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REMARKS ON THE ADJOINT OF A H - MAP

BY

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1. Introduction. One of the main features of the theory of H-cones is that it allows a good duality theory. Namely, the dual of a H-cone is also a H-cone. Moreover, for standard H-cones, there exists a common space of representation (for the cone and its dual), such that "many" elements of the dual can be represented as measures on this space.

As for the definition of the adjoint of a morphism, there are some difficulties and it must be accepted that *not* every morphism of H-cones has an adjoint.

The aim of this paper is to investigate the properties of those morphisms, which posses as an adjoint a H-map. The main results are th.1 and th.2 from Sec. 3, which give necessary, resp. sufficient conditions, in order that the adjoint of a morphism be a H-map.

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2. Preliminaries. All notions and notations connected with H-cones are used in accordance with [2]. The morphisms of H-cones are defined in [5]. We recall [5] that: φ^* is said to be *an* adjoint of the morphism φ if

$$\forall\mu \in T^*, \forall s \in S : \ \varphi^*(\mu)(s) = \mu(\varphi s)$$

Given S and T H-cones of functions on the sets X and Y, then we call a H-map (from S to T) any function $\varphi : Y \to X$, which has the following property:

$$\forall s \in S, s \text{ bounded } \Rightarrow s \circ \varphi \in T$$

Each H-map induces naturally a morphism of H-cones. Conversely, a morphism φ of H-cones, satisfying: $\varphi(1) = 1$ and $\forall s, t \in S : \ \varphi(s \wedge t) = \varphi s \wedge \varphi t$ is induced by a H-map.

In order to characterise the situation when φ^* is a H-map, we recall the next result. $Ex\ S$ stands for the set of all the elements of the H-cone S, belonging to an extreme ray.

Proposition 1. *Let S and T be H-cones and φ be a semi-finite morphism from S to T, possesing an adjoint. If:*

$$D(\varphi^*) \cap Ex\ T^* \text{separates } T$$

then the following properties are equivalent:

(i) $\forall s, t \in D(\varphi): \ \varphi(s \wedge t) = \varphi s \wedge \varphi t$.

(ii) $\forall \mu \in D(\varphi^*) \cap Ex\ T^* \Rightarrow \varphi^*(\mu) \in Ex\ S^*$.

Proof. (i)$\Rightarrow$ (ii) Let $\mu \in D(\varphi^*) \cap Ex\ T^*$ and $s, t \in D(\varphi)$. Using the characterisation of the elements belonging to an extreme ray [2, cor.2.3.5], we have:

$$\varphi^*(s \wedge t) = \mu[\varphi(s \wedge t)] = \mu[\varphi s \wedge \varphi t] = \min(\mu(\varphi s), \mu(\varphi t)) =$$
$$= \min(\varphi^*(\mu)(s), \varphi^*(\mu)(t))$$

(ii)$\Rightarrow$ (i) Let $s, t \in D(\varphi)$ and $\mu \in D(\varphi^*) \cap Ex\ T^*$. We can write:

$$\mu(\varphi(s \wedge t)) = \varphi^*(\mu)(s \wedge t) = \min(\varphi^*(\mu)(s), \varphi^*(\mu)(t)) =$$
$$= \min(\mu(\varphi s), \mu(\varphi t)) \geq \mu(\varphi s \wedge \varphi t)$$

The converse inequality is obvious. Since $D(\varphi^*) \cap Ex\ T^*$ separates T, it results that $\varphi(s \wedge t) = \varphi s \wedge \varphi t$.

An important rôle in the main results plays the condition that a morphism commutes with arbitrary infima. We recall the following characterisation [6,th.1]:

Proposition 2. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y. Let φ be a morphism from S to T, such that $\varphi(1) = 1$. The following properties are equivalent:*

(i) $\forall F \subseteq D(\varphi): \ \varphi(\bigwedge F) = \bigwedge \varphi(F)$.

(ii) $\forall A \subseteq X$ semi-polar $\Rightarrow \varphi^{-1}(A)$ is semi-polar in Y.

(iii) $\forall \mu \in D(\varphi^*)$ nearly continuous $\Rightarrow \varphi^*(\mu)$ is nearly continuous.

We recall also the following useful property of morphisms commuting with arbitrary infima [7,th.1]:

Proposition 3. *Let S and T be stonean cones on the compact spaces X and Y. Let φ be a morphism from S to T, having the property:*

$$\forall F \subseteq D(\varphi): \ \varphi(\bigwedge F) = \bigwedge \varphi(F)$$

There exists then a balayage B on S, uniquely defined by: $\varphi s = \varphi s' \Leftrightarrow Bs = Bs'$. Moreover, $\varphi(S)$ is canonically isomorphic with $B(S)$, hence $\varphi(S)$ is a H-cone.

There exists a large class of morphisms, namely the morphisms of Dirichlet cones, which commute with arbitrary infima. We recall next the necessary definitions and results.

We say that the H-cone S is associated with the Dirichlet space (H, H_+, T) [2,ch.7] if there exists an isomorphism between S and the cone P_T of potentials from H. In this case, S is called a Dirichlet cone.

Proposition 4. *Let S be a Dirichlet cone.*

(i) If $F \subseteq S$ is increasing to s, then:

$$\bigwedge_{t \in F} R(s - t) = 0.$$

(ii) If $F \subseteq S^*$ is decreasing, then:

$$\forall s \in S: \ (\bigwedge_{\mu \in F} \mu)(s) = \inf_{\mu \in F} \mu(s).$$

Proof. (i) Since $\|R(s - t)\| \leq M^2 . \|s - t\|$ and $\|s - t\| \to 0$, it follows that $R(s - t) \to 0$. But $(R(s - t))_{t \in F}$ is decreasing, hence:

$$0 = \lim_{t \in F} R(s - t) = \bigwedge_{t \in F} R(s - t)$$

(ii) Let us denote: $\mu_0(s) = \inf_{\mu \in F} \mu(s)$. μ_0 is an element of $S^\vee$. By [2,prop.7.2.7], μ_0 coincides with $\bigwedge_{\mu \in F} \mu$.

The morphisms of Dirichlet cones were defined and studied in [1], see also [8].

Let (H, H_+, T) and (V, V_+, T_1) be Dirichlet spaces, and:

$$\varphi : D(\varphi) \subseteq H \to V$$

be a densely defined, linear operator. We denote by:

$$D(\varphi^\vee) = \{y \in V \mid D(\varphi) \ni x \mapsto < T_1 \varphi x, y > \text{is continuous}\}.$$

and define $\varphi^\vee(y)$, for $y \in D(\varphi^\vee)$ as the unique element of H, which satisfies:

$$< T_1 \varphi x, y > = < x, T^\vee \varphi^\vee y >, \ \forall x \in D(\varphi)$$

We get thus a linear operator:

$$\varphi^\vee : D(\varphi^\vee) \subseteq V \to H$$

Let now denote by:

$$\tilde{D}(\varphi) = \{x \in H \mid D(\varphi^\vee) \ni y \mapsto < x, T^\vee \varphi^\vee y > \text{ is continuous}\}.$$

Clearly. $D(\varphi) \subseteq \tilde{D}(\varphi)$; moreover, since for any $x \in D(\varphi)$, φx is the unique element of V which satisfies:

$$< T_1 \varphi x, y > = < x, T^\vee \varphi^\vee y >, \ \forall y \in D(\varphi^\vee)$$

it follows that $\varphi^{\vee\vee}$, defined on $\tilde{D}(\varphi)$, is an extension of φ.

Following [3], we define now the notion of morphism of Dirichlet cones. Let S and S_1 be Dirichlet cones, associated with the Dirichlet spaces (H, H_+, T) and (V, V_+, T_1). φ is called a morphism of Dirichlet cones if:

$$\varphi : D(\varphi) \subseteq H \to V$$

is a linear operator, satisfying the next properties:
$P_T \subseteq D(\varphi)$
$\varphi(P_T) \subseteq P_{T_1}$
φ is increasing

If φ is a morphism of Dirichlet cones, then $\varphi^\vee$ is also a morphism of Dirichlet cones; φ^* exists and is related to $\varphi^\vee$ by the formula:

$$\varphi^\vee = (T^\vee)^{-1} \circ \varphi^* \circ T_1^\vee$$

φ is a closeable operator and $\varphi^{\vee\vee}$ is its closure. φ has the greates extension given by:

$$\tilde{\varphi} = \left[(\varphi\,|_{[S]})^\vee |_{[S_1^*]} \right]^\vee$$

We remark that the definition above produces only finite morphisms [5]. There exists a more general definition, giving semi-finite morphisms [8].

Proposition 5. *Let φ be a morphism of Dirichlet cones. For any decreasing family $F \subseteq D(\varphi)$, we have:*

$$\varphi(\bigwedge F) = \bigwedge \varphi(F)$$

Proof. There exists a decreasing sequence $(s_n)_n$ of F, such that $\bigwedge F = \bigwedge_n s_n$. Next, $s_n \to \bigwedge F$ in the norm of H and $\varphi(s_n) \to \bigwedge_n \varphi s_n$ in the norm of V, as decreasing sequences. Since φ is closeable, it follows that:

$$\bigwedge \varphi s_n = \varphi(\bigwedge s_n) = \varphi(\bigwedge F)$$

But $\bigwedge \varphi(F) \leq \bigwedge_n \varphi s_n$, hence:

$$\bigwedge \varphi(F) \leq \varphi(\bigwedge F)$$

The converse inequality is obvious.

3. The Main Results. We begin with a definition. Let φ be a morphism of H-cones, from S to T. We say that φ has a *specifically solid image in the strong sense* if:

there exists a dense, solid part $A \subseteq D(\varphi)$ such that:

$\forall s \in A$, $\forall t \in T$ with $t \preceq \varphi s$, $\exists s_1 \in S$, $s_1 \preceq s$ and $\varphi s_1 = t$.

We remark that the fact that the image is specifically solid does not insures that s_1 is specifically smaller than s.

Theorem 1. *Let φ be a semi-finite morphism of H-cones, having an adjoint. If φ has a specifically solid image in the strong sense, then:*

$$\forall \mu, \nu \in D(\varphi^*) : \quad \varphi^*(\mu \wedge \nu) = \varphi^*(\mu) \wedge \varphi^*(\nu)$$

Proof. Let $s \in A \subseteq D(\varphi)$; by [2,prop.2.2.4], we can find $t_1, t_2 \in T$ such that:

$$\varphi^*(\mu \wedge \nu)(s) = \mu(t_1) + \nu(t_2) \text{ and } \varphi s = t_1 + t_2$$

Now, there exists $s_1 \in S$ such that $s_1 \preceq s$ and $t_1 = \varphi s_1$. Let us denote by $s_2 = s - s_1 \in S$. Then: $t_2 = \varphi s_2$ and we can write:

$$\varphi^*(\mu \wedge \nu)(s) = \mu(\varphi s_1) + \nu(\varphi s_2) \geq [\varphi^* \mu \wedge \varphi^* \nu](s_1) + [\varphi^* \mu \wedge \varphi^* \nu](s_2) =$$
$$= (\varphi^*(\mu) \wedge \varphi^*(\nu))(s)$$

Since A is dense, we obtain: $\varphi^*(\mu \wedge \nu) \geq \varphi^*(\mu) \wedge \varphi^*(\nu)$. The converse inequality is obvious.

Corollary. *Let φ be a semi-finite morphism of H-cones, having an adjoint. If φ is injective and has a specifically solid image, then:*

$$\forall \mu, \nu \in D(\varphi^*) : \quad \varphi^*(\mu \wedge \nu) = \varphi^*(\mu) \wedge \varphi^*(\nu)$$

Proof. It remains to verify that φ has a specifically solid image in the strong sense. But, if $\varphi s = t_1 + t_2$, there exist $s_1, s_2 \in S$, such that $\varphi s_k = t_k$, $k = 1, 2$. Hence: $\varphi s = t_1 + t_2 = \varphi(s_1 + s_2)$. From the injectivity, we get now: $s = s_1 + s_2$ and thus $s_1 \preceq s$.

For the converse result, we consider the following situation. S and T are H- cones of functions on the sets X and Y. We suppose that $1 \in S$. $\varphi : Y \to X$ is a H-map, commuting with arbitrary infima. Moreover, we suppose that Y is endowed with a completely regular topology, such that:

the continuous elements of T form a dense part in T; and any bounded H-integral from T^* is a measure on Y. Then:

Theorem 2. *Under the above assumptions, φ^* has a specifically solid image in the strong sense.*

Proof. Let $\mu \in T^*$, $\mu(1) < +\infty$ and $\nu, \nu' \in S^*$ be such that: $\nu + \nu' = \varphi^*(\mu)$. We define first $\nu_0, \nu_0' : \varphi(S) \to \mathbf{R}_+$ as follows: $\nu_0(\varphi s) = \nu(s), \forall s \in D(\varphi)$ and analogously ν_0'. This definition is correct: if $\varphi s = \varphi s'$, then $\varphi(s \wedge s') = \varphi s$ and:

$$\nu(s \wedge s') + \nu'(s \wedge s') = \varphi^*(\mu)(s \wedge s') = \mu(\varphi(s \wedge s')) = \mu(\varphi s) =$$
$$= \varphi^*(\mu s) = \nu(s) + \nu'(s) \geq \nu(s \wedge s') + \nu'(s \wedge s')$$

hence: $\nu(s) = \nu(s \wedge s')$.

By prop.3, $\varphi(S)$ is a H-cone and ν_0, ν_0' result H-integrals on $\varphi(S)$. The continuity of ν_0 and ν_0' can be shown using the balayage associated with φ. Moreover, for any $t \in \varphi(S)$, $t = \varphi s$, we have:

$$\nu_0(t) + \nu_0'(t) = \nu_0(\varphi s) + \nu_0'(\varphi s) = \nu(s) + \nu'(s) = \varphi^*(\mu)(s) =$$
$$= \mu(\varphi s) = \mu(t)$$

We prove next that ν_0 and ν_0' can be extended to H-integrals $\tilde{\nu}_0, \tilde{\nu}_0' \in T^*$ such that: $\tilde{\nu}_0 + \tilde{\nu}_0' = \mu$. We use the idea from a lemma of Choquet [4,ch.IX,T3]. Let us denote by $C_b(Y)$ the space of real, bounded, continuous functions on Y and define the convexe and monotone functionals p, p' on $C_b(Y)$ by:

$$p(f) = \inf_{s \in [S'],\, s \geq f} \nu_0(s)$$

(where $[S']$ stands for the linear space of differences of continuous and bounded functions from $\varphi(S)$). Analogously define p'. $1 \in [S']$ shows that the definition is correct. Since $\varphi(S)$ is min-stable in T, it follows that: $\forall \varepsilon > 0$ and $f \in C_b(Y)$ there exists $u \in [S']$ such that $u \geq f$ and:

$$\mu(f) \leq \mu(u) = \nu_0(u) + \nu_0'(u) \leq p(f) + p'(f) + \varepsilon$$

Moreover, p and p' has the following property: if $f \in C_b(Y)$ and $s \in [S']$ then: $p(f + s) = p(f) + p(s)$. Indeed:

$$p(f+s) = \inf_{u \in [S'], u \geq f+s} \nu_0(u) = \inf_{v \in [S'], v \geq f} \nu_0(v + s) =$$
$$= \nu_0(s) + \inf_{v \in [S'], v \geq f} \nu_0(v) = p(s) + p(f)$$

Let now be $f \in C_b(Y) \setminus [S']$; we extend ν_0 and ν_0' to the linear space $[S'] + \mathbf{R}.f$ by:

$$\tilde{\nu}_0(u + \lambda.f) = \nu_0(u) + \lambda.x$$

$$\tilde{\nu}_0'(u + \lambda.f) = \nu_0'(u) + \lambda.x'$$

where $x, x' \in \mathbf{R}$ are choosen such that :

$$-p(-f) \leq x \leq p(f)$$

$$-p'(-f) \leq x' \leq p'(f)$$

$$x + x' = \mu(f)$$

This choice guarantees that:

$$\tilde{\nu}_0(u + \lambda.f) \leq p(u + \lambda.f)$$

$$\tilde{\nu}_0'(u + \lambda.f) \leq p'(u + \lambda.f)$$

$$\tilde{\nu}_0(u + \lambda.f) + \tilde{\nu}_0'(u + \lambda.f) = \mu(u + \lambda.f)$$

The possibility of such a choice results from the inequality:

$$-p(-f) - p'(-f) \leq \mu(f) \leq p(f) + p'(f)$$

and the following elementary:

Lemma. *Let $\alpha \leq \beta$, $\gamma \leq \delta$ and $a \in [\alpha + \gamma, \beta + \delta]$. Then, there exist $a' \in [\alpha, \beta]$, $a'' \in [\gamma, \delta]$ such that $a = a' + a''$.*

Using now the usual argument based on Zorn's lemma, we get linear forms $\tilde{\nu}_0$ and $\tilde{\nu}_0'$ on $C_b(Y)$, such that:

$$\tilde{\nu}_0 \leq p,\ \tilde{\nu}_0' \leq p',\ \tilde{\nu}_0 + \tilde{\nu}_0' = \mu \text{ on } C_b(Y)$$

If $f \in C_b(Y)$, $f \leq 0$, then $\tilde{\nu}_0(f) \leq p(f) \leq 0$ shows that $\tilde{\nu}_0$ and $\tilde{\nu}_0'$ are monotone. By [3,cor. to prop.5,Sec.5.2], $\tilde{\nu}_0$ and $\tilde{\nu}_0'$ result positive measures on Y. Finally, [3,prop.1.6,Sec.1.6] shows that $\tilde{\nu}_0$ and $\tilde{\nu}_0'$ are in fact H-integrals. The equality $\varphi^*(\tilde{\nu}_0) = \nu$ is obvious from the definition.

Remark. The situation considered in the theorem holds, for instance, if T is a standard H-cone of functions on the semi-saturated space Y, endowed with the natural topology.

In the case of Dirichlet cones, the conditions can be considerably relaxed. Let S and T be Dirichlet cones and cones of functions on the sets X and Y. We suppose morewer that X is "saturated" in the following sense: any $\mu \in Ex\ S^*$, $\mu(1) = 1$ is of the form ε_x, with $x \in X$.

Let φ be a morphism of H-cones from S to T

Corollary. *φ is H-map iff φ^* has a specifically solid image in the strong sense and $\varphi(1) = 1$.*

The proof can also be simplified, since the balayage associated with φ can be constructed using [2,prop.2.2.10]. Also, the use of the theory of measure on completely regular spaces is not necessary, the extensions being H-integrals by [2,prop.7.2.7].

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MORSE THEORY ON GRASSMANIANS

BY

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0.Introduction. The paper [13], while studying adiabatic deformation of Dirac operators on manifolds with boundary, we were led to the following finite dimensional dynamics problem.

Consider $\Lambda(n)$ the grassmannian of largrangian subspaces in the canonical symplectic space $E = \mathbb{R}^{2n}$. If $A : E \to E$ is a selfadjoint operator anticommuting with the canonical complet structure J on E then A belongs to the Lie algebra of the symplectic group $Sp(E)$ and thus e^{tA} is a flow of symplectic matrices. It induces a flow on $\Lambda(n)$ via the transitive action of $Sp(E)$ on this grassmanian. This flow presents many similarities with a gradient like flow. In particular it has a nice asymptotic behaviour. More precisely for $L \in \Lambda(n)$ $e^{At}L$ converges to some A - invariant lagrangian as t goes infinity. In fact when $n = 1$ so that $\Lambda(1) \sim S^1$ the phase portrait (see Fig.2 Sec.1) resembles the phase portrait of the gradient flow of a perfect Morse function on S^1. A natural question arises. Is the flow the gradient flow (in some appropriate metric) of a Morse function on $\Lambda(n)$?

This is one of the question we adress in these notes. The author was vary pleased to find out that this question has the best answer one can hope for. Indeed this is the gradient flow (in a natural metric) of some Morse function. This function which must depend on A has a simple description

$$f_A(L) = -tr(AP_L) \tag{0.1}$$

where P_L is the orthogonal projection onto L. Its critical points are the A - invariant lagrangian subspaces and they admit a very nice combinatoric description. For a particular choice of A this function is selfindexing and Z_2 - perfect. This allowed us to derive the Z_2 Poincaré polynomial of $\Lambda(n)$.