Acknowledgement. The author thanks Professor T. Precupanu for his suggestions and constant encouragement.

REFERENCES

1. A t t o u c h, H. – *Variational Convergence for Functions and Operators*, Pitman, Boston, 1984.
2. B a r b u, V., P r e c u p a n u, T. – *Convexity and Optimization in Banach Spaces*, (Second English Edition) Publ. House of Romanian Acad. and Riedel Publishing Comp., 1986
3. B e e r, G. – *Infima of convex functions*, Trans. Amer. Math. Soc. 315 (1989), 849 - 864.
4. B e e r, G. – *On the Mosco convergence of convex sets*, Bull. Australian Math. Soc. 38 (1988), 239 - 253.
5. B e e r, G. – *Support and distance functionals for convex sets*, Numer. Funct. Anal. and Optimiz. 10 (1989), 15 - 36.
6. B e e r, G. – *Three characterizations of the Mosco topology for convex functions*, to appear
7. B e e r, G., P a i, D. – *On the convergence of convex sets and relative Chebyshev centers*, J. Approx. Theory 62 (1990), 147 - 169.
8. M o s c o, U. – *Convergence of convex sets and solutions of variational unequalities*, Advance in Math. 3 (1969), 510 - 589.
9. M o s c o, U. – *On the continuity of Young - Fenchel transform*, J. Math. Anal. Appl. 35 (1971), 518 - 535.
10. P r e c u p a n u, T. – *Spaţii liniare topologice şi elemente de analiză convexă*, Ed. Acad. Române, Bucureşti, 1992.
11. S h u n m u g a r a j, P., P a i, D. – *On approximate minima of a convex functional and lower semicontinutity of metric projections*, J. Approx. Theory 62 (1991), 32 - 55.
12. S o n n t a g, Y. – *Convergence au sens de Mosco: théorie et applications à l'approximation des solutions d'inequations*, Thèse d'Etat, Univ. de Provance (1982).
13. S o n n t a g, Y., Z ă l i n e s c u, C. – *Set convergence. An attemps of calssification*, to appear in Trans. Amer. Math. Soc.

Received:28.VII.1993

Faculty of Mathematics
University "Al.I.Cuza" Iaşi
Iaşi - 6600
ROMANIA

ANALELE STIIŢIFICE ALE UNIVERSITAŢII "AL.I.CUZA" IAŞI
Tomul XL, s.I.a, Matematică, 1994, f1.

SEMI - SLANT SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

BY

NECULAI PAPAGHIUC

The differential geometry of CR- submanifolds in a Kaehlerian manifold has been investigated by many people (for references see A. B e j a n c u. [1]).

Recently in [2]. B.Y. C h e n introduced and studied the notion of slant submanifold in an almost Hermitian manifold. The particular classes of complex and totally real submanifolds are contained in the class of CR - submanifolds as well as in the class of slant submanifolds, but the class of proper CR - submanifolds is disjoint of the class of proper slant submanifolds.

The purpose of the present note is to introduce a class of submanifolds in an almost Hermitian manifold, called the class of semi - slant submanifolds, so that the class of proper CR - submanifolds and the class of slant submanifolds to appear as particular cases in the class of semi - slant submanifolds.

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J and almost Hermitian metric g, and let M be an m - dimensional Riemannian manifold isometrically immersed in $\bar{M}$. We shall denote by the same symbol g the Riemannian metric induced on M. Denote by TM and $TM^{\perp}$ respectively the tangent bundle of M and the normal bundle to M.

A differentiable distribution D on M is called a slant distribution if for any nonzero vector $X \in D_x$; $x \in M$, the angle $\theta(X)$ between JX and the vector space D_x is constant, i.e. $\theta(X)$ is independent of the choice of $x \in M$ and of $X \in D_x$. The constant angle θ is called the slant angle of the slant distribution D.

Remark that the complex and totally real distributions on M are slant distributions with $\theta = 0$ and $\theta = \pi/2$, respectively. A slant distribution is said to be proper if it is neither complex nor totally real.

Suppose there exist on M two differentiable orthogonal distributions D_1 and D_2 such that the following conditions are fulfilled:

(i) $TM = D_1 \oplus D_2$.

(ii) the distribution D_1 is a complex distribution, i.e. $J(D_{1x}) = D_{1x}$, for each $x \in M$.

(iii) the distribution D_2 is a slant distribution with the slant angle $\theta \neq 0$.

Definition. *A submanifold M of a Hermitian manifold $\tilde{M}$ endowed with the pair of orthogonal distributions (D_1, D_2) satisfying conditions (i), (ii), (iii) is called a semi - slant submanifold of $\tilde{M}$.*

Suppose the dimension of distribution D_1 (resp. D_2) is $2p$ (resp. q). Then it is easily seen that we have the following particular cases:

(a) when $q = 0$, then M is a complex submanifold.

(b) when $p = 0$ and $\theta = \pi/2$, then M is a totally real submanifold.

(c) when $pq \neq 0$ and $\theta = \pi/2$, then M is a proper CR - submanifold.

(d) when $p = 0$ and $\theta \neq \pi/2$, then M is a proper slant submanifold.

A semi - slant submanifold is called a proper semi - slant submanifold if $pq \neq 0$ and $\theta \neq \pi/2$. Remark that if the real dimension of the almost Hermitian manifold $\tilde{M}$ is $2n$ and M is a proper semi - slant submanifold of $\tilde{M}$, then we have $n \geq p + q$ and $q > 1$.

In order to get an example of proper semi - slant submanifold, we denote by E^{2n} the Euclidean $2n$ - space with the standard metric and with the almost complex structure J_0 defined by $J_0(a_1, ..., a_n, b_1, ..., b_n) = (-b_1, ..., -b_n, a_1, ..., a_n)$. Then (E^{2n}, J_0) is complex analitically isometric to the complex number space C^n with the standard flat Kahlerian metric. Consider $\tilde{M} = C^3 = (E^6, J_0)$ and for any $\theta \in (0, \pi/2)$ consider the immersion $f : E^4 \to E^6$ defined by $f(u_1, u_2, u_3, u_4) = (u_1, u_3 \cos\theta, u_3 \sin\theta, u_2, u_4, 0)$. Denote by D_1 the distribution on $M = f(E^4)$ spanned by $X_1 = (1, 0, 0, 0, 0, 0)$ and $Y_1 = (0, 0, 0, 1, 0, 0)$. Also, denote by D_2 the distribution on $M = f(E^4)$ spanned by $X_2 = (0, \cos\theta, \sin\theta, 0, 0, 0)$ and $Y_2 = (0, 0, 0, 0, 1, 0)$. The normal bundle to M is spanned by $Z_1 = (0, 0, 0, 0, 0, 1)$ and $Z_2 = (0, -\sin\theta, \cos\theta, 0, 0, 0)$. It is easily seen that $J_0(X_1) = Y_1$, $J_0(Y_1) = -X_1$, $J_0(X_2) = (\cos\theta)Y_2 + (\sin\theta)Z_1$, $J_0(Y_2) = (-\cos\theta)X_2 + (\sin\theta)Z_2$, hence we obtain that the submanifold $M = f(E^4)$ of (E^6, J_0) is a proper semi - slant submanifold with the complex distribution D_1 and the slant distribution D_2 with the slant angle θ.

In the following we are dealing only with semi - slant submanifolds of a Kaehlerian manifold.

Let M be a $(2p + q)$ - dimensional semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. The projection operators of TM on D_1 and D_2 are

denoted respectively by P and Q. Then we have

$$X = PX + QX \tag{1}$$

for any vector field X tangent to M.

Denote by $\Gamma(TM)$ the module of all differentiable vector fields on M and by $\Gamma(D_1)$ (resp. $\Gamma(D_2)$) the module of differentiable sections on D_1 (resp. D_2). For any $X \in \Gamma(D_2)$ we put

$$JX = tX + nX \tag{2}$$

where tX and nX are the tangential and the normal components of JX, respectively. From (1) and (2) we have

$$JX = JPX + tQX + nQX \tag{3}$$

for any $X \in \Gamma(TM)$.

For any vector field N normal to the submanifold M, we put

$$JN = BN + CN \tag{4}$$

where BN and CN are the tangential and the normal components of JN, respectively.

Let $\tilde{\nabla}$ and ∇ be the Levi - Civita connection on $\tilde{M}$ and M, respectively and denote by h the second fundamental form of M and by A_N the fundamental tensor of Weingarten with respect to the normal section N. Then the Gauss and Weingarten formulas of M in $\tilde{M}$ are given resectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, where $\nabla^{\perp}$ is the normal connection induced by $\tilde{\nabla}$ on $TM^{\perp}$. It is known that

$$g(h(X, Y), N) = g(A_N X, Y). \tag{7}$$

Now we state the follwing general lemma.

Lemma 1. *Let M be a semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. Then we have*

$$P(\nabla_X JPY) + P(\nabla_X tQY) = JP(\nabla_X Y) + P(A_{nQY} X), \tag{8}$$

$$Q(\nabla_X JPY) + Q(\nabla_X tQY) = tQ(\nabla_X Y) + Q(A_{nQY}X) + Bh(X,Y), \tag{9}$$

$$h(X, JPY) + h(X, tQY) + \nabla^{\perp}_X nQY = Ch(X,Y) + nQ(\nabla_X Y), \tag{10}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Substituting Y by JY in (5) and using (3) and $\tilde{\nabla}_X JY = J\tilde{\nabla}_X Y$, we obtain

$$\begin{aligned} &\nabla_X JPY + h(X, JPY) + \nabla_X tQP + h(X, tQY) - \\ &- A_{nQY}X + \nabla^{\perp}_X nQY = J\nabla_X Y + Jh(X,Y). \end{aligned} \tag{11}$$

Next, by using (1), (3) and (4) from (11) follows (8) - (10), by identifying the components from D_1, D_2 and $TM^{\perp}$.

By using Lemma 1, we can obtain the integrability conditions of distributions D_1 and D_2 which are involved in the definition of a semi - slant submanifold. We have

Proposition 2. *Let M be a semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. Then the complex distribution D_1 is integrable if and only if we have*

$$h(X, JY) = h(JX, Y) \tag{12}$$

for any $X, Y \in \Gamma(D_1)$.

Proof. For all $X, Y \in \Gamma(D_1)$, from (10) we get $h(X, JY) = Ch(X,Y) + nQ(\nabla_X Y)$. Since h is a symmetric morphism of vector bundles it follows that $h(X, JY) - h(JX, Y) = nQ([X,Y])$. So, $[X,Y] \in \Gamma(D_1)$ if and only if (12) is satisfied.

Proposition 3. *Let N be a semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. Then the slant distribution D_2 is integrable if and only if we have*

$$P(\nabla_X tY - \nabla_Y tX) = P(A_{nY}X - A_{nX}Y) \tag{13}$$

for any $X, Y \in (D_2)$.

Proof. From (8) we obtain $P(\nabla_X tY) = JP(\nabla_X Y) + P(A_{nY}X)$ for any $X, Y \in \Gamma(D_2)$. Therefore, we get

$$JP([X,Y]) = P(\nabla_X tY - \nabla_Y tX) + P(A_{nX}Y - A_{nY}X) \tag{14}$$

for any $X, Y \in \Gamma(D_2)$. Thus, the assertion of Proportion 3 follows from (14).

Remark. In the particular case when M is a CR - submanifold with the totally real distribution D_2, we have $tX = 0$ and $nX = JX$ for any $X \in \Gamma(D_2)$ and the relation (13) becomes $P(A_{JX}Y) = P(A_{JY}X)$ for any $X, Y \in \Gamma(D_2)$. But, in this case we have $A_{JX}Y = A_{JY}X$ for any $X, Y \in \Gamma(D_2)$; therefore the totally real distribution of a CR - submanifold of a Kaehlerian manifold is always integrable.

Now, let M be a semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. Then, from the decomposition (3) of JX for any $X \in \Gamma(TM)$, we have that JP and tQ are endomorphisms of the tangent bundle TM and nQ is a normal bundle valued 1 - form on TM. We denote by $F = JP + tQ$ and remark that when M is a proper semi - slant summanifold of $\tilde{M}$ then F is an automorphism of the tangent bundle TM. Concerning the endomorphism F and the normal bundle valued 1- form nQ, from Lemma 1 we get

Lemma 4. *Let M be a semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. Then we have*

$$(\nabla_X F)Y = Bh(X,Y) + A_{nQY}X, \tag{16}$$

$$(\nabla_X nQ) = Ch(X,Y) - h(X, FY), \tag{17}$$

for any $X, Y \in \Gamma(TM)$, *where* $(\nabla_X F)Y = \nabla_X FY - F\nabla_X Y$ *and* $(\nabla_X nQ))Y = \nabla^{\perp}_X nQY - nQ\nabla_X Y$.

From (16) and (17) we easily obtain

Lemma 5. *Let M be a semi - slant submanifold of a Kaehlarian manifold $\tilde{M}$. Then we have*

(i) the endomorphism F is parallel, i.e. $\nabla F = 0$, if and only if

$$A_{nQY}X = A_{nQX}Y \tag{18}$$

for any $X, Y \in \Gamma(TM)$.

(ii) the normal bundle valued 1 - form nQ is parallel, i.e. $\nabla(nQ) = 0$, if and only if

$$A_{CN}X = -A_N FX \tag{19}$$

for any $X \in \Gamma(TM)$ *and* $N \in (TM^{\perp})$.

Remark. If M is either a complex submanifold or a totaly real submanifold, or a CR - product, or a Kaehlerian slant submanifold of a Kaehlerian manifold $\tilde{M}$ then $\nabla F = 0$.

A proper semi - slant submanifold M of a Kaehlerian manifold $\tilde{M}$ is said to be a *Kaehlaerian semi - slant submanifold* if the automorphism F is parallel.

Now we state

Lemma 6. *Let M be a proper semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. Then the endomorphism F of TM is parallel if and only if the endomorphism tQ is parallel.*

Proof. Suppose F is parallel. Then (16) implies

$$Bh(X,Y) + A_{nQY}X = 0 \tag{20}$$

for any $X, Y \in \Gamma(TM)$. From (20) we get

$$PA_{nQY}X = 0 \tag{21}$$

for any $X, Y \in \Gamma(TM)$ and

$$Bh(X,Y) = 0; \quad X \in \Gamma(TM), \quad Y \in \Gamma(D_1). \tag{22}$$

By using (22) and (10) we obtain $nQ(\nabla_X Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$, i.e. we have

$$\nabla_X Y \in \Gamma(D_1); \quad X \in \Gamma(TM), \quad Y \in \Gamma(D_1). \tag{23}$$

From (23), by taking into account that ∇ is the Levi Civita connection on M and the distribution D_1 and D_2 are orthogonal, we get $\nabla_X Z \in \Gamma(D_2)$ for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D_2)$, i.e. we have

$$P(\nabla_X tQY) = 0; \quad X, Y \in \Gamma(TM). \tag{24}$$

Finally, by using (21), (23), (24) and (8) we get $\nabla_X JPY - JP\nabla_X Y = 0$ for any $X, Y \in \Gamma(TM)$, i.e. the endomorphism JP is parallel, therefore the endomorphism $tQ = F - JP$ is parallel too.

Conversely, suppose the endomorphism tQ is parallel, i.e. we have

$$\nabla_X tQY - tQ\nabla_X Y = 0; \quad X, Y \in \Gamma(TM). \tag{25}$$

From (25) we have $P(\nabla_X tQY) = 0$ for any $X, Y \in \Gamma(TM)$, i.e.

$$\nabla_X Z \in \Gamma(D_2); \quad X \in \Gamma(TM), \quad Z \in \Gamma(D_2). \tag{26}$$

Because ∇ is the Levi Civita connection on M and the distributions D_1 and D_2 are orthogonal, from (26) we get

$$\nabla_X Y \in \Gamma(D_1); \quad X \in \Gamma(TM), \quad Y \in \Gamma(D_1). \tag{27}$$

By using (25), (26) and (27) in (8) and (9) we obtain

$$\nabla_X JPY - JP\nabla_X Y = PA_{nQY}X, \tag{28}$$

$$QA_{nQY}X + Bh(X,Y) = 0 \tag{29}$$

for any $X, Y \in \Gamma(TM)$. On the other hand, by using (29) and (7) we get

$$PA_{nQY} = 0; \quad X, Y \in \Gamma(TM). \tag{30}$$

Then (28) and (30) implly that the endomorphism JP is parallel, i.e. $F = JP + tQ$ is parallel too and the proof is complete.

From Lemma 6 it follows that a Kaehlerian semi - slant submanifold M of a Kaehlerian manifold $\tilde{M}$ is a Kaehlerian manifold with respect to the induced metric and with the almost complex structure $\tilde{J}$ defined by $\tilde{J} = JP + \frac{1}{\cos\theta} tQ$, where θ denotes the slant angle of the slant distribution on M.

Finally, from the proof of Lemma 6 and by taking into account the definition of a Kaehlerian semi - slant submanifold we obtain

Proposition 7. *Let M be a proper semi - slant submanifold of a Kaehlerian manifold $\tilde{M}$. Then M is a Kaehlerian semi - slant submanifold if and only if M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of the complex distribution D_1 immersed in $\tilde{M}$ as a totally geodesic complex submanifold of $\tilde{M}$ and M_2 is a leaf of the slant distribution D_2 immersed in $\tilde{M}$ as a Kaehlerian slant submanifold of $\tilde{M}$.*

REFERENCES

1. B e j a n c u, A. – *Geometry of CR - submanifolds*, D.Reidel Publishing Co., Dordrecht - Boston - Lancaster - Tokyo, 1986, 169 p.
2. C h e n B.Y. – *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, 1990, 120 p.

Received:15.XII.1993

Department of Mathematics
Polytechnical Institute of Iaşi
ROMANIA