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Received:3.I.1994

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Tomul XL, s.I.a, Matematică, 1994, f1.

ON THE CONTINUITY OF MINKOWSKI FUNCTIONALS

BY

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1.Introduction. Let X be a real normed linear space. For every nonempty set A and $x \in X$ one defines the Minkowski functional by

$$p_A(x) = \begin{cases} \inf\{\lambda > 0; x \in \lambda A\}, \text{ if there exists} \\ \qquad\qquad \text{one positive } \lambda \text{such that } x \in \lambda A; \\ +\infty, \text{ otherwise.} \end{cases}$$

Also, we denote :

$$\alpha_x = \{\lambda > 0;\ x \in \lambda A\}$$
$$A_x = \{a \in A;\ \text{there exists } \lambda > 0 \text{with } x = \lambda a\}$$

It is obvious that $A_x = (1/\alpha_x)A$ and $A_{tx} = A_x, \forall t > 0$ whenever $A_x \neq \emptyset$, and $A_x = A \cap d_x$, where d_x is the radius $d_x = \{y \in X;$ there exists $\lambda > 0$ such that $x = \lambda y\}$; also, $p_A(x) = \inf \alpha_x$ (we consider $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$).

The Functional Analysis uses the Minkowski functionals to endow some "norms" on topological spaces with special properties. This suggests us to study the class of all convex closed bounded subsets $A \subset X$ which contain the origin. Denote this class by $Ca(X)$. We note that, if A is a convex closed bounded and absorbent set (for every $x \in X$ there exists one positive μ with $\mu x \in A$), then $A \in Ca(X)$.

For $A \in Ca(X)$, we shall prove that α_x is a real closed unbounded interval at the right hand.

It is interesting to investigate the behaviour of Minkowski functional as set function, $A \mapsto p_A(x)$, when we endow $Ca(X)$ with certain hyperspace topologies.

The aim of this paper is to find the weakest topology on $Ca(X)$ for which the function $A \mapsto p_A(x)$ is continuous for all $x \in X$. We shall also introduce some covergences on $Ca(X)$ which assure the sequential continuity of this function.

Similar problems were examined [3,4,5,6,7,8,9] for other function.

2.Preliminaires and notations. In the sequel, let $(X, \|\cdot\|)$ be a normed linear space. We denote the origin by 0 and $S(x,\varepsilon) = \{y \in X : \|y - x\| < \varepsilon\}$. Let $\bar{A}$ be the closedness of a set A.

The topological dual of X is denoted X^*. For $x^* \in X^*$ and $x \in X$ we set $< x, x^* >= x^*(x)$. If a sequence (x_n) converges weakly to x, then we write $x_n \xrightarrow{w} x$; the convergence in norm is denoted by $x_n \to x$. We define the radius

$$d_x = \{y \in X; \text{there exists } \lambda > 0 \text{ with} x = \lambda y\}.$$

Let $Cc(X)$ be the family of convex closed subsets of X. We recall the following convergences on $Cc(X)$:

Definition 2.1. *$(A^n) \subset Cc(X)$ is Mosco convergent to A, denoted $A^n \xrightarrow{M} A$, if:*

i) for each $a \in A$ there is a sequence (a_n) strongly convergent to a such that $a_n \in A^n$ for every n.

ii) whenever $n(k)$ is an increasing sequence of positive integer numbers and $a_k \in A^{n(k)}$ for every k, such that $a_k \xrightarrow{w} a$ then $a \in A$.

The Kuratowski convergences of (A^n) to A, $A^n \xrightarrow{K} A$, is similary defined, replacing in ii) the weak convergences by the norm convergences.

Definition 2.2. *A^n is called scalar (or weak) convergent to $A(A^n \xrightarrow{S} A)$ if, for all $x^* \in X^*$, $\sup\{x^*(a) : a \in A^n\} \to \sup\{x^*(a) : a \in A\}$ in R.*

In [5], was introduced a topology "hit and miss", τ_M, compatible with the Mosco convergence of the sequences, named Mosco topology.

We mentioned that, if A is a convex closed subset of X, then the following assertion are equivalent:

1) $0 \in A$;

2) $A = \cup_{\lambda \in [0,1]} \lambda A$ (or $A = \cup_{\lambda \in (0,1]} \lambda A$)

3) $\lambda A \subseteq A$, for all $\lambda \in (0,1]$

4) for every, λ_1, λ_2 with $0 < \lambda_1 \leq \lambda_2$ we have $\lambda_1 A \subseteq \lambda_2 A$.

Now , we give a result concerning the structure of the sets α_x and A_x which will be used in the next sections.

Lemma 2.1. *Let $A \in Ca(X)$.*

1) If $x = 0$ then $\alpha_x = (0, +\infty)$ and $A_x = \{0\}$.

2) If $x \neq 0$ then $\alpha_x = \emptyset$ (and $A_x = \emptyset$) or there is a $\lambda_x > 0$ such that $\alpha_x = [\lambda_x, +\infty)$ and $A_x = (0, (1/\lambda_x)x]$.

Remarks 2.1. We have $A_x = A \cap d_x$ and $\bar{A}_x = A_x \cup \{0\}$. If $A_x \neq \emptyset$ then A_x is a bounded 1 − dimensional set, so A_x is relatively compact set.

3. The convergence of Minkowski functional. We introduce the following convergence for obtain the sequential continuity of Minkowski functional:

Definition 3.1. *The sequence $(A^n) \subset Ca(X)$ is upper convergent in the direction $x \in X$ to $A^0 \in Ca(X)$, denoted $A^n \xrightarrow{u.x} A^0$, if*

$$\sup\{\|a\| : a \in A_x^n\} \to \sup\{\|a\| : a \in A_x^0\}.$$

Theorem 3.1. *Let $A^n, A^0 \in Ca(X)$, for all n, and $x \in X$ such that $A_x^n, A_x^0 \neq \emptyset$. The following statements are equivalent:*

(1) $p_{A^n}(x) \to p_{A^0}(x)$;

(2) $A^n \xrightarrow{u.x} A^0$;

$$(3) \begin{cases} \sup\{x^*(a) : a \in A_x^n\} \to \sup\{x^*(a) : a \in A_x^0\} \\ \qquad\qquad \text{for all } x^* \in X^* \text{ with } x^*(x) > 0; \\ \inf\{x^*(a) : a \in A_x^n\} \to \inf\{x^*(a) : a \in A_x^0\} \\ \qquad\qquad \text{for all } x^* \in X^* \text{ with } x^*(x) < 0; \end{cases}$$

(4) $\sup\{x^*(a) : a \in A_x^n\} \to \sup\{x^*(a) : a \in A_x^0\}$ for all $x^* \in X^*$ (or there exists $x^* \in X^*$) with $x^*(x) > 0$;

Proof. (1) $\Leftrightarrow$ (2)

If $x = 0$ then $p_A(x) = 0$ and $\sup\{\|a\| : a \in A_x\} = 0$.

If $x \neq 0$ then $0 \notin A_x$ and we use the following result: for every $\lambda \in \alpha_x$ there exists $a_\lambda \in A_x$ such that $x = \lambda a_\lambda$, the correspondence $\lambda \mapsto a_\lambda$ being a bijection; then $\lambda = \|x\|/\|a\|$ and $p_A(x) = \|x\|/\sup\{\|a\| : a \in A_x\}$.

(1) $\Leftrightarrow$ (3)

If $x = \lambda a$, for all $x^* \in X^*$ with $x^*(x) \neq 0$ we have $\lambda = x^*(x)/x^*(a)$ and

$$p_A(x) = \begin{cases} x^*(x)\sup\{x^*(a) : a \in A_x\}, if x^*(x) > 0 \\ -x^*(x)\inf\{x^*(a) : a \in A_x\}, if x^*(x) < 0 \end{cases}$$

(3) $\Leftrightarrow$ (4) is obvious.

Theorem 3.2. *Let $A^n, A^0 \in Ca(X)$, for all n, and $x \in X$. Then*

$$A_x^n \xrightarrow{S} A_x^0 \Leftrightarrow A^n \xrightarrow{u.x} A^0.$$

Proof. The first implication results from Definition 2.2. and Theorem 3.1

Conversely, if $A_x^n = \emptyset$ confinally then $A_x^0 = \emptyset$ and $\sup\{x^*(a)\} : a \in A_x^n = \sup\{x^*(a) : a \in A_x^0\} = -\infty$. If $A_x^n \neq \emptyset$ eventually then $A_x^0 \neq \emptyset$. Let

$x \in X$ such that $x^*(x) > 0$; the assertion follows from Definition 3.1. and Theorem 2.1. Now let $x \in X$ such that $x^*(x) \leq 0$, then $\sup\{x^*(a) : a \in A_x^n\} = \sup\{x^*(a) : a \in A_x^0\} = 0$.

Remark 3.1. We have $\sup\{\|a\| : a \in A_x\} = \sup\{\|a\| : a \in \bar{A}_x\}$. If $A_x \neq \emptyset$, the set $\bar{A}_x = A_x \cup \{0\}$ being compact, there exists $a_A^x \in \bar{A}_x$ such that $\sup\{\|a\| : a \in A_x\} = \|a_A^x\|$. Moreover, $a_A^x \in \partial A \cap d_x$ if $x \neq 0$, and $a_A^x = 0$ if $x = 0$.

In the sequel we find necessary conditions for the directional upper convergences:

Proposition 3.1. *Let $A^n, A^0 \in Ca(X)$, for all n, such that $A^n \xrightarrow{u.x} A^0$, for all $x \in X$. Then we have the following assertions:*

(1) $\sup\{\|a\| : a \in A^0\} \leq \liminf_n \sup\{\|a\| : a \in A^n\}$;

(2) $\sup\{(sgn x^*(a))x^*(a) : a \in A^0\} \leq$
$\liminf_n \sup\{(sgn x^*(a))x^*(a) : a \in A^n\}$.

Proof.

(1) If $A \neq \{0\}$, then $A\backslash\{0\} = \cup_{x\neq 0} A_x$ and so $\sup\{\|a\| : a \in A\} = \sup\{\|a\| : a \in A\backslash\{0\}\} = \sup\{\sup\{\|a\| : a \in A_x\} : x \neq 0\}$.

For $A^0 \neq \{0\}$, let $a_0 \in A^0\backslash\{0\}$. Then, from $0 < \sup\{\|a\| : a \in A_a^0\}$, using the upper convergence in the direction a_0 we have $\sup\{\|a\| : a \in A_a^n\} > 0$ eventually, so $A^n \neq \{0\}$ eventually.

Now, $\liminf_n \sup\{\|a\| : a \in A^n\} = \liminf_n \sup\{\sup\{\|a\| : a \in A_x^n\} : x \neq 0\} \geq \liminf_n \sup\{\|a\| : a \in A_x^n\} = \sup\{\|a\| : a \in A_x^0\}$, for all $x \neq 0$.

Passing to the supremum for $x \neq 0$, we obtain the desired inequality.

For $A^0 = \{0\}$, $\sup\{\|a\| : a \in A^0\} = 0$ and the inequality is automatically satisfied.

(2) is analogously.

Proposition 3.2. *Let $A^n, A^0 \in Ca(X)$, for all n, with $A^n \xrightarrow{u.x} A^0 \neq 0$. Then $\forall a_0 \in A^0 \exists a_n \in A^n (a_n \in A_{a_0}^n)$ such that $a_n \to a_0$.*

Proof. If $a_0 = 0$, then $a_n = 0 \in A_a^n$.

If $a_0 \neq 0$, using Remark 3.1., there exists $\bar{a}_0 \in A_{a_0}^0$ such that $\|\bar{a}\| = \sup\{\|a\| : a \in A_{a_0}^0\}$ and $\bar{a}_n \in A_{a_0}^n$ such that $\|\bar{a}_n\| = \sup\{\|a\| : a \in A_{a_0}^n\}$, where $\bar{a}_0$ is an element of the bound of A^0. So $a_0 = \lambda_0 \bar{a}_0$, $0 < \lambda_0 \leq 1$ and $a_0 = \lambda_n \bar{a}_n, \lambda_n > 0$.

Theorem 3.1. implies $\|\bar{a}_n\| \to \|\bar{a}_0\|$, i.e. $\lambda_n \to \lambda_0$. We denote $\mu_n = \lambda_n/\lambda_0$ and $a_n(1/\mu_n)a_0$; then $\mu_n \to 1, a_n \to a_0$. Since $\lambda_n = \lambda_0\mu_n \leq \mu_n$, we have $a_n \in (0, (1/\lambda_n)a_0] = A_{a_0}$. Hence the desired sequence is (a_n).

Proposition 3.2 suggest us to find sufficient conditions for the directional upper convergence (similar to Mosco or Kuratowski convergence). We introduce some definitions:

Definition 3.2. *The sequence $(A^n) \subset Ca(X)$ is Mosco (respectively Kuratowski) convergent in the direction $x \in X$ to $A^0 \in Ca(X)$ if:*

i) $\forall a_0 \in A_x^0 \exists a_n \in A_x^n$ *such that* $a_n \to a_0$;

ii) $\forall n(k) \nearrow \infty, \forall a_k \in A^{n(k)}$ *with* $a_k \xrightarrow{w} a_0$, *we have* $a_0 \in A^0$, *respectively*

iii) $\forall n(k) \nearrow \infty, \forall a_k \in A^{n(k)}$ *with* $a_k \to a_0$, *we have* $a_0 \in A^0$.

We shall denote $A^n \xrightarrow{M.x} A^0$, *respectively* $A^n \xrightarrow{K.x} A^0$.

We compare this notions with the standard nottions of Mosco and Kuratowski convergences in the following result:

Proposition 3.3. *If $A^n, A^0 \in Ca(X)$, for all n, then the following statements hold:*

(1) $A^n \xrightarrow{M.x} A^0, \forall x \in X$ *implies* $A^n \xrightarrow{M} A^0$, *respectively*
$A^n \xrightarrow{K.x} A^0, \forall x \in X$ *implies* $A^n \xrightarrow{K} A^0$.

(2) $\forall x \in X$, *if* $A^n \xrightarrow{M.x} A^0$, *we have* $A^n \xrightarrow{K.x} A^0$.

(3) $\forall x \in X$, *if* $A^n \xrightarrow{K.x} A^0$, *then* $\overline{A_x^n} \xrightarrow{K} \overline{A_x^0}$

and, since the Mosco and Kuratowski convergence of the one dimensional sets coincide, we have

$$A^n \xrightarrow{M.x} A^0, \textit{implies } \overline{A_x^n} \xrightarrow{M} \overline{A_x^0}.$$

Proof. (1) The second condition of Definition 2.1 and 3.2 is the same. To prove the first condition, let $a_0 \in A^0$; since $a_0 \in A_{a_0}^0$ there is $a_n \in A_{a_0}^n \subset A^n$, with $a_n \xrightarrow{w} a_0$ (respectively $a_n \to a_0$).

(2) It is clearly.

(3) Let $a_0 \in \bar{A}_x^0$. Then either $a_0 \in A_x^0$ (and the property follows from Definition 3.2) or $a_0 = 0$ (and the desired sequence is $a_n = 0 \in \bar{A}_x^n$.).

To prove iii) we choose $n(k) \nearrow \infty, a_k \in A_x^{n(k)}, a_k \to a_0$ (the case $a_k = 0$ is trivial); since $a_k = (1/\lambda_k)x, \lambda_k > 0$, we obtain that $(1/\lambda_k)$ is convergent, say $(1/\lambda_k) \to \mu_0, \mu_0 \geq 0$.

If $\mu_0 = 0$ we obtain $a_n \to 0$, thus $a_0 = 0 \in \overline{A_x^0}$. If $\mu_0 > 0$ we set $\bar{a}_0 = \mu_0 x \in A_x^0$; since $a_n \to \bar{a}_0$ we have $a_0 = \bar{a}_0 \in A_x^0$.

Remark 3.2. The reverse assertion of (1) is not generally true: Indeed, let

$$A^n = \{(x,y) \in R^2 : x \leq y, 0 \leq y \leq 1, x \geq \frac{1}{n}y\} \text{and}$$

$$A^0 = \{(x,y) \in R^2 : x \leq y, 0 \leq y \leq 1, x \geq 0\}$$

Then $A^n \xrightarrow{M} A^0$, but (A^n) is not Mosco convergent to A^0 in the direction (0,1).

From Theorem 3.2 and Proposition 3.3 we obtain:

Corollary 3.1. *Let $A^n, A^0 \in Ca(X)$ and $x \in X$ such that $A^n_x, A^0_x \neq \emptyset$. If $A^n \overset{K,x}{\to} A^0$ (or $A^n \overset{M,x}{\to} A^0$) then $A^n \overset{u,x}{\to} A^0$.*

We denote $c(A) = \{\lambda a; \lambda > 0, a \in A\}$.

Theorem 3.3. *Let be $A^n, A^0 \in C\ a\ (X)$ with $c(A^0) = c(A^n), \forall n$; if $p_{A^n}(x) \to p_{A^0}(X)$ uniform with respect to $x \in S(0,1) \cap c(A^0)$ then $A^n \overset{M}{\to} A^0$.*

Proof. The first condition of Mosco convergence follows from proposition 3.3. For the second condition, let $(n_k) \nearrow \infty$, $a_k \in A^{n_k}$, $a_k \overset{w}{\to} a$. For this a_k there exists $x_k \in S(0,1)$ and $\mu_k > 0$ such that $a_k = \mu_k x_k$. Since (a_k) is weakly convergent and $\|x_k\| = 1$, we have that (μ_k) is a bounded sequence, hence there exists $\lim_k(p_{A^k}(a_k) - p_{A^0}(a_k)) = \lim_k \mu_k(p_{A^k}(x_k) - p_{A^0}(x_k)) = 0$. p_{A^0} is l.s.c. function, hence $p_{A^0}(a_0) \leq \liminf_k p_{A^0}(a_k) \leq \limsup_k p_{A^{n_k}}(a_k) \leq 1$ because A^{n_k} is a closed set. It results that $a_0 \in A^0$.

4. The directional upper topology. Now we require the weakest topology on $Ca(X)$ for which the function $A \mapsto p_A(x)$ is continous for all $x \in X$. For every $x \neq 0$ we define the set

$$I_x(\eta) = d_x \backslash \overline{S(0,\eta)} = ((\eta/\|x\|)x, \infty),$$

which is an unbounded open interval 1 - dimensional.

Definition 4.1. *We call the directional upper topology on $Ca(X)$ the topology generated by the following two families of sets:*

$$(I_x(\eta))^- \quad \text{and} \quad (I_x(\eta))^\#, x \in X\backslash\{0\}, \eta > 0.$$

where $\begin{cases} V^- = \{A \in Ca(X) : V \cap A \neq \emptyset\} \\ V\# = \{A \in Ca(X) : d(V, A) > 0\}. \end{cases}$

Remark 4.1. If $A \in Ca(X)$ and $x \neq 0$ then

$A \cap (\eta/\|x\|)x, \infty) \neq \emptyset \Leftrightarrow \sup\{\|a\| : a \in A_x\} > \eta$.

$d(A, ((\eta/\|x\|)x, \infty)) > 0 \Leftrightarrow \sup\{\|a\| : a \in A_x\} < \eta$

Hence

$(I_x(\eta))^- = \{A \in Ca(X) : \sup\{\|a\| : a \in A_x\} > \eta\}$ and

$(I_x(\eta))^\# = \{A \in Ca(X) : \sup\{\|a\| : a \in A_x\} < \eta\}$.

We note that $(I_x(\eta))^\# = \overline{(I_x(\eta))}^\#$.

Theorem 4.1. *The weakest topology on $Ca(X)$ for which the application $A \mapsto p_A(x)$ is continuous for all $x \in X$ is the directional upper topology.*

Proof. If $x = 0$ or $\alpha_x = \emptyset$ the assertion is obvious.

If $x \neq 0$ and $\alpha_x \neq \emptyset$ then $p_A(x) = d(0, \alpha_x)$.

Using Proposition 6.1 from [13] we obtain that the weakest topology on $\{\alpha \subset R : \exists \lambda_\alpha$ such that $\alpha = [\lambda_\alpha, \infty)\}$ for which $\alpha \mapsto d(0, \alpha)$ is continuous is generated by the following families of sets: $(0, \varepsilon)^-$ and $(0, \varepsilon)^\#$ (which ensure the upper semicontinuity, respectively the lower semicontinuity of this application).

Therefore we have:

i) $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ with $\alpha^0_x \in (0, \delta)^-$ such that

$\forall \alpha_x \in (0, \delta)^- : d(0, \alpha_x) < d(0, \alpha^0_x) + \varepsilon$;

$\Leftrightarrow \forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ with $\exists 0 < \lambda_0 < \delta, \exists a_0 \in A^0, x = \lambda_0 a_0$ such that

$\forall A \in Ca(X)$ with $\exists 0 < \lambda < \delta, \exists a \in A, x = \lambda a : p_A(x) < p_{A^0}(x) + \varepsilon$.

But $\|a_0\| = (1/\lambda_0)\|x\| > (1/\delta)\|x\|, \|a\| = (1/\lambda)\|x\| > (1/\delta)\|x\|$; denoting $\eta(\varepsilon) = (1/\delta(\varepsilon))\|x\|$ we obtain:

$\forall \varepsilon > 0 \exists \eta(\varepsilon) > 0$ with $A^0_x \cap S(0,\eta)^c \neq \emptyset$ such that $\forall A \in Ca(X)$ with $A \cap S(0,\eta)^c \neq \emptyset$ implies $p_A(x) < p_{A^0}(x) + \varepsilon$.

But $A_x \cap S(0,\eta)^c \neq \emptyset \Leftrightarrow A \cap I_x(\eta) \neq \emptyset$, thus we obtain the first type of generators.

ii) $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ with $d(\alpha^0_x, (0,\delta)) > 0$ such that

$\forall \alpha_x$ with $d(\alpha_x, (0,\delta)) > 0 : d(0, \alpha_x) > d(0, \alpha^0_x) - \varepsilon$;

$\Leftrightarrow \forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ with $\exists \gamma_0 > \delta : \lambda \geq \gamma_0 \forall \lambda \in \alpha^0_x$ such that

$\forall \alpha_x$ with $\exists \gamma > \delta : \lambda \geq \gamma \forall \lambda \in \alpha_x$ implies $d(0, \alpha_x) > d(0, \alpha^0_x) - \varepsilon$.

If $\lambda \in \alpha_x$, $x = \lambda a$, then $\|a\| = (1/\lambda)\|x\| \leq (1/\gamma_0)\|x\| < (1/\delta)\|x\|$. We denote $\eta(\varepsilon) = (1/\delta(\varepsilon))\|x\|$ and obtain

$\forall \varepsilon > 0 \exists \eta(\varepsilon) > 0$ with $d(A^0_x, \overline{\overline{S(0,\eta)}^c}) > 0$ such that $\forall A \in Ca(X)$ with $d(A_x, \overline{\overline{S(0,\eta)}^c}) > 0$ implies $p_A(x) > p_{A^0}(x) - \varepsilon$.

We shall show that $d(A_x, \overline{\overline{S(0,\eta)}^c}) > 0$ is equivalent with $d(A, \overline{I_x(\eta)}) > 0$;

For the first implication suppose that $d(A, \overline{I_x(\eta)}) = 0$.

Then there exist two sequences $(a_m) \subset A, (u_m) \subset \overline{I_x(\eta)}$ such that $\|a_m - u_m\| \to 0$. But $A, \overline{I_x(\eta)}$ are closed sets with $A_\infty \cap \overline{I_x(\eta)})_\infty = \{0\}$ (where $C_\infty = \cap_{\varepsilon > 0}\varepsilon(C - c), c \in C$ arbitrary, is the asymptotical cone of convex closed set C) and $\overline{I_x(\eta)}$ is asymptotically compact (there exist $\varepsilon_0 > 0$ and a neighbohood V_0 of the origin such that $([0, \varepsilon_0]_A) \cap V_0$ is relatively compact). Using one closedness criterion (see, for ex., [2] or [10]) we have that $A - \overline{I_x(\eta)}$ is closed; the $0 \in A - \overline{I_x(\eta)}$, thus $d(A_x, A(0,\eta)^c = 0$, a contradiction.

Conversely, if suppose $d(A_x, S(0,\eta)^c) = 0 = d(\bar{A}_x, S(0,\eta)^c)$ then there exist $(a_m) \subset \bar{A}_x$ and $(u_m) \subset S(0,\eta)^c$ such that $\|a_m - u_m\| \to 0$; $\bar{A}_x$ and $S(0,\eta)^c$ are closed, $\bar{A}_x$ is asymtotically compact and $(\bar{A}_x)_\infty \cap (S(0,\eta)^c)_\infty = \{0\}$; hence, as above, it follows that $d(A, \overline{I_x(\eta)}) = 0$, which is a contradiction.

Thus the second type of generators is $(I_x(\eta))^\#$.

Acknowledgement. The author thanks Professor T. Precupanu for his suggestions and constant encouragement.

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Received:28.VII.1993

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Tomul XL, s.I.a, Matematică, 1994, f1.

SEMI - SLANT SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

BY

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The differential geometry of CR- submanifolds in a Kaehlerian manifold has been investigated by many people (for references see A. B e j a n c u. [1]).

Recently in [2], B.Y. C h e n introduced and studied the notion of slant submanifold in an almost Hermitian manifold. The particular classes of complex and totally real submanifolds are contained in the class of CR - submanifolds as well as in the class of slant submanifolds, but the class of proper CR - submanifolds is disjoint of the class of proper slant submanifolds.

The purpose of the present note is to introduce a class of submanifolds in an almost Hermitian manifold, called the class of semi - slant submanifolds, so that the class of proper CR - submanifolds and the class of slant submanifolds to appear as particular cases in the class of semi - slant submanifolds.

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J and almost Hermitian metric g, and let M be an m - dimensional Riemannian manifold isometrically immersed in $\bar{M}$. We shall denote by the same symbol g the Riemannian metric induced on M. Denote by TM and $TM^{\perp}$ respectively the tangent bundle of M and the normal bundle to M.

A differentiable distribution D on M is called a slant distribution if for any nonzero vector $X \in D_x; x \in M$, the angle $\theta(X)$ between JX and the vector space D_x is constant, i.e. $\theta(X)$ is independent of the choice of $x \in M$ and of $X \in D_x$. The constant angle θ is called the slant angle of the slant distribution D.

Remark that the complex and totally real distributions on M are slant distributions with $\theta = 0$ and $\theta = \pi/2$, respectively. A slant distribution is