Finalement, de (6), à cause de la parité des vecteurs $\chi_{-1}, \chi_0, \chi_2, \dots, \chi_{p-3}$, respectivement de l'imparité des vecteurs $\chi_1, \chi_3, \dots, \chi_{p-2}$ et de (i) - (iv) il résulte:

Théorème. *Les vecteurs* $(F_p \pm I)\chi_{2l}, l = \{0, 1, \dots, [\frac{p-1}{4}]\}$ *et les vecteurs* $(iF_p \pm I)\chi_{2l+1}, l \in \{0, 1, \dots, [\frac{p-3}{4}]\}$ *avec* $\chi_{\frac{p-1}{2}}$ *forment une base (de vecteurs propres pour* F_p*) dans* $L^2(Z_p)$*. Par rapport à cette base, la repésentation matricielle de* F_p *est diagonale, les ordres de multiplicité des valeurs propres étant*

	1	-1	i	$-i$
$p \equiv 1 (mod\ 4)$	$\frac{p+3}{4}$	$\frac{p-1}{4}$	$\frac{p-1}{4}$	$\frac{p-1}{4}$
$p \equiv 3 (mod\ 4)$	$\frac{p+1}{4}$	$\frac{p+1}{4}$	$\frac{p+1}{4}$	$\frac{p-3}{4}$

BIBLIOGRAPHIE

1. Auslander, L., Tolimieri, R. – *Is Computing with the Finite Fourier Transform Pure or Applied Mathematics ?*, Bulletin of the American Math. Society (New Series), Vol. No. 6 (1979), 847 - 897.

2. Auslander, L., Tolimieri, R. – *Ring Structure and the Fourier Transform*, Math. Intelligencer, Vol. 7, No. 3 (1985), 49 - 53.

Received:2.IV.1993

University of Braşov
Braşov
ROMANIA

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XL, s.I.a, Matematică, 1994, f.1.

ON SOME INEQUALITIES OF THE WEYL TYPE

BY

B.G. PACHPATTE

1. Introduction. One of the most interesting and fundamental mathematical discoveries of Hermann Weyl [8] (see, also [4, p. 165]) is the following inequality:

$$\int_0^\infty f^2 dx \le 2\left(\int_0^\infty x^2 f^2 dx\right)^{\frac{1}{2}} \left(\int_0^\infty f'^2 dx\right)^{\frac{1}{2}}, \tag{1}$$

where f is a continuously differentiable function on $(0, \infty)$ and the integrals on the right side exist. This inequality is remarkable in terms of its simplicity, the large number of results to which it leads, and the variety of applications which can be related to it. In the past few years some useful generalizations and extensions of the inequality (1) are established by the various investigators, see [2,3,6,7]. In view of the importance of this inequality in the literature, it is natural to expect that the inequalities of the type (1) would also be equally important in certain new applications. The aim of the present paper is to establish some new integral inequalities involving functions and their derivatives which claim their origin to the Weyl's inequality given in (1). The analysis used in the proofs is elementary and our results provide new estimates on these types of inequalities.

2. Statement of results. In this section we state our results to be proved in this paper. In what follows, we assume that all the integrals appearing below exist on the respective domains of their definitions.

Our main result is given in the following theorem.

Theorem 1. *Let* $\alpha \ge 0$, $p \ge 0$, $q \ge 1$ *be real constants and* f *be a real-valued continuously differentiable function defined on* $(0, b)$ *for fixed real number* $b > 0$. *Then*

$$\int_0^b t^\alpha |f(t)|^{p+q} dt \le \tag{2}$$

$$\leq M\left(\int_0^b t^{\alpha+1}|f(t)|^p\left(\frac{|f(t)|}{b}+|f'(t)|\right)^q dt\right)^{\frac{1}{q}}\left(\int_0^b t^{\alpha+1}|f(t)|^{p+q}dt\right)^{\frac{q-1}{q}};$$

$$\int_0^b t^\alpha |f(t)|^{p+q}dt \leq \tag{3}$$

$$\leq M\left(\int_0^b t^{(\alpha+1)q}|f(t)|^p\left(\frac{|f(t)|}{b}+|f'(t)|\right)^q dt\right)^{\frac{1}{q}}\left(\int_0^b |f(t)|^{p+q}dt\right)^{\frac{q-1}{q}},$$

where $M=\max\left\{\frac{\alpha+2}{\alpha+1},\frac{2(p+q)}{\alpha+1}\right\}$.

Remark 1. It is interesting to note that, if the function f is continuously differentiable on $(0,\infty)$, then letting $b\to\infty$ in (2),(3) we get respectively the following inequalities

$$\int_0^\infty t^\alpha|f(t)|^{p+q}dt \leq M\left(\int_0^\infty t^{\alpha+1}|f(t)|^p|f'(t)|^q dt\right)^{\frac{1}{q}}\cdot\left(\int_0^\infty t^{\alpha+1}|f(t)|^{p+q}dt\right)^{\frac{q-1}{q}}. \tag{4}$$

$$\int_0^\infty t^\alpha|f(t)|^{p+q}dt \leq M\left(\int_0^\infty t^{(\alpha+1)q}|f(t)|^p+|f'(t)|^q dt\right)^{\frac{1}{q}}\cdot\left(\int_0^\infty |f(t)|^{p+q}dt\right)^{\frac{q-1}{q}}. \tag{5}$$

In particular, if we take $\alpha=0$, $p=0$, $q=2$, then the inequalities obtained in (4),(5) reduce to the slight variants of the Weyl's inequality given in (1).

A slightly different version of Theorem 1 is established in the following theorem.

Theorem 2. *Let α,p,q,f be as defined in Theorem 1. Then*

$$\int_0^b t^\alpha|f(t)|^{p+q}dt \leq M^q\int_0^b t^{\alpha+q}|f(t)|^p\left(\frac{|f(t)|}{b}+|f'(t)|\right)^q dt. \tag{6}$$

$$\int_0^b t^\alpha|f(t)|^{p+q}dt \leq M^{p+q}\int_0^b t^{\alpha+p+q}\left(\frac{|f(t)|}{b}+|f'(t)|\right)^{p+q} dt. \tag{7}$$

where M is as defined in Theorem 1.

Remark 2. If the function f is continuously differential on $(0,\infty)$, then letting $b\to\infty$ in (6),(7) we get respectively the following inequalities

$$\int_0^\infty t^\alpha|f(t)|^{p+q}dt \leq M^q\int_0^\infty t^{\alpha+q}|f(t)|^p|f'(t)|^q dt, \tag{8}$$

$$\int_0^\infty t^\alpha|f(t)|^{p+q}dt \leq M^{p+q}\int_0^\infty t^{\alpha+p+q}|f'(t)|^{p+q}dt. \tag{9}$$

Here we note that the inequalities obtained in (8) and (9) are similar to that of the Opial's type inequality (see [5]) and those of the well known Hardy's inequality (see [4]).

We next establish the following inequality inspired by the interesting generalization of the inequality (1) given by Benson in [2, p.297].

Theorem 3. *Let α,p,q be as defined in Theorem 1 and let f be a real-valued continuously differentiable function defined on (a,b) for fixed real numbers $a<b$. Then*

$$\int_a^b |t|^\alpha|f(t)|^{p+q}dt \leq |H| + \left(\frac{p+q}{\alpha+1}\right)\left(\int_a^b |t|^{\alpha+1}|f(t)|^p|f'(t)|^q dt\right)^{\frac{1}{q}}\cdot\left(\int_a^b |t|^{\alpha+1}|f(t)|^{p+q}dt\right)^{\frac{q-1}{q}}, \tag{10}$$

$$\int_a^b |t|^\alpha|f(t)|^{p+q}dt \leq |H| + \left(\frac{p+q}{\alpha+1}\right)\left(\int_a^b |t|^{(\alpha+1)q}|f(t)|^p|f'(t)|^q dt\right)^{\frac{1}{q}}\left(\int_a^b |f(t)|^{p+q}dt\right)^{\frac{q-1}{q}}, \tag{11}$$

where

$$H = \frac{1}{\alpha+1}\left[b^{\alpha+1}(sgn\ b)^\alpha|f(b)|^{p+q} - a^{\alpha+1}(sgn\ a)^\alpha|f(a)|^{p+q}\right]. \tag{12}$$

Remark 3. We note that, in the special case when $q = 1$, the inequalities obtained in (10) and (11) reduces to the following inequality

$$\int_a^b |t|^\alpha |f(t)|^{p+1} dt \leq |H_0| + \left(\frac{p+1}{\alpha+1}\right) \int_a^b |t|^{\alpha+1} |f(t)|^p |f'(t)| dt. \tag{13}$$

where H is defined by the right side of (12) by taking $q = 1$. If we put $a = 0$ and hence $|t| = t$ and let $b \to \infty$ in (10),(11), then we obtain

$$\begin{aligned} \int_0^\infty t^\alpha |f(t)|^{p+q} dt \leq \left(\frac{p+q}{\alpha+1}\right) \left(\int_0^\infty t^{\alpha+1} |f(t)|^p |f'(t)|^q dt\right)^{\frac{1}{q}} \cdot \\ \cdot \left(\int_0^\infty t^{\alpha+1} |f(t)|^{p+q} dt\right)^{\frac{q-1}{q}}, \end{aligned} \tag{14}$$

$$\begin{aligned} \int_0^\infty t^\alpha |f(t)|^{p+q} dt \leq \left(\frac{p+q}{\alpha+1}\right) \left(\int_0^\infty t^{(\alpha+1)q} |f(t)|^p |f'(t)|^q dt\right)^{\frac{1}{q}} \cdot \\ \left(\int_0^\infty |f(t)|^{p+q} dt\right)^{\frac{q-1}{q}}. \end{aligned} \tag{15}$$

Here we have used the that that $\lim_{b \to \infty} b^{\alpha+1} |f(b)|^{p+q} = 0$, in order to observe that $H = 0$. This can be shown from the existence of the integrals (see [8, p.393]).

3. Proofs of Theorems 1-3. Integrating by parts, we have the following identity:

$$\int_0^b \left[t^{\alpha+1} - \frac{1}{b} t^{\alpha+2}\right] |f(t)|^{p+q-1} |f'(t)| sgn\ f(t) dt =$$

$$= \int_0^b \left[(\alpha+1)t^\alpha - \frac{1}{b}(\alpha+2)t^{\alpha+1}\right] \frac{|f(t)|^{p+q}}{p+q} dt.$$

Thus we have

$$(\alpha+1) \int_0^b t^\alpha |f(t)|^{p+q} dt = (\alpha+2) \int_0^b t^{\alpha+1} \frac{1}{b} |f(t)|^{p+q} dt -$$

$$- (p+q) \int_0^b t^{\alpha+1} \left[1 - \frac{1}{b} t\right] |f(t)|^{p+q-1} |f'(t)| sgn\ f(t) dt \leq$$

$$\leq (\alpha+2) \int_0^b t^{\alpha+1} \frac{1}{b} |f(t)|^{p+q} dt +$$

$$+ (p+q) \int_0^b t^{\alpha+1} \left[1 + \frac{1}{b} t\right] |f(t)|^{p+q-1} |f'(t)| dt.$$

On the other hand, we observe that

$$\begin{aligned} \int_0^b t^\alpha |f(t)|^{p+q} dt &\leq \left(\frac{\alpha+2}{\alpha+1}\right) \int_0^b t^{\alpha+1} \frac{1}{b} |f(t)|^{p+q} dt + \\ &+ \frac{2(p+q)}{\alpha+1} \int_0^b t^{\alpha+1} |f(t)|^{p+q-1} |f'(t)| dt \leq \\ &\leq M \int_0^b t^{\alpha+1} |f(t)|^{p+q-1} \left[\frac{|f(t)|}{b} + |f'(t)|\right] dt = \\ &= M \int_0^b \left[t^{(\alpha+1)\frac{1}{q}} |f(t)|^{\frac{p}{q}} \left(\frac{|f(t)|}{b} + |f'(t)|\right)\right] \cdot \\ &\cdot \left[t^{(\alpha+1)(\frac{q-1}{q})} |f(t)|^{p+q-1-\frac{p}{q}}\right] dt. \end{aligned} \tag{16}$$

Now by using the Hölder's inequality with indices q, $\frac{q}{q-1}$, we get the required inequality in (2).

In order to establish the inequality (3), we rewrite the last inequality in the following form

$$\int_0^b t^\alpha |f(t)|^{p+q} dt \leq$$

$$\leq M \int_0^b \left[t^{\alpha+1} |f(t)|^{\frac{p}{q}} \left(\frac{|f(t)|}{b} + |f'(t)|\right)\right] \left[|f(t)|^{p+q-1-\frac{p}{q}}\right] dt.$$

By using the Hölder's inequality with indices q, $\frac{q}{q-1}$, we get the desired inequality in (3). This completes the proof of Theorem 1.

By following the same arguments as in the proof of Theorem 1 we have the inequality (6). Rewriting the inequality (16) and using the Hölder's inequality with indices q, $\frac{q}{q-1}$, we have

$$\begin{aligned} \int_0^b t^\alpha |f(t)|^{p+q} dt \leq M \int_0^b &\left[t^{\alpha+1-\alpha(\frac{q-1}{q})} |f(t)|^{\frac{p}{q}} \left(\frac{|f(t)|}{b} + |f'(t)|\right)\right] \\ &\left[t^{\alpha(\frac{q-1}{q})} |f(t)|^{p+q-1-\frac{p}{q}}\right] dt \leq \\ &\leq M \left(\int_0^b t^{\alpha+q} |f(t)|^p \left(\frac{|f(t)|}{b} + |f'(t)|\right)^q dt\right)^{\frac{1}{q}} \cdot \\ &\cdot \left(\int_0^b t^\alpha |f(t)|^{p+q} dt\right)^{\frac{q-1}{q}}. \end{aligned}$$

If $\int_0^b t^\alpha |f(t)|^{p+q}dt = 0$, then (6) is trivially true, otherwise dividing both sides by $\left(\int_0^b t^\alpha |f(t)|^{p+q}dt\right)^{\frac{q-1}{q}}$ and then taking the qth power on both sides of the resulting inequality, we get the required inequality in (6).

Rewriting the inequality (16) and using the Hölder's inequality with indices $p+q$, $\frac{p+q}{p+q-1}$, we have

$$\int_0^b t^\alpha |f(t)|^{p+q}dt \le M\int_0^b \left[t^{\alpha+1-\alpha\left(\frac{p+q-1}{p+q}\right)}\left(\frac{|f(t)|}{b}+|f'(t)|\right)\right]\cdot$$

$$\cdot\left[t^{\alpha\left(\frac{p+q-1}{p+q}\right)}|f(t)|^{p+q-1}\right]dt \le$$

$$\le M\left(\int_0^b t^{\alpha+p+q}\left(\frac{|f(t)|}{b}+|f'(t)|\right)^{p+q}dt\right)^{\frac{1}{p+q}}\cdot$$

$$\cdot\left(\int_0^b t^\alpha |f(t)|^{p+q}dt\right)^{\frac{p+q-1}{p+q}}.$$

Now by following the arguments as in the last part of the proof of inequality (6) with suitable modifications, we get the required inequality in (7). The proof of Theorem 2 is complete.

By rewriting, and integrating by parts the integral on the left side of (10), we have

$$\int_a^b |t|^\alpha |f(t)|^{p+q}dt = \int_a^b t^\alpha (sgn\ t)^\alpha |f(t)|^{p+q}dt =$$

$$= H - \left(\frac{p+q}{\alpha+1}\right)\int_a^b t^{\alpha+1}|f(t)|^{p+q-1}f'(t)sgn\ f(t)(sgn\ t)^\alpha dt.$$

Thus we obtain

$$\int_a^b |t|^\alpha |f(t)|^{p+q}dt \le |H| + \left(\frac{p+q}{\alpha+1}\right)\int_a^b |t|^{\alpha+1}|f(t)|^{p+q-1}|f'(t)|dt \le$$

$$\le |H| + \left(\frac{p+q}{\alpha+1}\right)\int_a^b \left[|t|^{\frac{\alpha+1}{q}}|f(t)|^{\frac{p}{q}}|f'(t)|\right]\left[|t|^{(\alpha+1)\left(\frac{q-1}{q}\right)}|f(t)|^{p+q-1}\right]dt.$$

Now by using again the Hölder's inequality with indices q, $\frac{q}{q-1}$, we get the required inequality in (10).

The proof of the inequality (11) is similar to that of the proof of inequality (10) given above with suitable modifications, so we omit it here. This completes the proof of Theorem 3.

Remark 4. The multidimensional integral inequalities of the Weyl type are also established by the authors in [1,3,6,7] by using different techniques. Here we note that the technique employed to establish the inequalities given in Theorems 1-3 is quite elementary and our results yield in the different special cases various inequalities, most of which, we believe are new to the literature.

REFERENCES

1. A d a m s, R.A. – *Sobolev Spaces*, Academic Press, New York, 1975.
2. B e n s o n, D.C. – *Inequalities involving integrals of functions and their derivatives*, J Math. Anal. Appl. 17 (1967), 292-308.
3. B e r n i s, F. – *Compactness of the support for some nonlinear elliptic problems of arbitrary order in dimension N*, Comm. Partial Diff. Eqs. 9 (1984), 271-312.
4. H a r d y, G.H., L i t t l e w o o d, J.E. and P o l y a, G. – *Inequalities*, Cambridge Univ.Press, Cambridge, 1934.
5. M i t r i n o v i ć, D.S. – *Analytic Inequalities*, Springer-Verlag, Berlin, New York, 1970.
6. P a c h p a t t e, B.G. – *On Sobolev type integral inequalities*, Proc. Royal soc. Edinburgh, 103 A (1986), 1-14.
7. P a c h p a t t e, B.G. – *On some extensions of Rellich's inequality*, Tamkang Jour. Math. 22 (1991), 259-265.
8. W e y l, H. – *The theory of groups and Quantum Mechanics*, (English translation), Dover, New York, 1931.

Received: 10.VII.1993

Department of Mathematics
Marathwada University
Aurangabad 431004
(Maharashtra)
INDIA