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CONFORMAL CHANGE OF FINSLER SPACE WITH 1 - FORM METRIC

BY

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We shall consider an n - dimensional Finsler space F^n with the fundamental function $L(a^\alpha)$, $a^\alpha = a_i^\alpha(x)y^i$, where L is (1) p - homogeneous in $a^\alpha, \alpha = 1, ..., n$, and $a_i^\alpha(x)dx^i$ are n linearly independent differential 1 - forms. This Finsler metric is called a *1 - form Finsler metric*. Finsler spaces with 1 - form metric have been studied by the present author [7] with H. Shimada and have attracted several physicist's attention ([1], [2]).

The purpose of the present paper is to deal with such Finsler spaces from the point of conformal change of metric and to show a condition to be conformally flat in a special sense.

The author expresses his gratitude to Professor M. Hashiguchi, because Hashiguchi's lecture [4] drew his attention to the present subject.
The terminology and notation are referred to the author's book [6].

1. Preliminaries. We shall sketch the theory of Finsler spaces with 1 - form metric based on the 1 - form Finsler connection from a previous paper [7].

The *1 - form Finsler connection* $F1 = (\Gamma_j{}^i{}_k, \Gamma_0{}^i{}_j, C_j{}^i{}_k)$ is defined by

$$\Gamma_j{}^i{}_k = a_\alpha^i \partial_k a_j^\alpha, \tag{1.1}$$

$\Gamma_0{}^i{}_k = y^r\Gamma_j{}^i{}_k$ and an arbitrary tensor field $C_j{}^i{}_k$, where (a_α^i) is the inverse matrix of (a_i^α). Thus the deflection tensor D of $F1$ vanishes.

The torsion and curvature tensors of $F1$ are as follows: The (h) h - torsion tensor T is given by

$$T_j{}^i{}_k = a_\alpha^i T_j{}^\alpha{}_k, \quad T_j{}^\alpha{}_k = \partial_k a_j^\alpha - \partial_j a_k^\alpha. \tag{1.2}$$

The (v) h - and (v) hv - torsion tensors R^1, P^1 vanishes, and (h) hv - and (v) v - torsion tensors C, S^1 are of the general form. The h - curvature tensor R^2 vanishes and the hv - curvature tensor P^2 is written as $P_i{}^h{}_{jk} = -C_i{}^h{}_{k;j}$ where (;) stands for the h - covariant differentiation with respect to F1. Finally the v - curvature tensor S^2 is of the general form.

As for the h - covariant derivative of metric, we have $L_{;k} = 0$ and $g_{ij;k} = 0$ for the fundamental tensor $g_{ij}(x, y)$. If we are concerned with Cartan's C - tensor $C_j{}^i{}_k$, we have $C_j{}^i{}_{k;h} = 0$, so the hv - curvature tensor P^2 vanishes also.

Therefore, from the axiomatical viewpoint ([5], [6]) we have

Proposition 1. *$(\Gamma_j{}^i{}_k, \Gamma_0{}^i{}_j)$ of the 1 - form Finsler connection F1 are uniquely determined by the system of axioms of Berwald type*

$$L_{;k} = 0, \quad D^i_j = 0, \quad P^i{}_{jk} = 0, \quad T_j{}^i{}_k = a^i_\alpha T_j{}^\alpha{}_k, \tag{1}$$

or of Cartan type,

$$g_{ij;k} = 0, \quad D^i_j = 0, \quad T_j{}^i{}_k = a^i_\alpha T_j{}^\alpha{}_k. \tag{2}$$

If the (h) h - torsion tensor T vanishes, then the space is locally Minkowski and an adapted coordinate system $(\bar{x}^\alpha)$ [6] is given by $\partial\bar{x}^\alpha/\partial x^i = a^\alpha_i$. Thus $T = 0$ is a sufficient condition for the space to be locally Minkowski, but not necessary. Thus we shall say

Definition. *A 1 - form Finsler space is called T - Minkowski,* if the (h) h - torsion tensor T of the 1 - form Finsler connection vanishes.

2. Conformal change of metric. We deal with a conformal change $L \to \bar{L} = e^{\sigma(x)}L$ of a 1 - form metric $L(a^\alpha)$. On account of the homogeneity with respect to a^α we have $\bar{L} = L(e^{\sigma(x)}a^\alpha)$, so we get

$$\bar{L}(x, y) = L(\bar{a}^\alpha), \quad \bar{a}^\alpha = e^{\sigma(x)}a^\alpha, \tag{2.1}$$

similarly to the case of (α, β) - metric [4].

It is well - known that Cartan's C - tensor $C_j{}^i{}_k$ is conformally invariant [3]. Throughout the remainder of the paper we shall denote by $C_j{}^i{}_k$ Cartan's C - tensor alone. Then the 1 - form connection $(\Gamma_j{}^i{}_k, \Gamma_0{}^i{}_j, C_j{}^i{}_k)$ is called the *1 - form Cartan connection* and denoted by C1.

The quantities of the 1 - form Finsler space with the changed metric $L(\bar{a}^\alpha)$ will be indicated by putting bar.

If we put $\sigma_i = \partial_i\sigma(x)$, then (1.1) easily gives

$$\bar{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k + \delta^i_j\sigma_k, \tag{2.2}$$

which is of the form of *one - sided projective transformation.*

The 1 - form Cartan connection C1 has only the surviving torsion tensor T and the curvature tensor S^2. The latter is clearly a conformal invariant and (2.2) gives

$$\bar{T}_j{}^i{}_k = T_j{}^i{}_k + \delta^i_j\sigma_k - \delta^i_k\sigma_j. \tag{2.3}$$

Therefore, if we put

$$T_i = T_r{}^r{}_i, \tag{2.4}$$

then (2.3) gives

$$\sigma_k = (\bar{T}_k - T_k)/(n-1), \tag{2.5}$$

and (2.2) shows that the quantities

$$*\Gamma_j{}^i{}_k = \Gamma_j{}^i{}_k - \delta^i_j T_k/(n-1) \tag{2.6}$$

enable us to define a new connection $CC1 = (*\Gamma_j{}^i{}_k, *\Gamma_0{}^i{}_j, C_j{}^i{}_k)$ which is invariant under conformal change of metric.

Definition. *The conformally invariant Finsler connection $CC1 = (*\Gamma_j{}^i{}_k, *\Gamma_0{}^i{}_j, C_j{}^i{}_k)$ is called the 1 - form Cartan C - connection.*

The quantities of CC1 will be indicated by putting asterisk.

3. Conformal invariants. From (2.6) we get the (h) h - torsion tensor

$$*T_j{}^i{}_k = T_j{}^i{}_k - (\delta^i_j T_k - \delta^i_k T_j)/(n-1) \tag{3.1}$$

of CC1, which is obviously a conformal invariant and also obtained from (2.3) together with (2.5).

From (1.1) we have $a^\alpha_{i;j} = 0$ and (2.6) immediately gives

$$a^\alpha_{i|j} = a^\alpha_i T_j/(n-1), \tag{3.2}$$

where ($|$) stand for the h - covariant differentiation with respect to CC1.

We have the Ricci identity of CC1 [6]:

$$a^\alpha_{i|j|k} - a^\alpha_{i|k|j} = -a^\alpha_r * R_i{}^r{}_{jk} - a^\alpha_{i|r} * T_j{}^r{}_k - a^\alpha_i|_r * R^r{}_{jk}, \tag{3.3}$$

where ($|$) stands for the v - covariant differentiation with respect to CC1. From $a^\alpha_i = a^\alpha_i(x)$ we have

$$a^\alpha_i|_j = -a^\alpha_r C_i{}^r{}_j. \tag{3.4}$$

From (3.2) we get $a^{\alpha}_{i|j|k} - a^{\alpha}_{i|k|j} = a^{\alpha}_i(T_{j|k} - T_{k|j})/(n-1)$. Thus, by transvecting (3.3) by a^h_α we have

$$(3.5)\quad *R_i{}^h{}_{jk} - C_i{}^h{}_r * R^r{}_{jk} = -\delta^h_i(T_{j|k} - T_{k|j} + T_r * T_j{}^r{}_k)/(n-1).$$

On account of $*R^h{}_{jk} = y^i * R_i{}^h{}_{jk}$ the transvecting of (3.5) by y^i gives

$$(3.6)\quad *R^h{}_{jk} = -y^h(T_{j|k} - T_{k|j} + T_r * T_j{}^r{}_k)/(n-1).$$

Therefore (3.5) reduces to

$$(3.7)\quad *R_i{}^h{}_{jk} = -\delta^h_i(T_{j|k} - T_{k|j} + T_r * T_j{}^r{}_k)/(n-1).$$

In the right - hand sides of (3.6) and (3.7) we can interchange quantities of CC1 for those of C1: It is easy to show that $T_{j|k} - T_{k|j} = T_{j;k} - T_{k;j}$ and $T_r * T_j{}^r{}_k = T_r T_j{}^r{}_k$, and thus we get

$$(3.6')\quad *R^h{}_{jk} = -y^h(T_{j;k} - T_{k;j} + T_r T_j{}^r{}_k)/(n-1),$$

$$(3.7')\quad *R_i{}^h{}_{jk} = -\delta^h_i(T_{j;k} - T_{k;j} + T_r T_j{}^r{}_k)/(n-1).$$

On the other hand, the (v) hv - torsion tensor $*P^h{}_{jk} = \dot\partial_k *\Gamma_0{}^h{}_j - *\Gamma_k{}^h{}_j = 0$ is obvious and the hv - curvature tensor [6] $*P_i{}^h{}_{jk} = \dot\partial_k *\Gamma_i{}^h{}_j - C_i{}^h{}_{k|j} + C_i{}^r{}_h * P^r{}_{jk} = -C_i{}^h{}_{k|j} = 0$ from $C_i{}^h{}_{k;j} = 0$ and (2.6).

Consequently we have

Proposition 2. *The surviving torsion and curvature tensors of the 1 - form Cartan C - connection CC1 are* $*T_j{}^h{}_k, *R^h{}_{jk}, *R_i{}^h{}_{jk}$ *given by (3.1), (3.6'), (3.7') respectively and* $C_j{}^i{}_k, S_i{}^h{}_{jk}$.

4. Conformally T - Minkowski space. The discussion of the last section shows us that

$$(4.1)\quad T_j{}^i{}_k - (\delta^i_j T_k - \delta^i_k T_j)/(n-1) = 0,$$

$$(4.2)\quad T_{j;k} - T_{k;j} + T_r T_j{}^r{}_k = 0.$$

are necessary and sufficient for a Finsler space with 1 - form metric to have all the vanishing torsion and curvature tensors of the connection CC1 except $C_i{}^i{}_k$ and $S_i{}^h{}_{jk}$.

Theorem. *A Finsler space with 1 - form metric is conformal to a T - Minkowski space with 1 - form metric if and only if the (h) h -torsion tensor T of the 1 - form Cartan connection C1 satisfies the equations (4.1) and (4.2).*

Proof. If a Finsler space F^n with 1 - form metric is conformal to a T - Minkowski space with 1 - form metric $\bar F^n$, then we have $\bar T = 0$, so that $*\bar T = *\bar R^2 = 0$. Since these tensors are conformally invariant, we have $*T = *R^2 = 0$, and thus (4.1) and (4.2).

Conversely, assume that (4.1) and (4.2) hold in the space F^n. We first observe

$$\partial_k T_j - \partial_j T_k = (T_{j;k} + T_r\Gamma_j{}^r{}_k) - (T_{k;j} + T_r\Gamma_k{}^r{}_j)$$

which vanishes by (4.2). Thus $T_j = T_j(x)$ is locally a gradient vector field and we have a scalar field $\sigma(x)$ such that $\sigma_j = \partial_j\sigma$ is given by

$$(4.3)\quad \sigma_j = -T_j/(n-1).$$

Then we make the conformal change $L \to \bar L = e^\sigma L$ by means of this $\sigma(x)$. We now observe

$$\begin{aligned}\bar T_i{}^\alpha{}_j &= \partial_j \bar a^\alpha_i - \partial_i \bar a^\alpha_j = \partial_j(e^\sigma a^\alpha_i) - \partial_i(e^\sigma a^\alpha_j) = \\ &= e^\sigma(\partial_j a^\alpha_i - \partial_i a^\alpha_j + a^\alpha_i\sigma_j - a^\alpha_j\sigma_i).\end{aligned}$$

Owing to (4.3) this is equal to $e^\sigma a^\alpha_k\{T_i{}^k{}_j - (\delta^k_i T_j - \delta^k_j T_i)/(n-1)\}$ and vanishes by (4.1). Therefore (1.2) gives $\bar T_i{}^k{}_j = 0$ and the space with the 1 - form metric $\bar L = e^\sigma L$ is T - Minkowski.

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