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ON RICCI CURVATURES OF ALMOST COSYMPLECTIC MANIFOLDS

BY

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0. Introduction. The following two results are well known, that is,

Theorem A ([4]). *Let (M,φ,ξ,η,g) be an almost cosymplectic manifold of constant φ-sectional curvature H. Then the scalar curvature S and the φ-sectional curvature H satisfy the inequality $n(n+1)H \geq S$. Equality holds if and only if (M,φ,ξ,η,g) is cosymplectic.*

Theorem B ([4]). *Let (M,φ,ξ,η,g) be a conformally flat almost cosymplectic manifold with $\dim M \geq 5$. Then the scalar curvature of (M,φ,ξ,η,g) is non-positive and the manifold is cosymplectic if and only if it is locally flat.*

The purpose of this paper is to generalize Theorems A and B by using the Ricci curvature tensor in T a n n o's way [5].

1. Almost cosymplectic manifolds. Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional almost contact metric manifold, that is, M is a differentiable manifold and (ϕ,ξ,η,g) an almost contact metric structure on M, formed by tensor fields φ,ξ,η, of type $(1,1)$,$(1,0)$ and $(0,1)$, respectively, and a Riemannian metric g such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1,$$

$$\eta(X) = g(X,\xi), \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

On such a manifold we may always define 2-form Φ by $\Phi(X,Y) = g(\phi X, Y)$. Then (M,ϕ,ξ,η,g) is said to be an almost cosymplectic manifold if the forms Φ and η are closed, i.e., $d\Phi = 0$, $d\eta = 0$, where d is the operator of exterior differentiation. In particular, if the almost contact structure of an almost cosymplectic manifold is normal, then it is said to be a cosymplectic manifold. As it is known, an almost contact metric structure is cosymplectic if and only if both $\nabla\eta$ and $\nabla\phi$ vanish, where ∇ is the covariant differentiation with respect to g (see [1], also see [3] and [4]). However, if we have $\nabla\phi = 0$, by taking the covariant differentiation of $\phi\xi = 0$, we can see $\nabla\eta = 0$ easily.

We need in the sequel the following lemma.

Lemma 1.1. ([4]) *In an almost cosymplectic manifold M, we have*

$$\nabla_{\phi X}\xi = -\phi(\nabla_X \xi) \tag{1.1}$$

$$(\nabla_{\phi X}\phi)\phi Y + (\nabla_X\phi)Y - \eta(Y)\nabla_{\phi X}\xi = 0 \tag{1.2}$$

$$\begin{aligned} &g(R(X,Y)Z,\xi) - g(R(\phi X,\phi Y)Z,\xi) + g(R(\phi X,Y)\phi Z,\xi)+ \\ &+ g(R(X,\phi Y)\phi Z,\xi) + 2(\nabla_{\nabla_{\phi Z}\xi}\Phi)(X,Y) = 0, \end{aligned} \tag{1.3}$$

where R is the Riemannian curvature tensor of M.

Here, on an almost cosymplectic manifold we define an operator h by $h = -\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then we get the following two lemmas.

Lemma 1.2. *In an almost cosymplectic manifold M, h is symmetric.*

Proof. From (1.1) and (1.2), we have $\nabla_\xi\xi = 0$ and $\nabla_\xi\phi = 0$. Thus we find

$$\begin{aligned} -2g(hX,Y) &= g((\mathcal{L}_\xi\phi)X,Y) = \\ &= g(\nabla_\xi(\phi X) - \nabla_{\phi X}\xi - \phi(\nabla_\xi X) + \phi(\nabla_X\xi), Y) = \\ &= g(-\nabla_{\phi X}\xi + \phi(\nabla_X\xi), Y), \end{aligned}$$

which vanishes if either X or Y is ξ. For X and Y orthogonal to ξ, $d\eta = 0$ leads us to $\eta([\phi X,Y]) = \eta([X,\phi Y]) = 0$. Continuing the computation, we get

$$\begin{aligned} -2g(hX,Y) &= g((\mathcal{L}_\xi\phi)X,Y) = \eta(\nabla_{\phi X}Y) + \eta(\nabla_X(\phi Y)) = \\ &= \eta(\nabla_Y(\phi X)) + \eta(\nabla_{\phi Y}X) = g((\mathcal{L}_\xi\phi)Y,X) = -2g(hY,X), \end{aligned} \tag{1.4}$$

which completes the proof.

Lemma 1.3. *In an almost cosymplectic manifold M, the following relations are satisfied.*

$$\nabla_X\xi = \phi hX \tag{1.5}$$

$$h\xi = 0 \tag{1.6}$$

$$\phi h + h\phi = 0 \tag{1.7}$$

$$Tr\ h = 0, \quad \text{where Tr h is the trace of h.} \tag{1.8}$$

Proof. For an almost cosymplectic manifold M, we generally have (cf. [2])

$$\begin{aligned} 2g((\nabla_X\phi)Y,Z) &= -3d\Phi(X,\phi Y,\phi Z) + 3d\Phi(X,Y,Z)+ \\ &+ g\left(N^{(1)}(Y,Z),\phi X\right) + N^{(2)}(Y,Z)\eta(X)+ \\ &+ 2d\eta(\phi Y,X)\eta(Z) - 2d\eta(\phi Z,X)\eta(Y), \end{aligned} \tag{1.9}$$

where $N^{(1)}(X,Y) = N_\phi(X,Y) + 2d\eta(X,Y)\xi$ $(N_\phi(X,Y) = \phi^2[X,Y] + +[\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y])$ and $N^{(2)}(X,Y) = (\mathcal{L}_{\phi X}\eta)Y - (\mathcal{L}_{\phi Y}\eta)X$. So, in an almost cosymplectic manifold (1.9) turns out

$$2g((\nabla_X\phi)Y,Z) = g(N^{(1)}(Y,Z),\phi X) + N^{(2)}(Y,Z)\eta(X), \tag{1.10}$$

where $N^{(1)}(X,Y) = N_\phi(X,Y)$. Putting $Y = \xi$ into (1.10), we find

$$\begin{aligned} 2g((\nabla_X\phi)\xi,Z) &= g(N^{(1)}(\xi,Z),\phi X) + N^{(2)}(\xi,Z)\eta(X) = \\ &= g(N_\phi(\xi,Z),\phi X) - (\mathcal{L}_{\phi Z}\eta)(\xi)\eta(X) = \\ &= g(\phi^2[\xi,Z],\phi X) + g(\phi[\phi Z,\xi],\phi X) + \eta([\phi Z,\xi])\eta(X) = \\ &= g(\phi[\xi,Z],X) + g([\phi Z,\xi],X) = -g((\mathcal{L}_\xi\phi)Z,X). \end{aligned}$$

Thus, by lemma 1.2, we get

$$(\nabla_X\phi)\xi = hX, \tag{1.11}$$

from which $-\phi\nabla_X\xi = hX$. Therefore we obtain (1.5). From (1.2) and (1.11), we easily find (1.6). Next, we prove (1.7). By (1.4), for X and Y orthogonal to ξ, we see that

$$g(h\phi X,Y) = -\eta(\nabla_X Y) + \eta(\nabla_{\phi X}(\phi Y)) \quad \textit{and}$$

$$g(hX,\phi Y) = \eta(\nabla_{\phi Y}(\phi X)) - \eta(\nabla_Y X).$$

On the other hand, $d\eta = 0$ leads us to $\eta([X,Y]) = \eta([\phi X,\phi Y]) = 0$, from which

$$-\eta(\nabla_X Y) + \eta(\nabla_Y X) + \eta(\nabla_{\phi X}(\phi Y)) - \eta(\nabla_{\phi Y}(\phi X)) = 0.$$

Therefore we obtain (1.7), so that, taking ϕ-basis, we have (1.8).

Moreover, we prepare some relations which hold on an almost cosymplectic manifold. First of all, by (1.5), we get

$$\nabla_r\eta_i\nabla^r\eta_j = h_{ir}h^r{}_j. \tag{1.12}$$

The following lemma is gotten by (1.5) and Olszak's Theorem 4.2 ([6]):

Lemma 1.4. *For any almost cosymplectic manifold we have*

$$\frac{1}{2}(R(\xi,X)\xi - \phi R(\xi,\phi X)\xi) = h^2X, \tag{1.13}$$

$$g(Q\xi,\xi) = -\mathrm{Tr}\,h^2, \tag{1.14}$$

where Q is the Ricci operator.

We also have the following

Lemma 1.5. *The Ricci tensor on an almost cosymplectic manifold M satisfies the following*

$$R_{jr}\xi^r = \nabla_r\nabla_j\xi^r = \nabla^r\nabla_r\eta_j, \tag{1.15}$$

$$\begin{aligned} &\phi^s{}_j\nabla^r\nabla_r\phi_{ks} + \phi^s{}_k\nabla^r\nabla_r\phi_{js} = \\ &= -2\nabla_r\phi_{sj}\nabla^r\phi^s{}_k + R_{jr}\xi^r\eta_k + R_{kr}\xi^r\eta_j + 2h_{jr}h^r{}_k. \end{aligned} \tag{1.16}$$

Proof. Contracting $R^i{}_{rkt}\xi^r = \nabla_k\nabla_t\xi^i - \nabla_t\nabla_k\xi^i$ with respect to i and k, we obtain the first equality of (1.15). To verify the second equality we rewrite $\nabla^r\nabla_r\eta_j$ as $\nabla^r\nabla_r\eta_j = \nabla^r\nabla_j\eta_r$. Then we get (1.15), where we used $\nabla_i\eta_j - \nabla_j\eta_i = 0$. Next, operating $\nabla^r\nabla_r$ to $\phi^s{}_j\phi_{ks} = -g_{jk} + \eta_j\eta_k$, we get

$$\begin{aligned} &\phi^s_j\nabla^r\nabla_r\phi_{ks} + \phi^s{}_k\nabla^r\nabla_r\phi_{js} - 2\nabla_r\phi_{sj}\nabla^r\phi^s{}_k = \\ &= \nabla^r\nabla_r\eta_j\eta_k + \eta_j\nabla^r\nabla_r\eta_k + 2\nabla_r\eta_j\nabla^r\eta_k. \end{aligned}$$

Applying (1.12) (1.15) to the last equation, we obtain (1.16).

Here, we define R^*_{ij} by the same way as in the Kählerian case $2R^*_{ij} = -R_{irkt}\phi^r{}_j\phi^{kt}$. By the Bianchi identity R^*_{ij} is also written as $R^*_{ij} = -R_{ikrt}\phi^r{}_j\phi^{kt}$. We define S^* by $S^* = R^*_{ij}g^{ij}$.

Lemma 1.6. *R^*_{ij} satisfies the following in an almost cosymplectic manifold*

$$R^*_{ij} = R_{ij} + R_{rs}\phi^r{}_i\phi^s{}_j + \nabla_r\phi_{si}\nabla^r\phi^s{}_j + h_{ir}h^r{}_j.$$

Proof. By the Ricci identity for ϕ, we get

$$\nabla_t\nabla_k\phi^i{}_j - \nabla_k\nabla_t\phi^i{}_j = -R^i{}_{skt}\phi^s{}_j + R^s{}_{jkt}\phi^i{}_s.$$

Contracting the last equation with respect to i and k, we find

$$-\nabla_i\nabla_t\phi^i{}_j = -R_{st}\phi^s{}_j + R_{sjrt}\phi^{rs}. \tag{1.18}$$

Transvecting (1.18) by $-\phi^t{}_k$, we get

$$\phi^t{}_k\nabla_i\nabla_t\phi^i{}_j = R_{st}\phi^s{}_j\phi^t{}_k - R^*{}_{jk} \tag{1.19}$$

Transvecting (1.18) by $\phi^j{}_k$, we also have

$$-\phi^j{}_k\nabla_i\nabla_t\phi^i{}_j = R_{kt} - R_{ts}\xi^s\eta_k - R^*{}_{tk} \tag{1.20}$$

Changing t to j in (1.20), the result and (1.19) imply

$$\phi^r{}_k\nabla^i(\nabla_r\phi_{ij} - \nabla_j\phi_{ir}) = R_{jk} + R_{rs}\phi^r{}_j\phi^s{}_k - R_{js}\xi^s\eta_k - 2R^*{}_{jk}.$$

Since we have $\nabla_r\phi_{ij} + \nabla_r\phi_{jr} + \nabla_j\phi_{ri} = 0$, the above equation is written as

$$-\phi^r{}_k\nabla^i\nabla_i\phi_{jr} = R_{jk} + R_{rs}\phi^r{}_j\phi^s{}_k - R_{js}\xi^s\eta_k - 2R^*{}_{jk}.$$

Taking the symmetric part of the last equation and using (1.16), we obtain (1.17).

We define $(\nabla\phi)(X)$ for a vector field X by $(\nabla\phi)(X) = (\nabla_r\phi_{si}X^i)$. Then we see $\|(\nabla\phi)(X)\|^2 = (\nabla_r\phi_{si}\nabla^r\phi^s{}_jX^iX^j)$, where $\|T\| = T_{ij}T^{ij}$ for $T = (T_{ij})$ generally. It is also easy to verify

$$\|(\nabla\phi)(\xi)\| = \|h\|. \tag{1.21}$$

Thus, we have the following from (1.17).

Proposition 1.1. *An almost cosymplectic manifold is cosymplectic if and only if $R^*{}_{ij} + R^*{}_{ji} = R_{ij} + R_{rs}\phi^r{}_i\phi^s{}_j$.*

Remark 1.1. (1.14) and (1.17) give the Olszak's identity

$$S - S^* - g(Q\xi,\xi) + \frac{1}{2}\|\nabla\phi\|^2 = 0 \quad (S \text{ is the scalar curvature}). \tag{1.22}$$

2. Almost cosymplectic manifolds with constant ϕ-sectional curvature. By D we denote the distribution of an almost contact metric manifold M defined by $\eta = 0$.

M is said to be of constant ϕ-sectional curvature if at any point $x \in M$ the sectional curvature $K(X, \phi X)$ is independent of the choice of non-zero $X \in D_x$. In this case, the ϕ-sectional curvature K is a function on M.

Theorem 2.1. *Let M be a $(2n+1)$-dimensional almost cosymplectic manifold of constant ϕ-sectional curvature K. The Ricci curvatures satisfy*

$$Ric(X,X) + Ric(\phi X, \phi X) \leq (n+1)K \tag{2.1}$$

for each unit $X \in D_x$, $x \in M$. Equality holds for any point $x \in M$ and for any unit $X \in D_x$ if and only if M is cosymplectic, where Ric is the Ricci curvature tensor.

Proof. We define C and D by

$$C_{ijkl} = R_{arbs}\phi^r{}_c\phi^s{}_d(\delta^a{}_i - \xi^a\eta_i)(\delta^c{}_j - \xi^c\eta_j)(\delta^b{}_k - \xi^b\eta_k)(\delta^d{}_l - \xi^d\eta_l) =$$
$$= R_{irks}\phi^r{}_j\phi^s{}_l + R_{arbs}\xi^a\xi^b\phi^r{}_j\phi^s{}_l\eta_i\eta_k -$$
$$- R_{arks}\xi^a\phi^r{}_j\phi^s{}_l\eta_i - R_{irbs}\xi^b\phi^r{}_j\phi^s{}_l\eta_k,$$

$$D_{ijkl} = K(g_{ik} - \eta_i\eta_k)(g_{jl} - \eta_j\eta_l).$$

Then $K(X, \phi X) = K$ for any non-zero $X \in D_x$ is equivalent to

$$(C_{ijkl} - D_{ijkl})Y^iY^jY^kY^l = 0 \tag{2.2}$$

for any $Y \in T_x(M)$. Putting $L = C - D$, (2.2) is equivalent to

$$L_{ijkl} + L_{ijlk} + L_{ikjl} + L_{iklj} + L_{ilkj} + L_{iljk} +$$
$$+ L_{jikl} + L_{jilk} + L_{kijl} + L_{kilj} + L_{likj} + L_{lijk} = 0.$$

Transvecting the last equation by g^{jl}, we get

$$R_{ik} + R_{rs}\phi^r{}_i\phi^s{}_k - R_{aibk}\xi^a\xi^b - R_{arbs}\xi^a\xi^b\phi^r{}_i\phi^s{}_k +$$
$$+ 3R^*{}_{ik} + 3R^*{}_{ki} - R_{ir}\xi^r\eta_k - R_{kr}\xi^r\eta_i - 3R^*{}_{ri}\xi^r\eta_k - 3R^*{}_{rk}\xi^r\eta_i +$$
$$+ + R_{rs}\xi^r\xi^s\eta_i\eta_k - 4(n+1)K(g_{ik} - \eta_i\eta_k) = 0.$$

Let $X \in D_x$ such that $\|X\| = 1$. Transvecting the last equation by X^iX^k and applying (1.13) and 91.17), we obtain

$$Ric(X,X) + Ric(\phi X, \phi X) = (n+1)K - \frac{3}{4}\|(\nabla\phi)(X)\|^2 - \frac{5}{4}\|hX\|^2. \tag{2.3}$$

Therefore, we obtain (2.1). Equality of (2.1) for any $X \in D$ implies ... for any $X \in D_x$. Since $h\xi = 0$, $hX = 0$ for

any $X \in D$ implies $h = 0$. From this we obtain $(\nabla\phi)(\xi) = 0$ by (1.21). Therefore, $(\nabla\phi)(X) = 0$ for any $X \in D$ implies $\nabla\phi = 0$, so M is cosymplectic.

Remark 2.1. Let $\{e_\alpha, \phi e_\alpha, \xi;\ 1 \leq \alpha \leq n\}$ be a ϕ-basis of $T_x(M)$ of an almost cosymplectic manifold of constant ϕ-sectional curvature K. Since we have $\|h\phi X\| = \|\phi h\| = \|hX\|$, (2.3) gives $\|(\nabla\phi)(\phi X)\| = \|(\nabla\phi)(X)\|$ for $X \in D$. Thus, we obtain $\|\nabla\phi\|^2 = 2\sum_\alpha \|(\nabla\phi)(e_\alpha)\|^2 + \|h\|^2$ and $\|h\|^2 = 2\sum_\alpha \|he_\alpha\|^2$. Then, by (1.14) and (2.3), the scalar curvature S is given by

$$S = Ric(\xi,\xi) + \sum_\alpha Ric(e_\alpha, e_\alpha) + \sum_\alpha Ric(\phi e_\alpha, \phi e_\alpha) =$$
$$= -\|h\|^2 + (n+1)nK - \frac{3}{4}\sum_{\alpha=1}^{n}\|(\nabla\phi)(e_\alpha)\|^2 - \frac{5}{4}\sum_{\alpha=1}^{n}\|he_\alpha\|^2 \leq (n+1)nK.$$

The last inequality is due to O l s z a k [4].

3. Conformally flat almost cosymplectic manifolds. Let M be a conformally flat almost cosymplectic manifold. Then the Riemannian curvature tensor R is expressed as

$$R^i{}_{jkl} = \frac{1}{2n-1}(\delta^i{}_kR_{jl} - \delta^i{}_lR_{jk} + R^i{}_kg_{jl} - R^i{}_lg_{jk}) -$$
$$- \frac{S}{2n(2n-1)}(\delta^i{}_kg_{jl} - \delta^i{}_lg_{jk}).$$

Thus $R^*{}_{ij}$ is given by

$$R^*{}_{ij} = \frac{1}{2n-1}(R_{ij} + R_{rs}\phi^r{}_i\phi^s{}_j - R_{ir}\xi^r\eta_j) - \frac{S}{2n(2n-1)}(g_{ij} - \eta_i\eta_j).$$

Let $X \in D$ such that $\|X\| = 1$. Then

$$R^*{}_{ij}X^iX^j = \frac{1}{2n-1}(Ric(X,X) + Ric(\phi X, \phi X)) - \frac{S}{2n(2n-1)}.$$

On the other hand, (1.17) leads us to

$$R^*{}_{ij}X^iX^j = Ric(X,X) + Ric(\phi X, \phi X) + \|(\nabla\phi)(X)\|^2 + \|hX\|^2.$$

Combining the last two equations, we get

$$(2n-3)(Ric(X,X) + Ric(\phi X, \phi X)) = -\frac{S}{n} - (2n-1)(\|(\nabla\phi)(X)\|^2 + \|hX\|^2).$$

Therefore we obtain the following.

Theorem 3.1. *Let M be a conformally flat almost cosymplectic manifold of dimension $2n+1 \geq 5$. Then, for any unit $X \in D$,*

$$Ric(X,X) + Ric(\phi X, \phi X) \leq \frac{S}{(2n-3)n} \tag{3.2}$$

holds. Equality holds for any unit $X \in D$, if and only if M is cosymplectic.

Remark 3.1. Let $\{e_\alpha, \phi e_\alpha, \xi\}$ be a ϕ-basis of $T_x(M)$ of a conformally flat almost cosymplectic manifold. Then, by (1.14) and (3.1), we can see that the following relation is satisfied

$$\frac{2n-2}{2n-1}S + \frac{2n-3}{2n-1}\|\nabla\xi\|^2 + \frac{1}{2}\|\nabla\phi\|^2 = 0,$$

where we have used $\|\nabla\xi\|^2 = \|h\|^2$. This equation is due to O l s z a k [4], (6.4).

4. k-nullity distributions in an almost cosymplectic manifold.

Let k be a real number. By $N(k): x \to N_x(k)$ we denote the k-nullity distribution of a Riemannian manifold (M,g), where

$$N_x(k) = \{Z \in T_x(M);\ R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y),$$
$$X, Y \in T_x(M)\}.$$

Then we prove the following.

Proposition 4.1. *Let M be an an almost cosymplectic manifold. If ξ belongs to the k-nullity distribution, then $k \leq 0$. If $k < 0$, then M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{-k}$.*

Proof. By $\xi \in N(k)$ we can see $Ric(\xi,\xi) = 2nk$. Then (1.14) implies $k \leq 0$. Now we suppose $k < 0$. From $\xi \in N(k)$ and (1.13), we get $h^2 = k\phi^2$; Moreover, by (1.3), we obtain

$$(\nabla_X\phi)Y = -g(hX,Y)\xi + \eta(Y)hX \tag{4.1}$$

Since $h\xi = 0$ and h is symmetric, $h^2 = k\phi^2$ implies that the restriction $h|D$ of h to the distribution D has eigenvalues $\lambda = \sqrt{-k}$ and $-\lambda$. By $D(\lambda)$ and $D(-\lambda)$ we denote the distributions defined by the eigenspacesof h corresponding to λ and $-\lambda$, respectively. By $D(0)$ we denote the distribution defined by ξ. Then these three distributions are mutually orthogonal. Let $X \in D(\lambda)$. Then $hX = \lambda X$ and $\phi h = -h\phi$ imply $h(\phi X) = -\lambda(\phi X)$, and hence $\phi X \in D(-\lambda)$. This means that the dimension of $D(\lambda)$ and $D(-\lambda)$ are equal to n. We prove that $D(\lambda)$ (resp. $D(-\lambda)$) is integrable. Let $X, Y \in D(\lambda)$ (resp. $D(-\lambda)$). Then, we have $\nabla_X\xi = \phi hX = \pm\lambda\phi X$ and $\nabla_Y\xi = \phi hY = \pm\lambda\phi Y$. Therefore, $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X) = 0$ holds. Thus $d\eta(X,Y) = 0$ and $\eta([X,Y]) = 0$ follows. And $X, Y \in D$ and $\xi \in N(k)$ imply $R(X,Y)\xi = 0$. On the other hand,

$$0 = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi =$$
$$= \pm\lambda\nabla_X(\phi Y) \mp \lambda\nabla_Y(\phi X) - \phi h[X,Y] =$$
$$= \pm\lambda\{(\nabla_X\phi)Y + \phi(\nabla_X Y) - (\nabla_Y\phi)X - \phi(\nabla_Y X)\} - \phi h[X,Y] =$$
$$= \pm\lambda\{(\nabla_X\phi)Y - (\nabla_Y\phi)X + \phi[X,Y]\} - \phi h[X,Y].$$

By (4.1), the first term of the last line vanishes. And, so we obtain $\phi h[X,Y] = \pm\lambda\phi[X,Y]$, which together with $\eta([X,Y]) = 0$ implies $[X,Y] \in D(\lambda)$ (resp. $D(-\lambda)$).

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