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HYPERBOLIC MEAN VALUE THEOREMS OF DIFFERENTIAL FORM

BY

GH.BANTAŞ and M.TURINICI

1.Introduction. Let I and J be nonempty intervals of the real axis (R) and denote for simplicity $K = I \times J$. By a *standard rectangle* of K we shall mean, in the following, any cartesian product $\Delta = [a, b \times [c, d]$ where $[a, b], [c, d]$ are closed sub–intervals of I and J, respectively. In this case, the points

$$A = (a, c),\ \ B = (b, c),\ \ C = (b, d),\ \ D = (a, d)$$

are called the *vertices* of Δ; correspondingly, the rectangle in question will be represented as [ABCD].

Let $(X, \|\cdot\|)$ be a normed space and $f : X \to X$, a mapping. For each rectangle $\Delta = [ABCD]$ as above, denote

$$m_f(\Delta) = f(A) - f(B) + f(C) - f(D).$$

This will be referred to as the *hyperbolic measure* of Δ generated by f. Note that, when $X = R$ and

$$f(t, s) = ts,\quad t, s \in R,$$

this quantity reduces to

$$m(\Delta) = (b - a)(d - c) \text{ (the usual \textit{Lebesque measure} of } \Delta).$$

Finally, denote (again for any rectangle Δ as before)

$$R_f(\Delta) = \frac{m_f(\Delta)}{m(\Delta)},\quad S_f(\Delta) = \|R_f(\Delta)\|.$$

These will be reffered to as the *variational quotients* of f with respect to Δ.

Now, by a *mean value property* for f over Δ we mean an evaluation of $R_f(\Delta)$ or $S_f(\Delta)$ with the aid of some expressions depending on the objective to be attained. For example, we may distinguish between:

i) mean value theorems of non–differential (relative) form;

ii) mean value theorems of differential form.

Also, we may have mean value theorems of inequality (resp., equality) form. These correspond to the general case about X (respectively, $X = R$). In this direction, a first class of such results is that characterized by relations like

$$S_f(\Delta) \le S_f(\Delta') \ \ (resp., R_f(\Delta) = R_f(\Delta')),$$

where Δ' is a sub–rectangle of Δ (interior or not to Δ). A second class of these may be obtained through a limit process from the preceding one. It consists of results like

$$S_f(\Delta) \le \limsup_{h,k\to 0} \left\| \frac{f(u+h,v+k)-f(u,v+k)-f(u+h,v)+f(u,v)}{hk} \right\|$$

in the general (vector) case and, respectively

$$R_f(\Delta) = \limsup_{h,k\to 0} \frac{f(u+h,v+k)-f(u,v+k)-f(u+h,v)+f(u,v)}{hk}$$

is the real case. Here $P = (u,v)$ is a point in Δ which may be – or may be not – interior to Δ. The limit in the right number of this last relation is the so-called *hyperbolic derivative* of f at P; and the relation itself is a bi–dimensional version of the Lagrange mean value theorem.

From a historical perspective, a first systematic study of these properties has been made by B ö g e l [4]. His efforts were continued by C i o – r ă n e s c u [5] and N i c o l e s c u [11] among others. Their contributions are related to the hyperbolic derivative operator in the context of Taylor developments, analytic functions and hyperbolic polynomials. As a further step in this topic we may note the results obtained by D o b r e s c u [6] and M a r c u s [7]. These involve different aspects of the theory such as (hyperbolic) continuity and monotonicity . A special mention must be made about the results in N i c o l e s c u [9,10] which partially extend – to the bi–dimensional case – the ones in B ö g e l [op.cit.] by a procedure developed in A z i z and D i a z [1]. Other (non–differential) aspects were discussed in the paper by B a n t a ş and T u r i n i c i [3]; the method used there is taken from the uni–dimensional developments in this area due to the same authors [2]. It is the main aim of the present exposition to get a differential version of the above precised facts; details will be given in Section 3 (the general case) and Section 4 (the real case). All preliminary material is being collected in Section 2. Some further aspects will be considered elsewhere.

2. Preliminaries. Let I, J be real intervals, $K = I \times J$ and $(X, \|\cdot\|)$ be a normed space. For each rectangle Δ of K and each function $f : K \to X$, we construct the quantity $m_f(\Delta)$ as in the introductory part. It therefore appears as a mapping with two entries. Concerning its behaviour with respect to these (f and Δ) the main aspects were delineated in B a n t a ş and T u r i n i c i [3]. It is our aim in the following to expose them, without going into details.

By a *hyperbolic constant* over K we mean any map $h : K \to X$, of the form

$$h(t,s) = \varphi(t) + \psi(s), \ \ (t,s) \in K,$$

where $\varphi : I \to X$, $\psi : J \to X$ are given functions. This term is justified by

Proposition 1. *For each rectangle Δ of K and each hyperbolic constant h over K,*

$$m_h(\Delta) = 0 \ \ (hence\ m_{f+h}(\Delta) = m_f(\Delta)). \tag{2.1}$$

As an immediate consequence

$$R_h(\Delta) = 0 \ \ (\text{hence } R_{f+h}(\Delta) = R_f(\Delta)).$$

In other words, any property of $R_f(\Delta)$ (or $S_f(\Delta)$) my be transfered to the function $f + h$ which, in principle, fails to be entitled with the properties of the function f.

Now, by a *division* of the rectangle Δ we mean any finite decomposition $\Delta = \cup_r \Delta_r$ or Δ into (standard) rectangles of K, with the family $\{int(\Delta_r)\}$ being mutually disjoint. Among these, we distinguish the divisions of Δ obtained from divisions of the intervals generating Δ. Precisely, given a division

$$[a,b] = \cup_i [t_i, t_{i+1}] \ , \ [c,d] = \cup_j [s_j, s_{j+1}]$$

of these intervals, the division of Δ induced by this may be written as $\Delta = \cup_{i,j} \Delta_{i,j}$ where

$$\Delta_{i,j} = [t_i, t_{i+1}] \times [s_j, s_{j+1}], \text{ for all possible } (i,j);$$

it will be termed a *normal division* for Δ.

Proposition 2. *We have*

$$m(\Delta) = \sum_r m_f(\Delta_r) \tag{2.2}$$

for any division $\{\Delta_r\}$ *of the rectangle* Δ.

This property has a number of important consequences for our developments. Some of these were precised in the authors' paper. Other aspects – of differential type – may be given along the following lines. Remember that, given the interior point $P = (u, v)$ of K, the hyperbolic derivative of a function $f : K \to X$ at P was introduced by the formula

$$D_H f(P) = \lim_{h,k \to 0} \frac{f(u+h, v+k) - f(u+h, v) - f(u, v+k) + f(u, v)}{hk}.$$

The following statement – proved in the paper by N i c o l e s c u [9] – will be useful for us:

Proposition 3. *Suppose f is hyperbolic derivable at a point $P \in$ $\in \operatorname{int}(K)$. Let also $(\Delta_n = [A_n B_n C_n D_n])_{n \geq 1}$ be a sequence in K which converges to P, in the sense*

i) $P \in \Delta_n$, *for all* $n \geq 1$

ii) *all sequences* $(A_n), (B_n), (C_n), (D_n)$ *converge to* P.

Then, we necessarily have

$$D_H f(P) = \lim_{n \to \infty} R_f(\Delta_n).$$

We also need the following statement known as "the Bögel's Lemma" (see N i c o l e s c u [12, ch.19, Sec.2] for a direct proof).

Proposition 4. *Let the function f be hyperbolic derivable in the segment (interior to K)*

$$S = \{p\} \times [q, r] \quad (resp., S = [p, q] \times \{r\}).$$

(Here, p,q,r are numbers with $q < r$ (resp., $p < q$)).

Then, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$\|m_f(\Delta)\| < \varepsilon$, *for each rectangle Δ in $S_\delta = [p - \delta, p + \delta] \times [q, r]$*

(resp., in $S_\delta = [p, q] \times [r - \delta, r + \delta]$).

An immediate consequence of this is the following. Let $\Delta = [a, b] \times [c, d]$ be a rectangle. For each $P = (u, v) \in R^2$ and each $\rho, \sigma \in R$, denote

$$\Delta_{\rho\sigma}(P) = [(u, v), (u + \rho, v), (u, v + \sigma), (u + \rho, v + \sigma)].$$

Suppose $\rho, \sigma > 0$ are now such that

$$\Delta^* = [a, b + \rho] \times [c, d + \sigma] \text{ is interior to } K.$$

We introduce the function $g : \Delta \to X$ by

$$g(P) = m_f(\Delta_{\rho\sigma}(P)), \; P \in \Delta.$$

Proposition 5. *Suppose f is hyperbolic derivable over Δ^*. Then, g is continuous on the rectangle Δ.*

Proof. Let $P_0 = (t_0, s_0)$ be fixed in Δ, and $P = (t, s)$, another point in Δ, close to P_0. By Proposition 2, we have immediately

$$g(P) - g(P_0) = \sum_{i=1}^{4} \{m_f(\Gamma_i); \; \Gamma_i = \text{sub rectangle of } S_\delta^{(i)}\}.$$

Here $\{S_\delta^{(i)}; 1 \leq i \leq 4\}$ are the subsets of the form appearing in Proposition 4, determined by the segments generating the boundary of the rectangle $\Delta_{\rho\sigma}(P)$. But these are arbitrarily small in norm. Hence the result. q.e.d.

3. Main results (inequality form). Let the notations above be admitted. For each rectangle $\Delta = [a, b] \times [c, d]$ of K, denote

$$\operatorname{diam}(\Delta) = \max(b - a, d - c)$$

(the diameter of Δ in the supremum norm of R^2). Let also $(X, \|\cdot\|)$ be a normed space and $f : K \to X$, a mapping. We need in the following a non–differential mean value property established in B a n t a ş and T u r i n i c i [3]. Let (E_1, E_2, E_3, E_4) be four points on the boundary of Δ, distinct from the vertices of Δ. We term such a systems *admissible*, when there exists a sub-rectangle Δ' of Δ interior to Δ, with

a) each point of this system is on a certain exterior normal to Δ' in a vertex of Δ'

b) each vertex of Δ' generates, by this exterior normal construction, a point of the considered system.

(Here, e.g., by an *exterior normal* to $\Delta' = [A'B'C'D']$ at A' we mean each of the half-lines through A', opposite to (the half-lines) $A'B'$, respectively $A'D'$).

Lemma. *Let Δ be a rectangle in K. The following conclusions take place*

a) *for each $\varepsilon > 0$, there exists a sub-rectangle Δ_ε of Δ, with*

$$\operatorname{diam}(\Delta_\varepsilon) < \varepsilon \;, \; S_f(\Delta) \leq S_f(\Delta_\varepsilon),$$

b) *if, in addition*

$$(H_1)\ \begin{cases} f \text{ is continous at the vertices of } \Delta \text{ and normally continuous} \\ \text{at an admissible four points system of bd } (\Delta) \end{cases}$$

then Δ_ε may be taken interior to Δ in (3.1).
Here, "normally continuous" means "continuous after the normal at bd (Δ) (at the considered point)". Note that a sufficient condition for (H_1) is

(H_1') f is continuous over bd (Δ).

Some other aspects may be found in the above quoted paper.

Now, under these preliminaries, the first main result of our exposition is

Theorem 1. *Suppose that the rectangle Δ of K and the function $f : K \to X$ are taken as in (H_1). Then, there exists a point P of Δ, interior to Δ, such that*

$$S_f(\Delta) \leq \limsup_{h,k\to 0} S_f(\Delta_{hk}(P)).$$

Proof. By part b) of the Lemma, there exists a sub-rectangle $\Delta_1 = [A_1B_1C_1D_1]$ of $\Delta = [ABCD]$, interior to Δ, with

$$\text{diam } (\Delta_1) < 2^{-1} \text{ diam } (\Delta), S_f(\Delta) \leq S_f(\Delta_1).$$

Further, by part a) of the Lemma, there may be determined a sub-rectangle $\Delta_2 = [A_2B_2C_2D_2]$ of Δ_1, with

$$\text{diam } (\Delta_2) < 2^{-1} \text{ diam } (\Delta_1), S_f(\Delta_1) \leq S_f(\Delta_2).$$

(Note that, in general, Δ_2 fails to be interior to Δ_1.) This process may be continued. It gives us a sequence of sub-rectangles $(\Delta_n = [A_nB_nC_nD_n])_{n\geq 1}$, all interior to Δ, with

$$\Delta_1 \supset \Delta_2 \supset ...;\ \text{diam } (\Delta_n) < 2^{-n} \text{ diam } (\Delta),\ n \geq 1 \tag{3.3}$$

$$S_f(\Delta) \leq S_f(\Delta_1) \leq S_f(\Delta_2) \leq .... \tag{3.4}$$

By the Cantor's intersection theorem, there must be a single point $P = (u, v)$, interior to Δ, with

a) $P \in [A_nB_nC_nD_n]$, for all $n \geq 1$

b) $(A_n), (B_n), (C_n), (D_n)$ converge to P as $n \to \infty$.

Let us now construct parallels to the coordinate axes through P. The parallel to the x-axis intersects Δ_n at $\{E_n, F_n\}$ and the one to the y-axis, in $\{L_n, M_n\}$. (We make these notations, for each $n \geq 1$, in the sense of increasing values for x and y respectively.) Denote

$$\begin{cases} \Delta_n^{(1)} = [E_nPM_nD_n], \\ \Delta_n^{(2)} = [PF_nC_nM_n], \\ \Delta_n^{(3)} = [AL_nPE_n], \\ \Delta_n^{(4)} = [L_nB_nF_nP], \end{cases}$$

for each $n \geq 1$. One gets immediately (by Proposition 2 plus convexity arguments)

$$S_f(\Delta_n) \leq \max_{1\leq i\leq 4} (S_f(\Delta_n^{(i)}),\ n = 1, 2, ...$$

Hence, passing eventually to a subsequence, we may assume that, for at least one $i \in \{1, 2, 3, 4\}$,

$$\text{c)}\ \begin{cases} \Delta_n^{(i)} \text{ is non-degenerate, and } S_f(\Delta_n) \leq S_f(\Delta_n^{(i)}), \\ \text{for all } n \geq 1. \end{cases}$$

To make a choice, assume $i = 2$, that is

$$S_f(\Delta_n) \leq S_f([PF_nC_nM_n]),\ \ n \geq 1.$$

Since, on the other hand

$$F_n \to P \text{ and } M_n \to P \text{ as } n \to \infty,$$

it is clear that

$$S_f(\Delta) = \limsup_{n\to\infty} \left\|\frac{f(u+h_n, v+k_n) - f(u, v+k_n) - f(u+h_n, v) + f(u, v)}{h_nk_n}\right\|.$$

where $(h_n), (k_n)$ are the dimensions of $\Delta_n^{(2)} = [PF_nC_nM_n]$, for $n \geq 1$. This ends the argument. q.e.d.

Corollary 1. *Let the rectangle Δ and the function f be as in (H_1'). Then, conclusion of Theorem 1 is retainable.*

The used hypothesis (H_1) or its variant has nothing to do with the hyperbolic continuity properties of f over int(Δ). However, when f is hyperbolic discontinuous at a point $P \in int(\Delta)$ then (3.2) becomes trivial. Indeed, it would follow from

$$\limsup_{h,k\to 0} \|m_f(\Delta_{hk}(P))\| > 0$$

that (by definition itself)

$$\limsup_{h,k\to 0} S_f(\Delta_{hk}(P)) = \limsup_{h,k\to 0} \frac{1}{|hk|}\|m_f(\Delta_{hk}(P))\| = \infty.$$

proving the claim. Hence the natural framework of Theorem 1 is that of

(H_2) f is hyperbolic continuous over int(Δ)

being accepted. In particular, H_2) may be obtained as a consequence of

(H_3) f is hyperbolic derivable over int(Δ).

In this case, Theorem 1 gives

Corollary 2. *Suppose that Δ and f are as in $(H_1)+(H_3)$. Then,*

$$S_f(\Delta) \leq \|D_H f(P)\|, \quad \textit{for some } P \in \textit{int } (\Delta).$$

This result has an interesting consequnce which may be deemed as a reciprocal of Proposition 1. More precisely, we have

Corollary 3. *Suppose that the rectangle Δ of K and the function $f: K \to X$ are such that*

(H_1'') *f is continuous on Δ*

(H_3') *$D_H f(P)$ exists and equals zero, for all points $P \in$ int(Δ).*

Then, f is a hyperbolic constant (over Δ).

Proof. Let Δ be represented as [ABCD]. We take an arbitrary point Q of Δ and construct parallels to the x-axis (y-axis) through Q. Denote the intersection of these with Δ by $\{E, F\}$, respectively $\{L, M\}$. By the admitted hypotheses, conditions of Corollary 2 are fulfilled over $\Delta' = [ALQE]$. Hence

$$S_f(\Delta') = 0 \text{ (that is, f(Q) } - \text{ f(E) } - \text{ f(C) } + \text{ f(A) } = 0).$$

Putting $A = (a, c), Q = (x, y)$, this shows

$$f(x,y) = f(a,y) + f(x,c) - f(a,c), (x,y) \in \Delta.$$

Hence, f is a hyperbolic constant over Δ, as claimed. q.e.d.

And now, some bibliographical comments. The first main result extends a statement in this area due to B a n t a ş and T u r i n i c i [2] for the uni-dimensinal case; this is also valid for Corollary 1. On the other hand, Corollary 2 includes a similar result due to N i c o l e s c u [9]. Finally, Corollary 3 may be viewed as a vector extension of a mean value statement obtained by B ö g e l [4]; see also N i c o l e s c u [12, ch.19, Sec.2]

4. The real case. Now, the choice $X = R$ will be analyzed, from the perspective of the introductory part. So, let I, J be real intervals and

$K = I \times J$. We take a rectangle Δ of K and a function $f: K \to R$. The second main result of the present exposition is

Theorem 2. *Suppose that, either*

(H_2') *$D_H f(P)$ exists, for all $P \in \Delta$*

or (H_1'') plus

(H_2^o) *$D_H f(P)$ exists, for all $P \in$ int (Δ),*

are accepted. Then, f satisfies the mean value property of the introductory part: i.e.,

$$R_f(\Delta) = D_H f(P), \quad \textit{for some } P \in \textit{int } (\Delta). \tag{4.1}$$

Proof. Let us construct an equidistant (normal) division of Δ by
$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b, \rho = t_{i+1} - t_i < \varepsilon, 0 \leq i \leq n-1$
$c = s_0 < s_1 < \ldots < s_{n-1} < s_n = d, \sigma = t_{j+1} - s_j < \varepsilon, 0 \leq j \leq m-1$
Here, $\varepsilon > 0$ is arbitrary fixed and $n, m \geq 3$ are positive integers. Denote

$$\Psi(P) = R_f(\Delta_{\rho\sigma}(P)), \quad P \in \Delta^\varepsilon = [a, t_{n-1}] \times [c, s_{m-1}].$$

It follows by the assumptions above, plus Proposition 5 that $P \vdash \Psi(P)$ is continuous over Δ^ε. But then, the argument developed in B a n t a ş and T u r i n i c i [3] shows

$$R_f(\Delta) = \Psi(P_\varepsilon) = R_f(\Delta_{\rho\sigma}(P_\varepsilon)), \quad \textit{for some } P_\varepsilon \in \textit{int } (\Delta^\varepsilon).$$

We now develop an inductive approach as follows. Given $\varepsilon = 2^{-1}\text{diam}(\Delta)$, there exists by these facts, a sub-rectangle Δ_1 of Δ, interior to Δ, with

$$\text{diam}(\Delta_1) < 2^{-1}\text{diam}(\Delta), R_f(\Delta) = R_f(\Delta_1).$$

Further, for this Δ_1 and $\varepsilon = 2^{-2}\text{diam}(\Delta)$, there exists a sub-rectangle Δ_2 of Δ_1, interior to Δ_1 (hence to Δ) such that

$$\text{diam}(\Delta_2) < 2^{-2}\text{diam}(\Delta), R_f(\Delta_1) = R_f(\Delta_2).$$

We therefore get a sequence of sub-rectangles $(\Delta_n)_{n\geq 1}$ of Δ, with

$$\Delta_{n+1} \subset int(\Delta_n), \; \text{diam}(\Delta_n) < 2^{-n}\text{diam}(\Delta), n \geq 1 \tag{4.2}$$

$$R_f(\Delta) = R_f(\Delta_1) = R_f(\Delta_2) = \ldots. \tag{4.3}$$

By (4.2) plus the Cantor intersection theorem, (Δ_n) converges to a point $P = (u, v)$ interior to Δ. But, in the case, Proposition 3 tells us $D_H f(P) = \lim_{n\to\infty} R_f(\Delta_n)$. This, combined with (4.3) ends the argument. q.e.d.

Theorem 2 may be viewed as a bidimensional version of the Lagrange mean value theorem. It also gives a corresponding version of the Rolle (mean value) theorem:

Corollary 4. *Let the hypotheses in Theorem 2 be accepted. If, in addition $R_f(\Delta) = 0$ then, there may be determined a point $P \in int(\Delta)$ with $D_H f(P) = 0$.*

The above statement is also useful in giving us a "weak" counterpart of Corollary 2. Let K be as above and $(X, \|\cdot\|)$, a normed space. Denote by X^* the *topological dual* of X. Let us call the function $f : K \to X$, *weakly continuous* over a rectangle Δ in K when, for each x^* in X^*, the composed function $g = x^* \circ f$ (from K to R) is continuos (in the usual sense). Also, given $P \in int(K)$, denote by $D_H^* f(P)$ the *weak hyperbolic derivative* of f at P, introduced as

$$x^*(D_H^* f(P)) = D_H((x^* \circ f)(P)), x^* \in X^*.$$

And, call the function f, *weakly hyperbolic derivable* at P when this quantity exists. Of course, f admits such a property, whenever it is strongly derivable at P; the reciprocal is not valid, in general.

Corollary 5. *Let the rectangle Δ of K and the function $f : K \to X$ be such that*

(H_3^*) *f is weakly continuous over Δ*

(H_2^*) *f has a weakly hyperbolic derivative at any point of* int(Δ).

Then

$$S_f(\Delta) \leq \|D_H^* f(P)\|, \text{ for some } P \in \text{int } (\Delta). \tag{4.4}$$

Proof. By the Hahn–Banach theorem, we may find a linear continuos functional $x^* \in X^*$, with

$$\|x^*\| = 1, x^*(R_f(\Delta)) = S_f(\Delta).$$

The composed function (from K to R)

$$g(P) = x^*(f(P)), P \in K$$

fulfils, by $(H_3^*) + (H_2^*)$, conditions of Theorem 2. So, there exists a point P, interior to Δ, with $R_g(\Delta) = D_H g(P)$. But, evidently,

$$R_g(\Delta) = x^*(R_f(\Delta)) = S_f(\Delta):$$

and moreover $D_H g(P) = x^*(D_H^* f(P)) \leq \|D_H^* f(P)\|$. Combining these facts yields the desired conclusion. q.e.d.

The obtained corollary may by compared with a statement in this direction due to N i c o l e s c u [10]. For a number of historical aspects we refer to the survey paper by N a s h e d [8].

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Department of Mathematics
"Al.I.Cuza" University
11, Copou Boulevard
6600, Iaşi
ROMANIA