ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XL, s.I.a, Matematică, 1994, f1.

GALOIS CONNECTIONS BETWEEN THE LATTICES OF SUBMODULES

BY

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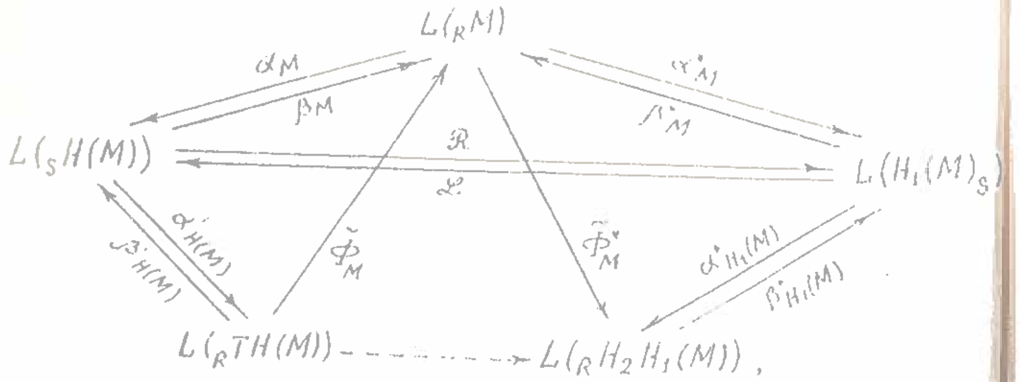
Let ${}_RU_S$ be an arbitrary $R-S$ - bimodule. We consider the following functors between the categories of modules:

$$ {}_S\mathcal{M} \underset{T=U\otimes_S -}{\overset{H=Hom_R(-,U)}{\rightleftarrows}} {}_R\mathcal{M} \underset{H_2=Hom_S(-,U)}{\overset{H_1=Hom_R(-,U)}{\rightleftarrows}} \mathcal{M}_S $$

Every of these functors determines a pair of mappings between the lattice of submodules $L(M)$ or an arbitrary module M and the lattice of submodules of its image. Namely, the functor H gives the mappings $L({}_RM) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} L({}_SH(M))$, where $M \in {}_R\mathcal{M}, \alpha_M(M') = \{f \in H(M) | Imf \subseteq M'\}$ and $\beta_M(N') = \Sigma\{Imf' | f' \in N'\}$ for every $M' \in L({}_RM)$ and $N' \in L({}_SH(M))$. Further, the functor T determines the pair $L({}_SN) \underset{\beta'_N}{\overset{\alpha'_N}{\rightleftarrows}} L({}_RT(N))$, where $N \in {}_S\mathcal{M}, \alpha'_N(N') = \Sigma\{U \otimes {}_Sn' \subseteq U \otimes {}_SN | n' \in N'\}$ and $\beta'_N(M') = \{n \in N | U \otimes {}_Sn \subseteq M'\}$ for every $N' \in L\,{}_S(N)$ and $M' \in L\,{}_RT(N))$. Finally, for the functor H_1 (similarly for H_2) there are the mappings $L({}_RM) \underset{\beta^*_M}{\overset{\alpha^*_M}{\rightleftarrows}} L(H_1(M)_S)$, defined by the rules: $\alpha^*_M(M') = \{f \in H_1(M) | Kerf \supseteq M'\}$, $\beta^*_M(N') = \cap\{Kerf' | f' \in N'\}$ ([1], [2], [3]).

In the articles [2], these pairs of mappings are examined separately and the conditions under which they determine (anti) isomorphisms of lattices of special submodules are shown. In the present report some questions related with the joint research of these pairs of mappings are considered. The natural transformations of functors, which are associated to the adjoint pairs (H,T) and (H_1, H_2) we denote by $\phi : TH \to 1_{{}_R\mathcal{M}}$ and $\phi^* : 1_{{}_R\mathcal{M}} \to H_2H_1$ (for example $\phi_M : TH(M) \to {}_RM$ is defined by the rule $(u \otimes f)\phi_M = uf$). For arbitrary module $M \in {}_R\mathcal{M}$ we have the following diagram:



in which $\mathcal{R}$ and $\mathcal{L}$ are the operators of right and left annihilation, the mappings $\tilde{\phi}_M$ and $\tilde{\phi}^*_M$ are induced by ϕ_M and ϕ^*_M, respectively. The upper triangle of this diagram is well known and was examined in connection with different questions, in particular, considering the "triangular Galois theories" [1], the rings of endomorphisms etc. Obviously, $\mathcal{R} = \alpha^*_M\beta_M$ and $\mathcal{L} = \alpha_M\beta^*_M$.

The mentioned pairs of mappings are closely connected between themselves. The following statements are easily proved by direct verification.

Proposition 1. $\alpha_M = \beta'_{H(M)} \cdot \tilde{\phi}_M^{-1}, \beta_M = \tilde{\phi}_M \cdot \alpha'_{H(M)}$.

Proposition 2. $\alpha^*_M = \beta^*_M \cdot \tilde{\phi}^*_M, \beta^*_M = (\tilde{\phi}^*_M)^{-1} \cdot \alpha'_{H_1(M)}$.

It is easy to see that the pairs (α^*_M, β^*_M), $(\mathcal{R}, \mathcal{L})$ and $(\alpha^*_{H_1(M)}, \beta^*_{H_1(M)})$ are the Galois connections. The pairs (α_M, β_m) and $(\alpha'_{H(M)}, \beta'_{H(M)})$ possess the analogous properties, but these mappings preserve the inclusion of submodules, in contrast to the previous ones.

Let $M \in {}_R\mathcal{M}$ and $X \in L({}_RM)$. We use the following notations:

$$X^a = \beta^*\alpha^*(X),\ C^a = \{X \in L({}_RM) | X^a = X\};$$

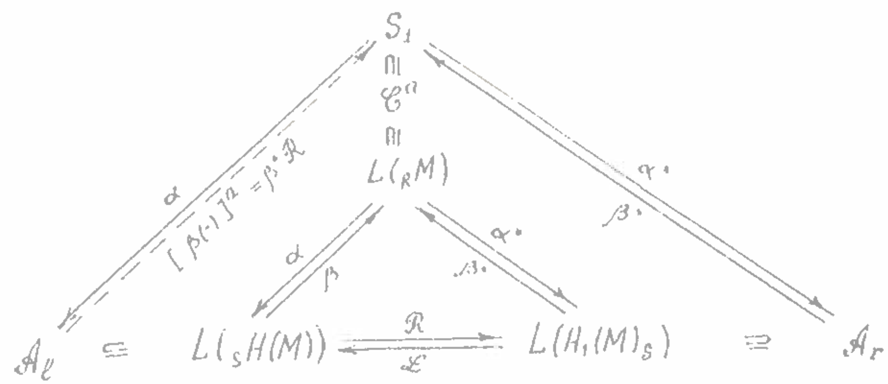
$$X^b = \beta\alpha(X),\ C^b = \{X \in L({}_RM) | X^b = X\};$$

$$\mathcal{A}_l = Im\mathcal{L},\ \mathcal{A}_r = Im\mathcal{R};$$

$$S_1 = \{X \in L({}_RM) | X^{ba} = X\} = Im\beta^*\mathcal{R};$$

$$S_2 = \{X \in L({}_RM) | X^{ab} = X\} = Im\beta\mathcal{L};$$

For the closure operator a, we obtain the following situation:



For simplicity, here and below we preserve the notations of mappings $\alpha, \beta, \ldots$ for their restrictions to the mentioned subsets.

Proposition 3. *(a) The mappings α^* and β^* establish one - to - one correspondences between S_1 and $\mathcal{A}_r$ (i.e. $\mathcal{A}_r \underset{\beta^*}{\overset{\alpha^*}{\rightleftarrows}} S_1$ is a duality).*

(b) The mappings α and $\beta^*\mathcal{R}$ establish one - to - one correspondences between S_1 and $\mathcal{A}_l$ (i.e. $\mathcal{A}_l \underset{\beta^*\mathcal{R}}{\overset{\alpha}{\rightleftarrows}} S_1$ is a projectivity).

Proof. (a) If $X \in S_1$, then $X = X^{ba} = \beta^*\alpha^*\beta\alpha(X)$, therefore $\alpha^*(X) = \alpha^*\beta^*\alpha^*\beta\alpha(X) = \alpha^*\beta\alpha(X) = \mathcal{R}\alpha(X) \in Im\mathcal{R} = \mathcal{A}_r$. Further, let $Z \in \mathcal{A}_r$. Then $Z = \mathcal{R}(Y)$ for some $Y \subseteq H(M)$. Denote $X = \beta^*(Z) = \beta^*\mathcal{R}(Y)$ and verify that $X \in S_1$, i.e. $X^{ba} = X$. We have: $X^{ba} = \beta^*\alpha^*\beta\alpha(X) = \beta^*\mathcal{R}\alpha(X) = \beta^*\mathcal{R}\alpha\beta^*(Z) = \beta^*\mathcal{R}\mathcal{L}(Z) = \beta^*\mathcal{R}\mathcal{L}\mathcal{R}(Y) = \beta^*\mathcal{R}(Y) = X$, therefore β^* reflects $\mathcal{A}_r$ in S_1.

Now we prove that $S_1 \underset{\beta^*}{\overset{\alpha^*}{\rightleftarrows}} \mathcal{A}_r$ are inverse bijections to one another. Let $X \in S_1$. Then $X = \beta^*\alpha^*\beta\alpha(X)$, hence $\beta^*\alpha^*(X) = \beta^*\alpha^*(\beta^*\alpha^*\beta\alpha(X)) = \beta^*\alpha^*\beta\alpha(X) = X$. Conversely, for every $Z \in \mathcal{A}_r$ we have $Z = \mathcal{R}(Y)$ for some $Y \subseteq {}_SH(M)$ and $\beta^*(Z) = \beta^*\mathcal{R}(Y) = \beta^*\alpha^*\beta(Y)$, therefore $\alpha^*\beta^*(Z) = \alpha^*(\beta^*\alpha^*\beta(Y)) = \alpha^*\beta(Y) = \mathcal{R}(Y) = Z$.

(b) If $X \in S_1$, then $X = \beta^*\alpha^*\beta\alpha(X)$, hence $\alpha(X) = \alpha\beta^*\beta\alpha(X) = \mathcal{L}\alpha^*\beta\alpha(X) \in Im\mathcal{L} = \mathcal{A}_l$. Similarly, if $Y \in \mathcal{A}_l$, then its image $[\beta(Y)]^a = \beta^*\alpha^*\beta(Y)$ lies in S_1, since $[\beta^*\alpha^*\beta(Y)]^{ba} = \beta^*\alpha^*\beta\alpha(\beta^*\alpha^*\beta(Y)) = \beta^*\mathcal{R}\mathcal{L}\mathcal{R}(Y) = \beta^*\mathcal{R}(Y) = \beta^*\alpha^*\beta(Y)$.

Let us make sure now that the mappings $\mathcal{A}_l \underset{\beta^*\mathcal{R}}{\overset{\alpha}{\rightleftarrows}} S_1$ are bijections. Let $Y \in \mathcal{A}_l$. Then $Y \in \mathcal{L}(Z)$ for some $Z \subseteq H_1(M)_S$ and $\alpha([\beta(Y)]^a) = \alpha(\beta^*\alpha^*\beta(Y)) = \mathcal{L}\mathcal{R}(Y) = \mathcal{L}\mathcal{R}\mathcal{L}(Z) = \mathcal{L}(Z) = Y$. Similarly, if $X \in S_1$, then

$X = [\beta\alpha(X)]^a$ and $\beta^*\mathcal{R}(\alpha(X)) = [\beta\alpha(X)]^a = X$.

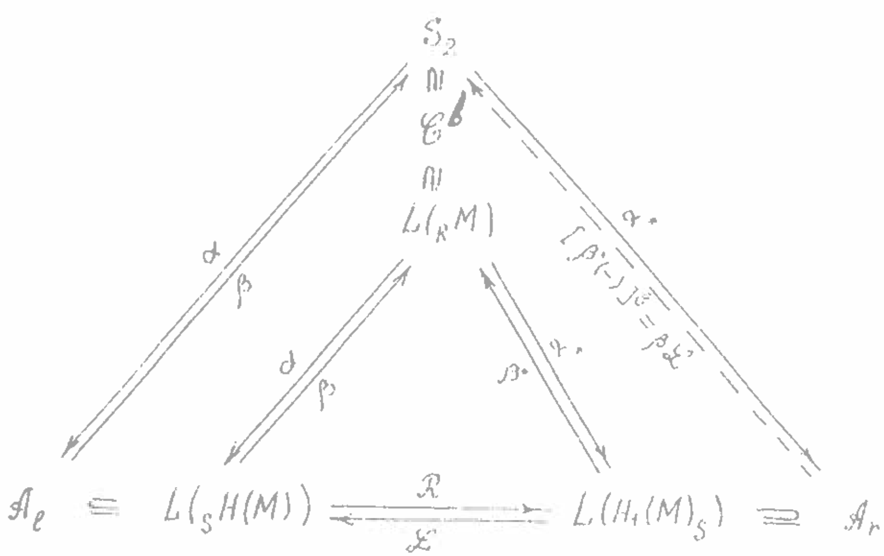
Corollary. *The following conditions are equivalent:*
1) $C^a = S_1$;

2) the mappings α^* *and* β^* *establish a duality* $C^a \underset{\beta^*}{\overset{\alpha^*}{\rightleftharpoons}} \mathcal{A}_r$;

3) the mappings α *and* $\beta^*\mathcal{R}$ *establish a projectivity* $\mathcal{A}_l \underset{\beta^*\mathcal{R}}{\overset{\alpha}{\rightleftharpoons}} S_1$;

A similar result was obtained by S.M. K h u r i ([3]), theorem 2.5) for the particular case $U = M$.

The symmetrical situation takes place for the pair (α, β) with the associated operator b:



Proposition 4. *(a) the mappings* α *and* β *establish one - to - one correspondences between* $\mathcal{A}_l$ *and* S_2 *(i.e.* $\mathcal{A}_l \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} S_2$*; is a projectivity).*

(b) The mappings α^* *and* $\beta\mathcal{L}$ *establish one - to - one correspondences between* S_2 *and* $\mathcal{A}_r$ *(i.e.* $S_2 \underset{\beta\mathcal{L}}{\overset{\alpha^*}{\rightleftharpoons}} \mathcal{A}_r$*; is a duality).*

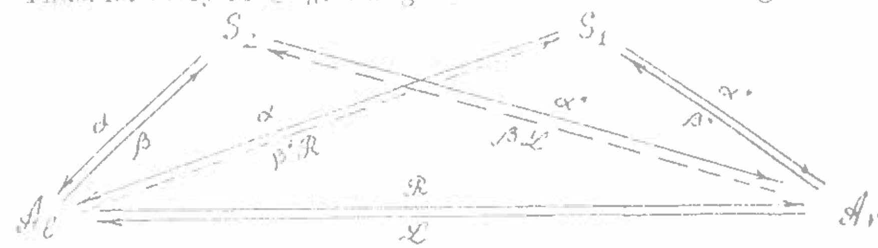
Proof. Is analogous to the proof of proposition 3.

Corollary. *The following conditions are equivalent:*
1) $C^b = S_2$;

2) the mappings α *and* β *establish a projectivity* $\mathcal{A}_l \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} C^b$;

3) the mappings α^* *and* $\beta\mathcal{L}$ *establish a duality* $C^b \underset{\beta\mathcal{L}}{\overset{\alpha^*}{\rightleftharpoons}} \mathcal{A}_r$;

Thus, for every $M \in {}_R\mathcal{M}$ in general case we have the diagram:



where all pairs are bijections. It is interesting to elucidate the question when $S_1 = S_2$ (in this case we obtain the maximum triangular Galois theory $(S_1 = S_2, \mathcal{A}_l, \mathcal{A}_r)$).

Proposition 5. *The following conditions are equivalent:*
1) $S_1 = S_2$;
2) $\beta = \beta^*\mathcal{R}$ *(as restriction on* $\mathcal{A}_l$*);*
3) $\beta^* = \beta\mathcal{L}$ *(as restriction on* $\mathcal{A}_r$ *).*

Proof. 1) $\Rightarrow$ 2), 3). From the condition $S_1 = S_2$ and bijectivity of pairs (α, β) and $(\alpha, \beta^*\mathcal{R})$ it follows that $\beta = \beta^*\mathcal{R}$. Similarly 1) induces $\beta^* = \beta\mathcal{L}$.

2) $\Rightarrow$ 1) (3) $\Rightarrow$ 1)). If $\beta = \beta^*\mathcal{R}$, then from Im $\beta = S_2$ and Im $\beta^*\mathcal{R} = S_1$ it follows $S_1 = S_2$.

REFERENCES

1. B r o d s k i, G.M. – *Functors Hom and lattices of submodules,*(in Russian) Trans. Moscow Math. Soc. 46 (1983), 154 - 186
2. K a s h u, A.I. – *Principal functors of category of modules and lattices of submodules,* (in Russian) I, II, III, Math. Researches. Kishinev, "Shtiintsa", 105 (1988), 53 - 80; 111 (1989), 55 - 81; 118 (1990), 42 - 57.
3. K h u r i, S.M. – *Correspondence theorems for modules and their endomorphism rings,* J. Algebra, 122 (1989), 380 - 396.

Received:2.IV.1993

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