g^{-1}(z))$ is a y-set in X. Hence $g \circ f$ is β -perfect, by Theorem 3.11.

4. β -perfect functions in some particularized settings. In this section we tabulate and illustrate the special behaviours of our unfied theory under certain well-known sets of conditions. We start with the collection of open sets in a topological space X, and put $\mathcal{B} = \cup_{x \in X} \mathcal{B}_x$, where the members of $\mathcal{B}_x$ are open neighbourhoods of x (for $x \in X$). We discuss below the behaviours of β -perfect functions between topological spaces X, Y and the associated notions for three well-known situations for different interpretations of the operators α and β.

When α stands for identity (closure, interior-closure) operator, the operator β becomes the closure (resp. θ -closure [11], δ -closure [11]) operator. Then the notion of β -adherent point of a filterbase reduces to its adherent point (resp. θ -adherent point [3,2], δ -adherent point [11,7]). A filterbase is β -directed towards a set A or a point x becomes equivalent to it being directed [12] (resp. θ -directed [2], δ - directed [7]) towards A or x. β -closure of a set takes the meaning of its closure (resp. θ closure [11] or almost closure [3], δ -closure [11]) and β -convergence of a filterbase becomes its ordinary convergence (resp. θ -convergence [11] or almost convergence [3], δ -convergence [11]). β -continuity in the three respective cases stands for continuity, θ -continuity [4] and δ -continuity [8] respectively. In the first two cases α - separated spaces become identical with Hausdorff and Urysohn spaces respectively, while in the third case β -closed functions are nothing but the δ -closed ones [11]. Finally, the three widely studied types of functions viz. perfect, θ -perfect and δ -perfect functions are achieved as special cases of β -perfect functions in the three respective cases under consideration.

We sum up and summarize the above discussions in the form of the following table:

Table - A

When α stands for	*β stands for*	*β − set becomes*	*y − set becomes*	*β − perfect function becomes*
Identity operator	*Closure operator*	*Compact set*	*Compact set*	*Perfect function* [12, 10]
Closure operator	*θ − closure operator*	*θ − rigid set*	*QHC −set*	*θ − perfect function* [2, 3]
Interior closure operator	*δ − closure operator*	*N − set*	*N − set*	*δ − perfect function*[7]

We now tabulate below certain well -known results on perfect, θ -perfect and δ -perfect functions, which follows as special cases of the results that we have derived here by taking the operator α to stand for the identity or the closure or the interior - closure operator. In the following table, the theorems, propositions etc., in column - I refer to those of this paper, while the theorems, corollaries etc. shown in column - II denote the results that are the particular forms of the corresponding ones in column-I.

Table - B

I		II
Proposition 2.10	:	Result 2(d) [12], Proposition 2.9 [3] (with A a singleton)
Theorem 2.12 along with Theorem 2.10	:	Result 2 (e) [12], Theorem 3.1 [3], Theorem 3.1 [7]
Proposition 2.14	:	Theorem 2.2 [7], Theorem 3.1.1 ((a) and (b)) [3]
Proposition 3.4	:	Corollary 2.10.1 [3], Theorem 2.2 ((1) $\Rightarrow$ (3)) [8] and a well - known result on continuity
Theorem 3.7	:	Theorem 3.4 [3]
Theorem 3.9	:	Theorem 2.6 [3], Theorem 2 [5], Lemma 3.3[7] and some well - known characterizations of compact sets.
Theorem 3.10	:	Corollary 2 [12], Theorem 3.4 [7] and Theorem 3.1.1 (c) [3].
Theorem 3.11 together with Theorem 2.12 (a) $\Leftrightarrow$ b))	:	Corollary 1 [2], Theorem 3 [12] and Theorem 3.5 [7].
Theorem 3.13	:	Theorem 3.7 [3]
Corollary 3.14	:	Theorem 3.7.1 [3]
Theorem 3.15	:	Theorem c (1.8)[10], Theorem 3.1 [2].

Apart from those shown in the above table, we derive certain new results on δ -perfect functions from Theorem 3.7 and Corollary 3.14, when α becomes the interior - closure operator. These are formulated as follows:

Theorem 4.1 *Let X, Y be topological spaces and $f : X \to Y$ be δ -continuous with Y_s (the semiregularization space of Y [10]) a Hausdorff space. The f is δ -perfect iff for any filterbase $\mathcal{F}$ on X, $f(\mathcal{F})$ δ -converges to $y(\in Y)$ implies δ -ad $\mathcal{F} \neq \emptyset$.*

Theorem 4.2. *Let X be a nearly compact space and Y be a topological space with Y_s a T_2 - space. If a function $f : X \to Y$ is δ -continuous, then f is δ -perfect.*

Moreover, Theorem 3.2 shows that in the converse of Theorem 3.4 of Dickman and Porter [3], the condition of θ -continuity is superfluous, also the condition of almost closedness on f can be weakened by the condition that $[f(A)]_\theta \subset f([A]_\theta)$.

We conclude with the remark that many other perfect - like functions than those considered so far, could be accomplished by assigning different interpretations to the operators α and β and / or by replacing the collection $\mathcal{B}$ of open sets some other collection like α -open sets [6] or regularity open sets etc.

REFERENCES

1. Carnahan, D. - *Locally nearly compact spaces*, Boll. Un. Mat. Ital. (4)6(1972), 146 - 153.
2. D'Aristotile, A.J. - *On θ -perfect mappings*, Boll. Un. Mat. Ital. 9(1974), 655-661.
3. Dickman, R.F.Jr., Porter, J.R. - *θ -perfect and θ -absolutely closed functions*, Illinois Jour. Math. 21(1977), 42-60.
4. Fomin, S. - *Extensions of topological spaces*, Annals of Math. 44(3)(1943), 471-480.
5. Mathur, A. - *Some weaker forms of compactness*, Proc. Kanpur topological conference (1968), 181-191.
6. Njastad, O. - *On some classes of nearly open sets*, Pacific Jour. Math. 15(1965), 961 - 970.
7. Noiri, T. - *A generalization of perfect functions*, Jour. London Math. Soc. 17(2)(1978), 540- 544.
8. Noiri, T. - *On δ -continuous functions*, Jour. Korean Math. Soc. 16(1980), 161 - 166.
9. Porter, J., Thomas, J.D. - *On H -closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc. 138(1969), 159 - 170.
10. Porter, J.R., Woods, R.G. - *Extensions and absolutes of Hausdorff spaces*, Springer - Verlag New-York Inc. (1988).
11. Velicko, N.V. - *H -closed topological spaces*, Amer. Math. Soc. Transl. 78(1968), 103-118.
12. Whyburn, G.T. - *Directed families of sets and closedness of functions*, Proc. Nat. Acad. Sci. USA 54(1965), 688 - 692.

Received:19.XI.1992

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta - 700 019
INDIA

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXIX, s.I.a, Matematică, 1993, f1

THE REFLEXIVE PARADOXES IN SET THEORY: THE LOGICAL RELATION OF THE ABIAN PARADOX TO THE CURRY PARADOX

BY

S.M. KIM

The Abian paradox, formulated in 1985, is a generalization of the Russel paradox because it applies to both sets and classes.

We formulate the Abian paradox as

$$(*) \qquad (Ax)(A \in x \longleftrightarrow x \notin x)$$

To prove that the Abian paradox applies to both sets and classes, we show that A in $(*)$ cannot be a class, regardless of whether quantification in $(*)$ is over sets or classes. Indeed replacing x in $(*)$ by Φ (which is both a set and a class), we arrive at the contradiction $A \in \Phi \longleftrightarrow \Phi \notin \Phi$ since $A \in \Phi$ is always false and $\Phi \notin \Phi$ is always true. It is not difficult to show that the Abian paradox does not totally depend upon the existence of Φ.

The Russel paradox is weaker than the Abian paradox, since the Russel paradox fails in cases where the Abian paradox holds. The reflexive paradoxes have been analyzed by Goddard, Johnston [1], Goddard [2] as arising in quantified formulas of the form $\neg(\exists)(\forall y)[f(y,x) \longleftrightarrow \neg f(y,y)]$, where a property f presented in the formula entails the negation of an quantifier instantiation for the formula containing presuppose any such condition(s). They argue that resolution of such reflexive paradoxes is achieved only if the value chosen to instantiate the existential quantifier is some how removed from the range of the universal quantifier. The Abian paradox exhibits this structure of reflexive paradoxes presented by Goddard and Johnston and requires the same conditions for its resolution.

The present status of research described above justifies in raising the open question in the context of Russel's paradox: How to define a paradoxical class (or to point to some already defined one), the paradoxality of which cannot be generated by the uniform principle which is, in fact, the application of Cantor's theorem to the universe of all classes. This question has not been answered completely but this deserves answer, and giving this answer is the subject of this project.

Definition. *A class A for which there is an infinite progression of classes $A_1, A_2, \ldots$ (not necessarily all distinct) such that $\ldots \in A_2 \in A_1 \in A$ is said to be groundless.*