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A NEW CRITERION FOR MEROMORPHIC p-VALENT STARLIKE FUNCTIONS

BY

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1. Introduction. Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \dots\}) \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. The Hadamard product or convolution of two functions $f, g \in \Sigma_p$ will be denoted by $f * g$. Let

$$D^n f(z) = \frac{1}{z^p(1-z)^{n+1}} * f(z), \quad n \geq 0 \tag{1.2}$$

$$= \frac{1}{z^p}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)}, \tag{1.3}$$

$$= \frac{1}{z^p} + \frac{(n+p)!}{n!p!} a_0 + \frac{(n+p+1)!}{n!(p+1)!} a_1 z + \cdots \tag{1.4}$$

In this paper along with other things we shall show that a function $f \in \Sigma_p$ which satisfies one of the conditions

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - (p+1)\right\} < -p\frac{n}{n+1}, \ |z| < 1, \ n \in N_0 = N \cup \{0\} \tag{1.5}$$

is meromorhically p-valent starlike in U^*. More precisely, it is proved that, for the classes M_n of functions in Σ_p satisfying (1.5),

$$M_{n+1} \subseteq M_n \tag{1.6}$$

holds. Since M_0 equals Σ_p^* (the class of meromorphically p-valent starlike functions), the starlikeness of members of M_n is a consequence of (1.6). Further for $c > 0$, let

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt, \tag{1.7}$$

it is shown that $F \in M_n$ whenever $f \in M_n$. Also it is shown that if $f \in M_n$ then

$$F(z) = \frac{n+1}{z^{n+p+1}} \int_0^z t^{n+p} f(t)dt \tag{1.8}$$

belongs to M_{n+1} for $F \neq 0$ in U^*. Some known results of Bajpai [1], Goel and Sohi [3] and Ganigi and Uralegaddi [2] are extended. In [5] Ruscheweyh obtained the new criteria for univalent functions.

2. Properties of the class M_n. In proving our main results (Theorem 1 and Theorem 2 below), we shall need the following lemma due to I.S. Jack [4].

Lemma. *Let w be non-constant regular in $U = \{z : |z| < 1\}, w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r > 1$ at z_0, we have $z_0 w'(z_0) = \xi w(z)$ where ξ is a real number, $\xi \geq 1$.*

Theorem 1. *$M_{n+1} \subset M_n$ for each integer $n \in N_0$.*

Proof. Let $f \in M_{n+1}$. Then

$$\operatorname{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (p+1)\right\} < -p\frac{n+1}{n+2}. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)} - (p+1)\right\} < -p\frac{n}{n+1}. \tag{2.2}$$

Define $w(z)$ in $U = \{z : |z| < 1\}$ by

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} - (p+1) = -p\left[\frac{n}{n+1} + \frac{1}{n+1}\,\frac{1-w(z)}{1+w(z)}\right]. \tag{2.3}$$

Clearly w is regular and $w(0) = 0$. Equation (2.3) may be written as

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} = \frac{(n+1)+(n+2p+1)w(z)}{(n+1)(1+w(z))}. \tag{2.4}$$

Differentiating (2.4) logarithmically and using the identity

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - (n+p+1)D^n f(z), \tag{2.5}$$

we obtain

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (p+1) + p\frac{n+1}{n+2} = \frac{(n+1)+(n+2p+1)w(z)}{(n+2)(1+w(z))} - \\ -\frac{n+p+1}{n+2} + \frac{2pzw'(z)}{(n+2)(1+w(z))[n+1+(n+2p+1)w(z)]}$$

that is

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (p+1) + p\frac{n+1}{n+2} = \\ = \frac{p}{n+2}\left\{-\frac{1-w(z)}{1+w(z)} + \frac{2zw'(z)}{(1+w(z))[n+1+(n+2p+1)w(z)]}\right\}. \tag{2.6}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists z_0 in U such that

$$z_0 w'(z_0) = \xi w(z_0) \tag{2.7}$$

where $|w(z_0)| = 1$ and $\xi \geq 1$. From (2.6) and (2.7), we obtain

$$\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - (p+1) + p\frac{n+1}{n+2} = \\ = \frac{p}{n+2}\left\{-\frac{1-w(z_0)}{1+w(z_0)} + \frac{2\xi w(z_0)}{(1+w(z_0))[n+1+(n+2p+1)w(z_0)]}\right\}. \tag{2.8}$$

Thus

$$\operatorname{Re}\left\{\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - (p+1) + p\frac{n+1}{n+2}\right\} \geq \frac{p}{2(n+2)(n+p+1)} > 0$$

which contradicts (2.1). Hence $|w(z)| < 1$ in U and from (2.3) it follows that $f \in M_n$.

Theorem 2. *Let $f \in \Sigma_p$ satisfy the condition*

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - (p+1)\right\} < p\left\{\frac{1-2n(c+p)}{2(n+1)(c+p)}\right\} \quad (z \in U) \tag{2.9}$$

for a given $n \in N_0$ and $c > 0$. Then

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)\,dt \tag{2.10}$$

belongs to M_n.

Proof. From the definition of F, we have

$$z(D^n F(z))' = cD^n f(z) - (c+p)D^n F(z). \tag{2.11}$$

Using (2.11) and the identity (2.5), the condition (2.9) may be written as

$$\operatorname{Re}\left\{\frac{(n+2)\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)} - (n+2-c)}{(n+1)-(n+1-c)\dfrac{D^n F(z)}{D^{n+1}F(z)}} - (p+1)\right\} < p\left\{\frac{1-2n(c+p)}{2(n+1)(c+p)}\right\}. \tag{2.12}$$

We have to prove that (2.12) implies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+1}F(z)}{D^n F(z)} - (p+1)\right\} < -p\frac{n}{n+1}.$$

Define w in U by

$$\frac{D^{n+1}F(z)}{D^n F(z)} - (p+1) = -p\left[\frac{n}{n+1} + \frac{1}{n+1}\,\frac{1-w(z)}{1+w(z)}\right]. \tag{2.13}$$

Clearly w is regular and $w(0) = 0$. The equation (2.13) may be written as

$$\frac{D^{n+1}F(z)}{D^n F(z)} = \frac{(n+1)+(n+2p+1)w(z)}{(n+1)(1+w(z))}. \tag{2.14}$$

Differentiating (2.14) logarithmically and using (2.5), we obtain

$$\begin{aligned}(n+2)\frac{D^{n+2}F(z)}{D^{n+1}F(z)} - (n+1)\frac{D^{n+1}F(z)}{D^n F(z)} - 1 &= \\ &= \frac{2pzw'(z)}{(1+w(z))[n+1+(n+2p+1)w(z)]}.\end{aligned} \tag{2.15}$$

The above equation may be written as

$$\begin{aligned}\frac{(n+2)\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)} - (n+2-c)}{(n+1)-(n+1-c)\dfrac{D^n F(z)}{D^{n+1}F(z)}} - (p+1) &= \frac{D^{n+1}F(z)}{D^n F(z)} - (p+1) + \\ &+ \left[\frac{2pzw'(z)}{(1+w(z))[n+1+(n+2p+1)w(z)]}\right]\left[\frac{1}{(n+1)-(n+1-c)\dfrac{D^n F(z)}{D^{n+1}F(z)}}\right]\end{aligned} \tag{2.16}$$

which, by using (2.13) and (2.14), reduces to

$$\begin{aligned}\frac{(n+2)\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)} - (n+2-c)}{(n+1)-(n+1-c)\dfrac{D^n F(z)}{D^{n+1}F(z)}} - (p+1) &= \\ = -p\left[\frac{n}{n+1} + \frac{1}{n+1}\,\frac{1-w(z)}{1+w(z)}\right] &+ \\ + \frac{2pzw'(z)}{(n+1)(1+w(z))[c+(2p+c)w(z)]}&.\end{aligned} \tag{2.17}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists a z_0, $|z_0| < 1$ such that

$$z_0 w'(z_0) = \xi w(z_0) \tag{2.18}$$

where $|w(z_0)| = 1$ and $\xi \geq 1$. Combining (2.17) and (2.18), we obtain

$$\begin{aligned}\frac{(n+2)\dfrac{D^{n+2}F(z_0)}{D^{n+1}F(z_0)} - (n+2-c)}{(n+1)-(n+1-c)\dfrac{D^n F(z_0)}{D^{n+1}F(z_0)}} - (p+1) &= \\ = -p\left[\frac{n}{n+1} + \frac{1}{n+1}\,\frac{1-w(z_0)}{1+w(z_0)}\right] + \frac{2p\xi w(z_0)}{(n+1)(1+w(z_0))[c+(2p+c)w(z_0)]}&.\end{aligned}$$

Thus

$$\operatorname{Re}\left\{\frac{(n+2)\frac{D^{n+2}F(z_0)}{D^{n+1}F(z_0)}-(n+2-c)}{(n+1)-(n+1-c)\frac{D^{n}F(z_0)}{D^{n+1}F(z_0)}}-(p+1)\right\}>p\left\{\frac{1-2n(c+p)}{2(n+1)(c+p)}\right\}$$

which contradicts (2.9). Hence $|w(z)|<1$ in U and from (2.13) it follows that $F\in M_n$.

Putting $p=1$ and $n=0$ in Theorem 2, we obtain the following result of Goel and Sohi [3].

Corollary. *If*

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k z^k \tag{2.19}$$

and satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<\frac{1}{2(c+1)}\quad (c>0) \tag{2.20}$$

then

$$F(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt \tag{2.21}$$

belongs to Σ^* *for* $F(z)\neq 0$ *in* $0<|z|<1$.

For $c=1$, the above Corollary extends a result of Bajpai [1].

Theorem 3. *If* $f\in M_n$, *then*

$$F(z)=\frac{n+1}{z^{n+p+1}}\int_0^z t^{n+p}f(t)dt \tag{2.22}$$

belongs to M_{n+1} *for* $F(z)\neq 0$ *in* $0<|z|<1$.

Proof. We have

$$cD^n f(z)=(n+1)D^{n+1}F(z)-(n+1-c)D^nF(z) \tag{2.23}$$

and

$$cD^{n+1}f(z)=(n+2)D^{n+2}F(z)-(n+2-c)D^{n+1}F(z). \tag{2.24}$$

Taking $c=n+1$ in the above relations we obtain

$$\frac{(n+2)D^{n+2}F(z)-D^{n+1}F(z)}{(n+1)D^{n+1}F(z)}=\frac{D^{n+1}f(z)}{D^nf(z)} \tag{2.25}$$

which reduce to

$$\frac{(n+2)}{(n+1)}\frac{D^{n+2}F(z)}{D^{n+1}F(z)}-\frac{1}{n+1}=\frac{D^{n+1}f(z)}{D^nf(z)}. \tag{2.26}$$

Thus

$$\begin{aligned}\operatorname{Re}\left\{\frac{n+2}{n+1}\frac{D^{n+2}F(z)}{D^{n+1}F(z)}-\frac{1}{n+1}-(p+1)\right\}&=\\ =\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}-(p+1)\right\}&<-p\frac{n}{n+1}\end{aligned} \tag{2.27}$$

from which it follows that

$$\operatorname{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-(p+1)\right\}<-p\frac{n+1}{n+2}. \tag{2.28}$$

Remark. Taking $p=1$ in the above theorems, we get the results obtained by Ganigi and Uralegaddi [2].

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