Or

$$(6.11)\qquad g(Y,V)[2rA(V)-\mathrm{d}r(V)+rC(V)]=-rB(Y)\quad [g(V,V)=1]$$

Since $r\neq 0$ and $B(Y)$ is assumed to be no–where zero, it follows from (6.11) that neither $g(Y,V)=0$ or $2rA(V)-\mathrm{d}r(V)+rC(V)=0$. Thus we see that V cannot be orthogonal to ρ, λ or μ for otherwise $g(\rho,V)=0$ or $g(\lambda,V)=0$ or $g(\mu,V)=0$. This leads to the following theorem.

Theorem 7. *If a $G(PWRS)_n$ admits a unit parallel vector field V, then V cannot be orthogonal to any of the basic vector fields ρ, λ or μ. Further, the scalar $2rA(V)-\mathrm{d}rV+rC(V)$ cannot be zero.*

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SOME RESULTS ON CR–SUBMANIFOLDS OF A NEARLY COSYMPLECTIC MANIFOLD

BY

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0. Introduction. CR–submanifolds of a Kaehler manifold has been defined and studied by A. B e j a n c u [1] and since then many papers have appeared on this topic. On the other hand, CR–submanifolds of an almost contact manifold were studied by M. K o b a y a s h i [6], K. Y a n o and M. K o n [8], and the present authors [8]. In [4], D. B l a i r defined a nearly cosymplectic structure on an odd dimensional differentiable manifold and CR–submanifold of such manifolds were studied by I a n u s [5]. The aim of this note is to continue the study of CR–submanifolds of a nearly cosymplectic manifold.

1. Preliminaries. Let $\overline{M}$ be an n–dimensional almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g) satisfying

$$(1.1)\qquad \phi^2X=-X+\eta(X)\xi,\ \eta(\xi)=1,\ \phi\circ\xi=0$$

$$(1.2)\qquad g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\ g(X,\xi)=\eta(X)$$

where X and Y are vector fields on $\overline{M}$. Further, $\overline{M}$ is said to be a nearly cosymplectic manifold if

$$(1.3)\qquad (\overline{\nabla}_X\phi)(Y)+(\overline{\nabla}_Y\phi)(X)=0$$

for any vector fields X and Y on $\overline{M}$, where $\overline{\nabla}$ denotes the Riemannian connection of g on $\overline{M}$.

Let M be an m-dimensional isometrically immersed submanifold of $\overline{M}$ and let us denote by the same g the Riemannian metric tensor field induced on M from that of $\overline{M}$.

A submanifold M of $\overline{M}$ is called a CR-submanifold if M is tangent to ξ and there exists a differentiable distribution $D : M \ni x \to D \subset_x TM_x$ such that

(i) *the distribution D is invariant under ϕ i.e. $\phi D_x \subset D_x$,*

(ii) *the orthogonal complementary distribution $D^\perp$ of D is anti-invariant under ϕ, i.e. $\phi D_x^\perp \subset \nu_x$, where ν_x is the normal space at $x \in M$.*

The distribution D (resp. $D^\perp$) is called the horizontal (resp. vertical) distribution. Also, the pair $(D, D^\perp)$ is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^\perp$).

For a vector field X tangent to M, we put

$$X = PX + QX \tag{1.4}$$

where PX and QX belong to the distributions D and $D^\perp$.

For a vector field N normal to M, we put

$$\phi N = BN + CN \tag{1.5}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

We denote by μ the complementary orthogonal vector bundle of $\phi D^\perp$ in ν i.e. $\nu = \phi D^\perp \oplus \mu$. Then it is easy to see that μ is invariant under ϕ.

The Gauss and Weingarten formulae are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{1.6}$$

where $X, Y \in TM$, $N \in \nu$. Here ∇ is the Levi–Civita connection of M, h is the second fundamental form of M and $\nabla^\perp$ is the normal connection induced by $\overline{\nabla}$ on ν. Moreover, we have

$$g(h(X,Y), N) = g(A_N X, Y) \tag{1.7}$$

for all $X, Y \in TM$ and $N \in \nu$.

Definition. *A CR-submanifold of a nearly cosymplectic manifold is said to be D-totally geodesic ($D^\perp$-totally geodesic) if $h(X,Y) = 0$, $\forall X, Y \in D$ ($h(W,Z) = 0$, $\forall W, Z \in D^\perp$) and mixed totally geodesic if $h(X,Z) = 0$, where $X \in D$, $Z \in D^\perp$.*

It is known that the five dimensional sphere S^5, as a totally geodesic hypersurface of S^6, carries a nearly cosymplectic structure which is not cosymplectic [4]. Each real hypersurface of nearly cosymplectic manifold which is tangent to ξ is typical example of CR-submanifold of $\overline{M}$.

2. Parallel horizontal distribution of CR-submanifolds. First we have the following lemma which follows from the definition of nearly cosymplectic manifold by equating horizontal, vertical and normal components.

Lemma 1. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\overline{M}$. Then*

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QY} X) - P(A_{\phi QX} Y) = \\ = \phi P \nabla_X Y + \phi P \nabla_Y X \tag{2.1}$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QY} X) - Q(A_{\phi QX} Y) = 2Bh(X,Y) \tag{2.2}$$

$$h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX = \\ = \phi Q \nabla_X Y + \phi Q \nabla_Y X + 2Ch(X,Y) \tag{2.3}$$

$\forall X, Y \in TM$.

Definition. *The distribution D is said to be D-parallel if $\nabla_X Y \in D$ for $X, Y \in D$.*

Proposition 2. *Let M be a ξ-vertical CR-submanifold of a nearly cosymplectic manifold $\overline{M}$. Then the distribution D is parallel if*

$$h(X, \phi Y) = h(Y, \phi X), \ \forall X, Y \in D. \tag{2.4}$$

Proof. As the distribution D is parallel, so we have

$$\nabla_X \phi Y \in D, \ \nabla_Y \phi X \in D, \quad \forall X, Y \in D.$$

Now from (2.2), we have $Bh(X,Y) = 0$, $\forall X, Y \in D$.
Again, since $\phi h(X,Y) = Bh(X,Y) + Ch(X,Y)$, so we have

$$\phi h(X,Y) = Ch(X,Y), \quad \forall X, Y \in D,$$

which together with (2.3) asserts

$$h(X,\phi Y)+h(Y,\phi X)=2Ch(X,Y)=2\phi h(X,Y).$$

Putting $X=\phi X$ where $X\neq\xi$ in (2.4) and using the fact that $\eta(X)=0$ for $\xi\in D^{\perp}$, we have

$$h(\phi X,\phi Y)+h(Y,\phi^2X)=2\phi h(\phi X,Y)\quad\text{or,}$$

(2.5) $$h(\phi X,\phi Y)-h(Y,X)=2\phi h(\phi X,Y).$$

Similarly, we have

(2.6) $$h(\phi X,\phi Y)-h(X,Y)=2\phi h(X,\phi Y).$$

Thus from (2.5) and (2.6), we get the result.

The following is an easy consequence of (1.7).

Lemma 3. *Let M be a CR-submanifold of a nearly cosymplectic manifold $\overline{M}$. Then M is mixed totally geodesic if and only if $A_NX\in D$, $\forall X\in D$.*

Definition. *A normal vector field $N\neq 0$ is D-parallel normal section if $\nabla^{\perp}_XN=0$, $\forall X\in D$.*

Proposition 4. *Let M be a mixed totally geodesic CR-submanifold of a nearly cosymplectic manifold $\overline{M}$. Then the normal section $N\in\phi D^{\perp}$ is D-parallel if and only if $\nabla_X\phi N\in D$, $\forall X\in D$.*

Proof. From (2.2), we have $Q(\nabla_Y\phi X)=0$, $\forall X\in D$, $Y\in D^{\perp}$. In particular, we have $Q(\nabla_YX)=0$. Now, using (2.3), we get

(2.7) $$\nabla^{\perp}_X\phi QY=\phi Q\nabla_XY,\quad \nabla^{\perp}_XN=-\phi Q\nabla_X\phi N.$$

If the normal section $N\neq 0$ is D-parallel, then $\nabla_X\phi N\in D$, $\forall X\in D$. Converse part easily follows from (2.7), which completes the proof.

Definition 1. *A CR-submanifold whose distribution D is integrable (i.e. involutive) is called foliate CR-submanifold.*

Finally, we prove

Theorem 5. *Let M be a mixed totally geodesic foliate CR-submanifold of a nearly cosymplectic manifold $\overline{M}$. If $N\neq 0$ is D-parallel normal section, then so is ϕN.*

Proof. Let $N\neq 0$ be a D-parallel normal section in μ. Then for each $X\in D$, using the definition of nearly cosymplectic manifold, we have

(2.8) $$(\overline{\nabla}_X\phi)(N)+(\overline{\nabla}_{\phi X}\phi)(N)=0\quad\text{or,}$$

(2.9) $$\nabla^{\perp}_X\phi N=\phi(\nabla^{\perp}_{\phi X}\phi N).$$

Suppose $\{N_1,\dots,N_s,\phi N_1,\dots,\phi N_s\}$ be a local frame in μ. Let us assume that $\nabla^{\perp}_XN_1=0$. Then our aim is to show that $\nabla^{\perp}_X\phi N_1=0$. Now, $g(\phi N_1,N_1)=0$ and $\nabla^{\perp}_XN_1=0$. Then these two equations give

(2.10) $$g(\nabla^{\perp}_X\phi N_1,N_1)=0.$$

Moreover,

$$g(N_1,N_j)=0,\quad j=2,\dots,s\quad\text{and}\quad\nabla_XN_1=0$$

which give $g(N_1,\nabla^{\perp}_XN_j)=0$.

Now, using (2.9) in the form $\nabla^{\perp}_XN=\phi(\nabla^{\perp}_{\phi X}N)$, we get

$$g(N_1,\phi(\nabla^{\perp}_{\phi X}N_j))=0\quad\text{and consequently}\quad g(\nabla^{\perp}_{\phi X}N_j,\phi N_1)=0.$$

This implies that

(2.11) $$g(\nabla^{\perp}_XN_j,\phi N_1)=0,\quad j=2,\dots,s.$$

Again, from $g(\phi N_1,N_j)=0$, $j=2,\dots,s$ and (2.11), we get

(2.12) $$g(\nabla^{\perp}_X\phi N_1,N_j)=0,\quad j=2,\dots,s.$$

From (2.10), (2.11) and (2.12), we have

(2.13) $$\nabla^{\perp}_X\phi N_1\perp N_i,\quad i=1,2,\dots,s.$$

Thus

$$\nabla^{\perp}_X\phi N_1=\sum_{i=1}^{s}\alpha_i\phi N_1+\sum_{k=1}^{q}\beta_k\phi E_k,$$

where $\{E_1,\dots,E_q\}$ is a local frame of D.

From (2.9), we have

$$\nabla^{\perp}_{\phi X}\phi N_1 = \sum_{i=1}^{s} \alpha_i N_1 + \sum_{k=1}^{q} \beta_k E_k. \tag{2.14}$$

Since $\nabla^{\perp}_{\phi X}\phi N_1$ is normal to M, we have $\beta_k = 0$, $k = 1, ..., q$.

Also, from (2.13), we have

$$\nabla^{\perp}_{\phi X}\phi N_1 \perp N_i.$$

Making use of it in (2.14), we get $\alpha_i = 0$, $i = 1, ..., s$. Hence $\nabla^{\perp}_{\phi X}\phi N_1 = 0$, i.e. $\nabla^{\perp}_{X}\phi N_1 = 0$, which completes the required proof.

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ABOUT $\mathcal{M}$ - TANGENT CONES IN LOCALLY CONVEX SPACES

BY

CONSTANTIN BĂCUŢĂ

1. Introduction. Let C be a closed subset of a Banach space E and $x \in C$. The *Clarke tangent cone* , $T(C,x)$ is the set of all $y \in E$ such that

$$\begin{cases} \forall\, Y \in \mathcal{V}(y)\ ,\ \exists X \in \mathcal{V}(x), \lambda > 0 \text{ such that} \\ \forall\, x' \in X \cap C\ ,\ \mu \in (0,\lambda) : (x' + \mu Y) \cap C \neq \emptyset, \end{cases} \tag{1.1}$$

where $\mathcal{V}(x)$ is the collection of all neighborhoods of x, respectively y. The *contingent cone* $K(C,x)$, is the set of all y such that

$$\forall\ \ Y \in \mathcal{V}(y),\ \lambda > 0 : [x + (0,\lambda)Y] \cap C \neq \emptyset. \tag{1.2}$$

J.S.T r e i m a n [8] gave a new chracaterization of the tangent cone, derived using the method of B i s h o p and P h e l p s [2]. Also is proved the following inclusion:

$$\liminf_{x' \to x} K(C,x') \subset T(C,x). \tag{1.3}$$

where $B(y,\varepsilon)$ is the open ε - ball around y.

To obtain similar results and their consequences in the case of a locally convex spaces, we used a new concept of limit for the multifunction. This concept (for a locally convex space E) uses a class of bounded subsets of E, and tries to substitute the fact that the neighborhoods of a point are not always bounded.

2. Limit concept. Throught this paper, the topology of the linear space E is assumed to be locally convex and separated (Hausdorff). Let S