ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XL, s.I.a, Matematică, 1994, f.1.

ON THE EKELAND AND BORWEIN - PREISS PRINCIPLES IN FINITE DIMENSIONS

BY

AURELIAN BEJANCU

1. Introduction and statements. Let (X,d) be a metric space and $f : X \to (-\infty, +\infty]$ a bounded from below function, not identically $+\infty$. Many times, instead of looking for the existence of a *solution* to the *minimization problem*

$$f(x) = \inf_X f, \tag{1}$$

one has to study *approximation solutions*, i.e. to consider for some $\varepsilon > 0$ the points x_ε for which

$$f(x_\varepsilon) \le \varepsilon + \inf_X f. \tag{2}$$

In analogy with the problem of searching necessary properties (like Fermat's principle) inherited by a minimum point, the question is now to find what conditions are necessarily satisfied by the approximate solutions.

An answer was provided by the celebrated Ekeland's variational principle: whenever (X,d) is complete, we always can find approximate solutions with the property that they are *strict solutions* of certain *perturbed minimization problems*. For a general and short proof, as well as for connected results and literature we refer the reader to [3].

Here is the statement of Ekeland's principle in the Euclidean case $(X = \mathbb{R}^n)$:

Theorem 1. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a lower semicontinuous (l.s.c.) function, not identically $+\infty$, bounded from below. Then for every $\varepsilon > 0, \lambda > 0$ and every approximate solution x_ε satisfying (2), there exists an approximate solution x'_ε such that:*

$$f(x'_\varepsilon) \le f(x_\varepsilon), \tag{3}$$

(4) $$\|x'_\varepsilon - x_\varepsilon\| \leq \lambda,$$

(5) $$f(x'_\varepsilon) < f(x) + (\varepsilon/\lambda)\|x - x'_\varepsilon\| \text{ for any } x \in I\!R^n, x \neq x'_\varepsilon.$$

In this context, Hiriart - Urruty [2] used a specific finite - dimensional argument (contained in Lemma 1 below) by means of which he proved only a weaker conclusion.

Namely, instead of the strict inequality in (5) he obtained just a non - strict one for all $x \in I\!R^n$. His proof relies on the minimization of the associated perturbed function

(6) $$g(x) := f(x) + (\varepsilon/\lambda)\|x - x_\varepsilon\|, \quad x \in I\!R^n$$

by using:

Lemma 1. *Let $g : I\!R^n \to (-\infty, +\infty]$ be a l.s.c. function, not identically $+\infty$, which verifies*

(7) $$\lim_{\|x\| \to +\infty} g(x) = +\infty.$$

Then g admits an absolute minimizer (i.e. on $I\!R^n$).

In this note we will modify the Hiriart - Urruty's approach to give a new and complete proof for Theorem 1.

The idea of our proof is to consider not just a single minimum point of g (like in [2]), but the set A of all absolute minimizers of g. The solution x'_ε of (5) will then be obtained as an optimal element for the original function f on this set A (see relation (8) below).

In addition, by using the same Lemma 1, we give a special proof for a finite - dimensional variant of the Borwein - Preiss' "smooth" variational principle (see [1] for the general statement). This principle is a smooth analogue of Ekeland's principle in which the perturbations are p th - powers of norms, $p \geq 1$. This is our precise statement:

Theorem 2. *Let $f : I\!R^n \to (-\infty, +\infty]$ be a l.s.c. function and $\varepsilon > 0, \lambda > 0, p \geq 1$ be given constants. Suppose that x_ε satisfies*

(2') $$f(x_\varepsilon) < \varepsilon + \inf_{I\!R^n} f.$$

Then there exists a neighborhood U of x_ε such that for any $x'_\varepsilon \in U$ there exists $x''_\varepsilon \in I\!R^n$ with the properties:

(3') $$f(x''_\varepsilon) < \varepsilon + \inf_{I\!R^n} f.$$

(4') $$\|x''_\varepsilon - x_\varepsilon\| < \lambda,$$

(5') $$f(x''_\varepsilon) + (\varepsilon/\lambda^p)\|x''_\varepsilon - x'_\varepsilon\|^p \leq f(x) + (\varepsilon/\lambda^p)\|x - x'_\varepsilon\|^p \text{ for any } x \in I\!R^n.$$

Here we will minimize the associated perturbed function given by the right - hand side of (5'). Unlike Theorem 1, the problem lies no more in proving (5'), but rather in obtaining the strict inequalities (3') and (4'). The proof will use essentially the same estimates from [1] (Theorem 5.2), but the finite - dimensional setting allows one to obtain the stronger conclusion with the neighborhood U.

2. The proofs.

Proof of Theorem 1. Given ε, λ and x_ε, we consider the perturbed function (6). By the hypotheses of the theorem, it follows that $g : I\!R^n \to (-\infty, +\infty]$ is l.s.c., not identically $+\infty$ and (7) holds. Then Lemma 1 ensures that the set A of all absolute minimizers of g is nonvoid. But $A = \{x \in I\!R^n : g(x) = \inf_{I\!R^n} g\} = \{x \in I\!R^n : g(x) \leq \inf_{I\!R^n} g\}$.

So g l.s.c. implies A is closed and (7) implies A is bounded in $I\!R^n$. Hence A is a compact set and by the Weierstrass theorem applied to the original l.s.c. function f, there exists $x'_\varepsilon \in A$ with:

(8) $$f(x'_\varepsilon) = \inf_A f.$$

We show that this is the required element of the conclusion.

Indeed, $x'_\varepsilon \in A$ implies:

(9) $$f(x'_\varepsilon) + (\varepsilon/\lambda)\|x'_\varepsilon - x_\varepsilon\| \leq f(x) + (\varepsilon/\lambda)\|x - x_\varepsilon\|, \ (\forall)\, x \in I\!R^n.$$

Taking $x := x_\varepsilon$ in (9) we get $f(x'_\varepsilon) + (\varepsilon/\lambda)\|x'_\varepsilon - x_\varepsilon\| \leq f(x_\varepsilon)$. Hence (3) follows and by (2) we deduce:

$$\|x'_\varepsilon - x_\varepsilon\| \leq (\lambda/\varepsilon)[f(x_\varepsilon) - f(x'_\varepsilon)] \leq \leq (\lambda/\varepsilon)[\varepsilon + \inf_{I\!R^n} f - f(x'_\varepsilon)] \leq \lambda,$$

i.e. (4). To prove (5), we distinguish two cases:

(i) If $x \neq x'_\varepsilon$ and $x \in A$, then by (8)

$$f(x'_\varepsilon) \leq f(x) < f(x) + (\varepsilon/\lambda)\|x - x'_\varepsilon\|.$$

(ii) If $x \neq x'_\varepsilon$ and $x \notin A$, then $g(x'_\varepsilon) < g(x)$, i.e. we have strict inequality in (9). By triangle inequality, it follows that:

$$f(x'_\varepsilon) \leq f(x) + (\varepsilon/\lambda)(\|x - x_\varepsilon\| - \|x'_\varepsilon - x_\varepsilon\|) \leq$$
$$\leq f(x) + (\varepsilon/\lambda)\|x - x'_\varepsilon\|.$$

Thus (5) is true in any case and the proof is complete.

Proof of Theorem 2. Fix constants ε_1 and ε_2 with

$$f(x_\varepsilon) - \inf_{I\!R^n} f < \varepsilon_2 < \varepsilon_1 < \varepsilon.$$

We will prove the theorem for $U = \{x \in I\!R^n : x \text{ satisfies (10) and (11) }\}$, where

$$\|x - x_\varepsilon\| \leq \lambda[1 - (\varepsilon_1/\varepsilon)^{1/p}] \tag{10}$$

and

$$\|x - x_\varepsilon\| \leq \lambda[(\varepsilon_1 - \varepsilon_2)/\varepsilon]^p. \tag{11}$$

(Note that (when ε_1 is fixed) we may choose ε_2 sufficiently close to ε_1 so that (11) $\Rightarrow$ (10)).

Fix now any $x'_\varepsilon \in U$ and consider the perturbed function

$$h(x) := f(x) + (\varepsilon/\lambda^p)\|x - x'_\varepsilon\|^p, \quad x \in I\!R^n.$$

Because of the strict inequality in (2′) (whence f is bounded from below and not identically $+\infty$). h verifies the hypotheses of Lemma 1. Hence there exists $x''_\varepsilon \in I\!R^n$ with property (5′). For this x''_ε we prove now (3′) and (4′).

Firstly, by (10)

$$\begin{aligned} f(x_\varepsilon) + (\varepsilon/\lambda^p)\|x'_\varepsilon - x_\varepsilon\|^p &\leq \\ \leq f(x_\varepsilon) + \varepsilon[1 - (\varepsilon_1/\varepsilon)^{1/p}]^p &< \varepsilon + \inf_{I\!R^n} f, \end{aligned} \tag{12}$$

because $f(x_\varepsilon) - \inf_{I\!R^n} f < \varepsilon_1 \leq \varepsilon\{1 - [1 - (\varepsilon_1/\varepsilon)^{1/p}]^p\}$ (the last inequality is an easy calculus exercise). On the other hand, for $x := x_\varepsilon$ in (5′) we get

$$\begin{aligned} f(x_\varepsilon) + (\varepsilon/\lambda^p)\|x_\varepsilon - x'_\varepsilon\|^p &\geq \\ \geq f(x''_\varepsilon) + (\varepsilon/\lambda^p)\|x''_\varepsilon - x'_\varepsilon\|^p &\geq f(x''_\varepsilon), \end{aligned} \tag{13}$$

so (12) and (13) imply (3′).

Secondly, (11) and the first part of (13) imply

$$f(x_\varepsilon) + \varepsilon_1 - \varepsilon_2 \geq f(x''_\varepsilon) + (\varepsilon/\lambda^p)\|x''_\varepsilon - x'_\varepsilon\|^p \geq$$
$$\geq \inf_{I\!R^n} f + (\varepsilon/\lambda^p)\|x''_\varepsilon - x'_\varepsilon\|^p.$$

hence

$$(\varepsilon/\lambda^p)\|x''_\varepsilon - x'_\varepsilon\|^p \leq f(x_\varepsilon) - \inf_{I\!R^n} f + \varepsilon_1 - \varepsilon_2 < \varepsilon_1$$

and

$$\|x''_\varepsilon - x'_\varepsilon\| < \lambda(\varepsilon_1/\varepsilon)^{1/p}. \tag{14}$$

From (14) and (10) we have now

$$\|x''_\varepsilon - x_\varepsilon\| \leq \|x''_\varepsilon - x'_\varepsilon\| + \|x'_\varepsilon - x_\varepsilon\| <$$
$$< \lambda[1 - (\varepsilon_1/\varepsilon)^{1/p}] + \lambda(\varepsilon_1/\varepsilon)^{\frac{1}{p}} = \lambda,$$

i.e. (4′) and the theorem is proved.

Aknowledgement. A first version of this note was completed while the author attended the Summer Courses in Mathematics, Perugia 1993.

REFERENCES

1. B o r w e i n, J.M., P r e i s s, D. – *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions.* Research Report, Dept. of Math., Dalhousie Univ., Halifax, Canada, 1986, Trans. Amer. Math. Soc., 303, 1987, 517 - 527.
2. H i r i a r t - U r r u t y, J.B.– *A short proof of the variational principle for approximate solutions of a minimization problem,* Am. Math. Mon. 90, 1983, 206 - 207.
3. P e n o t, J.-P. – *The Drop Theorem, the Petal Theorem and Ekeland's variational principle.* Nonlin. Analysis T.M.A., 10 (9), 1986, 813 - 822.

Received:28.X.1993
Revised:28.II.1994

"Al.Myller" Mathematical Seminar
University of Iaşi
6600 Iaşi
ROMANIA