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SPECTRUM OF POSITIVE DEFINITE FUNCTIONS ON HYPERGROUPS

BY

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1.Introduction. There has recently been considerable interest shown by some harmonic analysts in the question of which topological spaces have enough structure so that a convolution on the corresponding space of all finite regular Borel measures can be defined. D u n k l [3], J e w e t t [5] and S p e c t o r [7] have all considered this question and they given axioms which are essentially the same. These objects were called Hypergroups. We note that some methods of proof used in the group case are not available for hypergroups.

Let K be a locally compact Hausdorff space, $M(K)$ denote the space of all bounded Radon measures, $M^1(K)$ be the subset of all probability measures and P_x be the point measure of $x \in K$. The support of a measure μ is denoted by *supp* μ. $C(K)$ denotes the space of continuous functions on K. The space K is called a hypergroup if the following conditions are satisfied:

(H1) There exists a map : $K \times K \to M^1(K), (x,y) \to P_x * P_y$, called convolution, which is continuous, where $M^1(K)$ bears the vague topology. The linear extension to $M(K)$, see [5,Lemma 2.4B], satisfies

$$P_x * (P_y * P_z) = (P_x * P_y) * P_z$$

(H2) $supp\ P_x * P_y$ is compact.

(H3) There exists a homeomorphism $K \to K, x \to \bar{x}$, called involution, such that $x = \bar{\bar{x}}$ and $(P_x * P_y)^- = P_{\bar{y}} * P_{\bar{x}}$.

(H4) There exists an element $e \in K$, called unit element, such that $P_e * P_x = P_x * P_e = P_x$.

(H5) $e \in supp\ P_x * P_{\bar{y}}$ if and only if $x = y$.

(H6) The map $(x,y) \to supp\, P_x * P_y$ of $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter space with the topology as given in [5,7].

Now, let K be a hypergroup and $P(K)$ be the convex set of all continuous positive - definite functions φ on K with $\varphi(e) = 1$. The spectrum $sp\,\varphi$ of $\varphi \in P(K)$ can be defined as the set of all indecomposable $\psi \in P(K)$ which are limits, in the sense of the topology of uniform convergence on compact subsets of K, of functions of the form

$$x \to \sum_{i,j=1}^{n} P_{x_i} * P_{x_j}(\varphi(x)) c_i \bar{c}_j,$$

where $c_1, ..., c_n \in C, x_1, ..., x_n \in K$.

If π_φ denotes the cyclic unitary representation of K associated with φ, then $sp\,\varphi$ consists of all $\psi \in P(K)$ for which π_ψ is irreducible and weakly contained in π_φ (see [2], Chapter 18). If f is a positive function on K with compact support, $\tilde{f}(x) = \overline{f(x^{-1})}$ and $*$ is the convolution, it is easy to see that the function $\tilde{f} * f$ is positive - definite.

Our main subject here is to prove that exponentially bounded hypergroups and solvable discrete hypergroups satisfy the following property (which we denote by (P)):

(P) If $\varphi \in P(K)$ and if φ is positive in usual sense, then the constat positive - definite function 1 on K, 1_K, belongs to $sp\,\varphi$.

For connected hypergroups we show that the condition that the hypergroup is amenable is equivalent to the following weaker version (P^*) of P :

(P^*) If $\varphi \in P(K)$ and if φ is positive then $1_K \in sp_d(\varphi)$, where $sp_d(\varphi)$ is the spectrum of φ when the domain of φ is K_d (the discrete hypergroup).

2.Exponentially bounded hypergroups. Let π be a continuous unitary representation of the hypergorup K in the Hilbert space $(H_\pi, <.,.>)$. A unit vector $\xi \in H_\pi$ will be called a positive vector for π, if $Re\,\langle \pi(x)\xi, \xi\rangle \geq 0$ for all $x \in K$. So, $Re\,\langle \pi(.)\xi, \xi\rangle \in P(K)$.

Now, it is easy to translate (P) into a property of unitary representations with positive vectors. In fact, consider the following property (P') of K which is formally stronger than (P) :

(P$'$) If π is a unitary representation of K with a positive vector, then π contains weakly 1_K.

Proposition 2.1 *(P) and (P') are equivalent for every hypergroup K.*

Proof. Let π be a unitary representation of K with a positive vector [illegible] $\in K$. If (P) holds, then 1_K is weakly

contained in π_φ which is the subrepresentation of $\pi \oplus \pi$. Thus 1_K is weakly contained in $\pi \oplus \pi$ and this implies that 1_K is weakly contained in π.

A locally compact hypergroup is called Exponentially bounded if $\lim_n |G^n|^{1/n} = 1$ for each compact neighbourhood G of e, where $|.|$ denotes the Haar measure and $G^n = \{g_1...g_n; g_i \in G\}$. Exponentially bounded hypergroups are amenable [4].

Theorem 2.2 *Exponentially bounded hypergroups satisfy property (P).*

Proof. Let K be an exponentially bounded hypergroups and let $\varphi \in P(K)$ with $\varphi \geq 0$. Let G be a compact neighbourhood of e with $G = G^{-1}$, and $\varepsilon > 0$. Then there is an $n \in N$ such that

$$(1) \qquad \int_{G^{n+1}\times G^{n+1}} P_y * P_z(\varphi)dydz \leq (1+\varepsilon)\int_{G^n \times G^n} P_y * P_z(\varphi)dydz,$$

where dy and dz are Haar measures on K.

In fact, otherwise

$$|G^{n+1}|^2 \geq \int_{(G^{n+1})^2} P_y * P_z(\varphi)dydz > (1+\varepsilon)^n \int_{G\times G} P_y * P_z(\varphi)dydz,$$

for all $n \in N$. Since

$$\int_{G\times G} P_y * P_z(\varphi)dydz > 0,$$

this would be a contradiction with

$$\lim_n |G^n|^{1/n} = 1.$$

Now choosen $n \in N$ such that (1) holds. Let $f = \chi_{G^n}$ be the characteristic function of G^n. Let π be the unitary representation of K associated to φ with Hilbert space H_π. Let $\xi \in H_\pi$ be such that $\varphi(x) = \langle \pi(x)\xi, \xi\rangle, x \in K$. Then

$$\|\pi(f)\xi\|^2 = \int_K f^* * f(x)\varphi(x)dx > 0,$$

since $f^* * f(e)\varphi(e) > 0$ and $f^* * f(x)\varphi(x) \geq 0$ for all $x \in K$.

Now let

$$\psi(x) = \frac{1}{\|\pi(f)\xi\|^2}\langle \pi(x)\pi(f)\xi, \pi(f)\xi\rangle, x \in K.$$

Then ψ is associated to π. Moreover, for each $x \in K$

$$|\psi(x) - 1|^2 = \frac{1}{\|\pi(f)\xi\|^4}|\langle \pi(xf - f)\xi, \pi(f)\xi\rangle|^2 \leq$$
$$\leq \frac{\|\pi(xf - f)\xi\|^2}{\|\pi(f)\xi\|^2} = \int_{K\times K}(xf - f)(y)(xf - f)(z)P_y * P_{\bar z}(\varphi)dydz /$$
$$/ \int_{K\times K} f(y)f(z)P_y * P_{\bar z}(\varphi)dydz =$$
$$= \int_{(xG^n\Delta G^n)^2} P_y * P_{\bar z}(\varphi)dydz / \int_{(G^n)^2} P_y * P_{\bar z}(\varphi)dydz,$$

where Δ is the summetric difference. Now (1) implies that for $x \in G$

$$\int_{(xG^n\Delta G^n)^2} P_y * P_{\bar z}(\varphi)dydz \leq \int_{(G^{n+1})^2\setminus(G^n)^2} P_y * P_{\bar z}(\varphi)dydy+$$
$$+ \int_{(G^n\setminus xG^n)^2} P_y * P_{\bar z}(\varphi)dydy \leq \varepsilon \int_{(G^n)^2} P_y * P_{\bar z}(\varphi)dydz+$$
$$+ \int_{(x^{-1}G^n\setminus G^n)^2} P_y * P_{\bar z}(\varphi)dydz \leq 2\varepsilon \int_{(G^n)^2} P_y * P_{\bar z}(\varphi)dydz,$$

since $x^{-1} \in G$. Hence $|\psi(x) - 1|^2 \leq 2\varepsilon$ for all $x \in G$.

It is to be noted that theorem 2.2 can be reformulate in the form: "If φ is positive and $\varphi \in P(K)$ where K is an exponentially bounded hypergroup, then the constant function 1_K is the uniform limit on compact subsets of K of functions of the form

$$x \to \sum_{i,j=1}^{n} P_{x_i} * P_{\bar x_j}(\varphi(x))c_i\bar c_j,$$

where $c_l \geq 0$ and $x_l \in K$ for all $1 \leq l \leq n$.

Theorem 2.3. *Discrete solvable hypergroups satisfy property (P).*

Proof. Let K be a discrete solvable hypergroup and let $\varphi \in P(K)$ with $\varphi \geq 0$. Let $K = K_n \supseteq K_{n-1} \supseteq \ldots \supseteq K_0 = \{e\}$, be a composition series with abelian factor $K_i/K_{i-1}, 1 \leq i \leq n$. First we show by induction on i that :

for each $0 \leq i \leq n$ there is a net $(\psi_\alpha)_\alpha$ in $P(K)$ with $\psi \geq 0$ such that $\lim \psi_\alpha(x) = 1$ for all $x \in K_i$ and such that π_{ψ_α} is weakly contained in π for all α.

For $i = 0$, the assertion is trivial (take $\psi_\alpha = \varphi$).

For any i suppose that a net $(\psi_\alpha)_{\alpha\in N}$ exists. Let ψ be a limit point of $\{\psi_\alpha\}_{\alpha\in N}$ in teh weak $*$ -topology $\sigma(l^\infty(K), l^1(K))$. Then $\psi \in P(K)$ and $\psi \geq 0$. Moreover

$$\psi(x) = \lim_\alpha \psi_\alpha(x) = 1 \quad forall \quad x \in K_i.$$

Hence $\psi|K_{i-1}$ factors to a positive definite function of K_{i+1}/K_i. Thus by Theorem 2.2 in its reformulated form there is a net $(\psi'_\beta)_\beta$ in $P(K_{i+1}/K_i)$ of the form

$$\psi'_\beta(x) = \sum c_k c_l P_{x_k} * P_{\bar x_l}(\psi(x)), \quad x \in K_{i+1},$$

where all $c_k \geq 0$ and $x_k \in K_{i+1}$, such that

$$\lim \psi'_\beta(x) = 1 \quad for\ all \quad x \in K_{i+1}.$$

It is clear that $\psi'_\beta \in P(K)$ and $\psi'_\beta \geq 0$. Moreover $\pi_{\psi'_\beta} = \pi_\psi$. Hence each $\pi_{\psi'_\beta}$ is weakly contained in $\{\pi_{\psi_\alpha} | \alpha \in A\}$ which is weakly contained in π_φ. So, we get a net $(\psi_\alpha)_\alpha \in P(K)$ such that $\lim \psi_\alpha(x) = 1$ for all $x \in K_n = K$ and such that each π_{ψ_α} is weakly contained in π_φ. Hence 1_K is weakly contained in π_φ.

Now we reformulate property (P^*), defined earlier, as follows:
If π is a unitary representation of K with positive vectors, then 1_K is weakly containde in π when π and 1_K is viewed as representation of the discrete hypergroup K.

Theorem 2.4. *For a connected hypergroup K the following statements are equivalent:*

i) K has property (P^*).

ii) K is amenable.

Proof. Suppose K is amenable. Let N be the closure of the commutative subhypergroup of K, by [1] proposition 3, N has polynomial growth hence it is exponentially bounded [4]. Let $\varphi \in P(K), \varphi \geq 0$. By Theorem 2.2 in its reformulated form there is a net $(\psi_\alpha)_\alpha$ in $P(K)$ with $\psi_\alpha \geq 0$ such that $\lim \psi_\alpha(x) = 1$ for all $x \in N$ such that π_{ψ_α} is weakly contained in π_φ for all α. Considering K as a discrete group we can apply the method of proof of theorem (2.4) to get some $\psi \in P(K), \psi \geq 0$ with $\psi|N = 1$ and such that π_ψ is weakly contained in π_φ. Since K/N is abelian, 1_K is weakly contained in π_ψ and the result follows.

Now if K has property (P^*), then 1_k is weakly contained in the regular representation λ_k, when both representation are considered as representations of K. This is equivalent to the amenability of K [4].

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ON RICCI CURVATURES OF ALMOST COSYMPLECTIC MANIFOLDS

BY

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0. Introduction. The following two results are well known, that is,

Theorem A ([4]). *Let (M,ϕ,ξ,η,g) be an almost cosymplectic manifold of constant ϕ-sectional curvature H. Then the scalar curvature S and the ϕ-sectional curvature H satisfy the inequality $n(n+1)H \geq S$. Equality holds if and only if (M,ϕ,ξ,η,g) is cosymplectic.*

Theorem B ([4]). *Let (M,ϕ,ξ,η,g) be a conformally flat almost cosymplectic manifold with $\dim M \geq 5$. Then the scalar curvature (M,ϕ,ξ,η,g) is non-positive and the manifold is cosymplectic if and only if it is locally flat.*

The purpose of this paper is to generalize Theorems A and B by using the Ricci curvature tensor in T a n n o's way [5].

1. Almost cosymplectic manifolds. Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional almost contact metric manifold, that is, M is a differentiable manifold and (ϕ,ξ,η,g) an almost contact metric structure on M, formed by tensor fields ϕ,ξ,η, of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively, and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1,$$

$$\eta(X) = g(X,\xi), \qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

On such a manifold we may always define 2-form Φ by $\Phi(X,Y) = g(\phi X, Y)$. Then (M,ϕ,ξ,η,g) is said to be an almost cosymplectic manifold if the forms Φ and η are closed, i.e., $d\Phi = 0$, $d\eta = 0$, where d is the operator of exterior differentiation. In particular, if the almost contact structure of an almost cosymplectic manifold is normal, then it is said to be a cosymplectic manifold. As it is known, an almost contact metric structure is cosymplectic if