ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XL, s.I.a, Matematică, 1994, f1.

DISCONTINUOUS AUTOMORPHISMS IN COMPACT RINGS

BY

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It is known that for some classes of compact rings, like: a) the class of semisimple compact rings [1]; b) the class of left noetherian compact rings [2]; c) the class of totally disconnected compact topologically finitely generated rings [3]* (see also [4]), the compact topology is unique. The question about uniqueness of compact topologies on rings is closely related to the question about the existence of the discontinuous automorphisms in compact rings. Namely, if the topology of a compact ring is unique then each of its automorphisms is continuous. The author does not know the answer to the following question: Has a compact ring R whose topology is not unique a discontinuous automorphism ?

In the paper, the sufficient conditions for the existence of a discontinuous automorphism for a compact ring are given: Let R be a compact ring. If all idempotents of R are central and the component of zero is different from $\{0\}$, then R has a discontinuous automorphism.

All topological rings in discussion are assumed to be Hausdorff and associative. The set of positive integers will be denoted by N. The component of zero of a topological ring R will be denoted by R_0. If x is an element of R, then the subgroup of the additive group of R generated by x is denoted by $< x >^+$.

Lemma 1. *If R is a compact ring, $R_0 \neq \{0\}, R/\bar{R}^2$ is totally disconnected and R has an element x of infinite order, then R has a discontinous automorphism.*

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* Every compact left Noetherian ring is totally disconnected and topologically finitely generated.

Proof. We claim that there exists at least one discontinuous homomorphism $\alpha : R/\bar{R}^2 \to R_0$. Assume the contrary. If $a \in R_0$, then the map $x \mapsto a$ can be extended to a homomorphism $\varphi : < x >^+ \to < a >^+$. Since R_0 is a divisible group, the homomorphism φ can be extended to certain homomorphism $\Phi : R/\bar{R}^2 \to R_0$. Then $\overline{< a >^+} \subset \overline{\Phi(< x >^+)}$ and so $\overline{< a >^+}$ is totally disconnected. We obtain that R_0 is also totally disconnected, which is a contradiction. Now we try to obtain a discontinuous automorphism δ, using α and the canonical homomorphism $\beta : R \to R/\bar{R}^2$. Then $\gamma = \alpha \circ \beta$ is a ring homomorphism, having the following properties:

1) $\gamma(\bar{R}^2) = \{0\}$;

2) γ is discontinous;

3) $\gamma^2 = 0$;

4) $\gamma(R) \subset$ Ann (R).

Here, 2) follows from the fact that β is open, and 1) follows from the construction. It is obvious that $\gamma(R) \subset R_0$. Since $R/\bar{R}^2$ is totally disconnected and β is continuous, $\beta(\gamma(R)) \subset \beta(R_0) = \{0\}$, or $\beta(\gamma(R)) = \{0\}$, therefore $\alpha(\beta(\gamma(R))) = \{0\}$, i.e. $\gamma^2(R) = \{0\}$. 4) follows from [5] and the inclusion $\gamma(R) \subset R_0$.

The map $\delta = 1 + \gamma$ is a discontinuous endomorphism of the additive group of R. If $x, y \in R$, then $\delta(xy) = xy + \gamma(xy) = (x + \gamma(x))(y + \gamma(y)) = \delta(x) \cdot \delta(y)$, i.e. δ is a ring endomorphism. Since $\gamma^2 = 0, \delta = 1 + \gamma$ is a ring automorphism.

Lemma 2. *If D is an Abelian torsion - free divisible group, then it can be equipped in a unique manner with a structure of a vector space over Q. Subspaces are exactly divisible subgroup of D.*

Lemma 3. *Let D_1 and D_2 be two vector spaces over Q and $\varphi : D_1 \to D_2$ a group homomorphism. Then φ is a linear map.*

Bellow we shall use the following fact from the theory of compact groups: If $f : A \to B$ is a surjective continuous homomorphism, then $f(A_0) = B_0$.

Lemma 4. (see [6], 25.23; we give here another proof). *Every nonzero compact connected Abelian group C contains a subgroup which is isomorphic (in the algebraic sense) to the group $\mathbb{R}$ of reals.*

Proof. Let χ be a nontrivial character of the group C. Then $\chi : C \to T$ is a surjective map (where $T = \mathbb{R}/\mathbb{Z}$). If $t(T)$ is the subgroup of the elements of finite order of T, then $T/t(T)$ is algebraically isomorphic to $\mathbb{R}$. We get a sequence $C \to^{\chi} T \to^{k} T/t(T)$ of homomorphisms, where k is the canonical homomorphism of T onto T/t (T). The group C is divisible. Let $C = A \oplus t(C)$ be the direct sum, where t(C) is the subgroup of the elements of finite order of C. Let $\lambda = k \cdot \chi$. Then $\lambda(C) = \lambda(A) + \lambda(t(C)) = \lambda(A)$. Therefore $\lambda|_A$ is a homomorphism from A onto T / t(T). By the lemma 3, $\lambda|_A$ is a linear map from the vector Q - space A onto the vector Q - space T /t(T), which is isomorphic to the Q - space $\mathbb{R}$. The last part is obvious.

Lemma 5. *If R is a compact ring and $R/\bar{R}^2$ is not totally disconnected, then R has a discontinuous automorphism.*

Proof. If $\varphi : R \to R/\bar{R}^2$ is the canonical homomorphism, then $\varphi(R_0) = (R/\bar{R}^2)_0$. By the lemma 4, there exists a subgroup $L \subset (R/\bar{R}^2)_0$ isomorphic to $\mathbb{R}$. Therefore there exist two subgroups $A \subset L$ and $B \subset L$, algebraically isomorphic to Q and a subgroup C of $R/\bar{R}^2$ such that $R/\bar{R}^2 = A \oplus B \oplus C$ (direct sum).

Let $\Pi : R/\bar{R}^2 \to A \oplus B$ be the projection of $R/\bar{R}^2$ onto $A \oplus B$ and $\lambda : A \oplus B \to B$ the projection of $A \oplus B$ onto B. The homomophism $\Pi \circ \varphi : R \to A \oplus B$ is a surjection.

The group R_0 is divisible and so $R_0 = D \oplus K$, where D is a divisible torsion - free subgroup and K is a torsion divisible subgroup. Since $A \oplus B \subset (R/\bar{R}^2)_0$, it follows that $A \oplus B = \Pi((R/\bar{R}^2)_0) = \Pi \circ \varphi(R_0)$. Therefore $A \oplus B = \Pi \circ \varphi(R_0) = \Pi \circ \varphi(D \oplus K) = \Pi \circ \varphi(D)$.

The groups D and $A \oplus B$ are divisible and torsion - free. By the lemmas 2 and 3, D and $A \oplus B$ are vector Q - spaces and $\Pi \circ \varphi|_D$ is a linear map from the vector Q - space D onto the vector Q - space $A \oplus B$. Since A and B are divisible groups, by lemma 2, they are Q - subspaces. It is obvious that A and B are one - dimensional over Q. Therefore there exists an one - dimensional Q - subspace E in D such that $\Pi \circ \varphi(E) = A$. Obviously, $\Pi \circ \varphi|_E$ is an isomorphism. There exists a group isomorphism $\gamma : B \to^{on} E$.

We obtained a sequence of ring homomophisms $R \to^{\varphi} R/\bar{R}^2 \to^{\pi} A \oplus B \to^{\lambda} B \to^{\gamma} E \subset R_0 \subset$ Ann (R), where homomorphisms $\varphi, \Pi, \lambda, \gamma$ are onto. Let $\alpha = \gamma \circ \lambda \circ \Pi \circ \varphi : R \to R$. We list here some properties of the ring endomorphism α :

1) α is discontinuous.

Indeed, $\alpha(R) = E$. E is countable and there exist no compact countable groups.

2) $\alpha(R) \subset$ Ann (R).

3) $\alpha(\bar{R}^2) = \{0\}$.

4) $\alpha^2 = 0$.

Indeed, from $\alpha(R) = E$, it follows that $\Pi \circ \varphi \circ \alpha(R) = \Pi \circ \varphi(E) = A$. Then $\lambda \circ \Pi \circ \varphi \circ \alpha(R) = \lambda(A) = \{0\}$ and $\gamma \circ \lambda \circ \Pi \circ \varphi \circ \alpha(R) = \{0\}$ or $\alpha^2(R) = \{0\}$.

As in the proof of the lemma 1, $1 + \alpha$ is a discontinuous ring automorphism.

Theorem 1. *If in a compact ring R all idempotents are central and $R_0 \neq \{0\}$, then R has a discontinuous automorphism.*

Proof. We do not lose the generality, if we assume that R is quasiregular. Indeed, let $\varphi : R \to R/J(R)$ be the canonical homomorphism, where J(R) is the Jacobson radical of R. If e_1 is the identity of the ring R/J(R), then there exists (in that case a unique) idempotent $e \in R$ such that $\varphi(e) = e_1$.

Let R = Re ⊕ R(1-e) be the Pierce decomposition of the ring R. It is a topological decomposition and, moreover, $R(1-e) = \{x \in R : xe = 0\} \subset J(R), R_0 \subset R(1-e)$. Therefore it is sufficient to prove the theorem for the ring R (1-e) and so we may assume that R is quasiregular.

Since R is pronilpotent [7], it is true that $R \neq \bar{R}^2$. If $R/\bar{R}^2$ is not totally disconnected, the theorem follows from the lemma 5.

Now let $R/\bar{R}^2$ be totally disconnected. If $R/\bar{R}^2$ contains an element of infinite order, the theorem follows from the lemma 1.

Assume that $R/\bar{R}^2$ is a torsion group. Then there exists $k \in I\!N$ such that $kR \subset \bar{R}^2$. By induction, $k^n R \subset \overline{R^{n+1}}$, for every $n \in I\!N$.

Since R is a pronilpotent ring, it follows that for every $r \in R$, $lim_{n\to\infty} k^n r = 0$. Let χ be a character of the group R such that $\chi(R) = T(= I\!R/Z\!\!Z)$. If p is a prime number, $p > k$ then there exists a subgroup K of T such that $|K| = p$. We have $k^m x = 0$ for some $m \in I\!N$ and px = 0, for every $x \in K$. Then $x = 0$ which is a contradiction.

Corollary 1. *If in a compact ring R all idempotents are central and the topology is unique, then R is totally disconnected.*

Question: Is the theorem true for every compact ring with nontrivial component of zero ?

In the connection with the theorem 1, it is interesting to give examples of compact rings, all idempotents of which are central and for which the component of zero is not an algebraic direct summand. We will show such examples in the class of nilpotent of index three compact rings.

First we shall indicate a construction in the category of topological rings. Let A be a topological ring, B a topological Abelian group and $\varphi : A \to B$ a continuous homomorphism. On the direct product $R = A \times B$ of topological groups A and B, we define the multiplication in the following way: $(a,b)(a',b') = (0, \varphi(aa')), a, a' \in A, b, b' \in B$. It is a routine to prove that R is a topological ring. $R^3 = \{0\}, R^2 \subset 0 \times B, (\{0\} \times B)R = R(\{0\} \times B) = \{0\}$.

We claim that if $\varphi(A^2) \neq 0$, then there exists no subring S of R such that $R = S + \{0\} \times B$ and $S \cap (\{0\} \times B) = \{0\}$. Assume the contrary. Then S is (algebraically) isomorphic to $R/\{0\} \times B$ and therefore $S^2 = \{0\}$. By the equality $R = S + \{0\} \times B$, it follows $R^2 = \{0\}$. On the other hand, there exist $a, a' \in A$ such that $\varphi(a,a') \neq 0$. Then $(a,0)(a',0) = (0, \varphi(aa')) \neq 0$, i.e. $R^2 \neq \{0\}$– a contradiction.

In particular, let F be an arbitrary finite ring with identity and $\chi : F \to T$ a character of the additive group of F such that $\chi(1) \neq 0$. Then, on the group $F \times T$, the structure of a compact ring is defined and its component $\{0\} \times T$ is not an algebraic direct summand.

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Received:2.IV.1993

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