ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f1.

MEROMORPHICALLY STARLIKE FUNCTIONS WITH POSITIVE AND FIXED MANY COEFFICIENTS

BY

M.K. AOUF and H.E. DARWISH

1. Introduction. Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.1}$$

which are analytic in $U^* = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there.

A function $f \in \Sigma$ is said to be meromorphically starlike of order α if it satisfies the condition

$$\left| \left(\frac{zf'(z)}{f(z)} + 1 \right) \left(\frac{zf'(z)}{f(z)} - 1 \right)^{-1} \right| < \alpha, \ 0 < \alpha \leq 1. \tag{1.2}$$

The class of all such functions is denoted by $\Sigma[\alpha]$.

For the class $\Sigma[\alpha]$ U r a l e g a d d i [3] showed the following lemma.

Lemma 1. *A function f defined by* (1.1) *is in the class $\Sigma[\alpha]$ if and only if*

$$\sum_{n=1}^{\infty} (n + 1 + n\alpha - \alpha) a_n \leq 2\alpha. \tag{1.3}$$

In view of Lemma 1. all functions belonging to the class $\Sigma[\alpha]$ satisfy the coefficient inequality

$$a_n \leq \frac{2\alpha}{n + 1 + n\alpha - \alpha} \quad (n \geq 1). \tag{1.4}$$

Making use of (1.4), we now introduce the following class of functions: Let $\Sigma_{p_k}[\alpha]$ denote the subclass of $\Sigma[\alpha]$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\alpha p_i}{i+1+i\alpha-\alpha} z^i + \sum_{n=k+1}^{\infty} a_n z^n, \tag{1.5}$$

where $a_n \geq 0$, $0 \leq p_i \leq 1$, and $0 \leq \sum_{i=1}^{k} p_i \leq 1$.

For $k=1$, the class $\Sigma_{p_1}[\alpha] = \Sigma_p[\alpha]$ was studied by Uralegaddi [3].

In this paper we obtain coefficient estimates and closure theorems for the class $\Sigma_{p_k}[\alpha]$. Further the radius of convexity is obtained for the class $\Sigma_{p_k}[\alpha]$. Techniques used are similar to those of Silverman and Silvia [2], Uralegaddi [3] and Owa and Srivastava [1].

2. Coefficient Estimates.

Theorem 1. *Let the function f be defined by (1.5). Then f is in the class $\Sigma_{p_k}[\alpha]$ if and only if*

$$\sum_{n=k+1}^{\infty} (n+1+n\alpha-\alpha) a_n \leq 2\alpha \left(1 - \sum_{i=1}^{k} p_i\right), \tag{2.1}$$

where $0 \leq p_i \leq 1$ and $0 \leq \sum_{i=1}^{k} p_i \leq 1$. The result (2.1) is sharp.

Proof. Putting

$$a_i = \frac{2\alpha p_i}{i+1+i\alpha-\alpha} \quad (i = 1, 2, \ldots, k), \tag{2.2}$$

in Lemma 1, we have

$$\sum_{i=1}^{k} 2\alpha p_i + \sum_{n=k+1}^{\infty} (n+1+n\alpha-\alpha) a_n \leq 2\alpha, \tag{2.3}$$

which clearly implies (2.1). Further, by taking the function f of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\alpha p_i}{i+1+i\alpha-\alpha} z^i + \frac{2\alpha\left(1 - \sum_{i=1}^{k} p_i\right)}{n+1+n\alpha-\alpha} z^n \tag{2.4}$$

for $n \geq k+1$, we can see that the result (2.1) is sharp.

Corollary 1. *Let the function f defined by (1.5) be in the class $\Sigma_{p_k}[\alpha]$. Then*

$$a_n \leq 2\alpha \left(1 - \sum_{i=1}^{k} p_i\right) \cdot (n+1+n\alpha-\alpha)^{-1} \quad (n \geq k+1). \tag{2.5}$$

The result (2.5) is sharp for the function f given by (2.4).

3. Closure Theorems.

Theorem 2. *Let the functions*

$$f_j(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\alpha p_i}{i+1+i\alpha-\alpha} z^i + \sum_{n=k+1}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0) \tag{3.1}$$

be in the class $\Sigma_{p_k}[\alpha]$ for every $j = 1, 2, \ldots, m$. Then the function F defined by

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) \quad (d_j \geq 0) \tag{3.2}$$

is also in the same class $\Sigma_{p_k}[\alpha]$, where

$$\sum_{j=1}^{m} d_j = 1. \tag{3.3}$$

Proof. Combining the definitions (3.1) and (3.2), we have

$$F(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\alpha p_i}{i+1+i\alpha-\alpha} z^i + \sum_{n=k+1}^{\infty} \left(\sum_{j=1}^{m} d_j a_{n,j}\right) z^n, \tag{3.4}$$

where we have also used the relationship (3.3). Since $f_j \in \Sigma_{p_k}[\alpha]$ for every $j = 1, 2, \ldots, m$, Theorem 1 yields

$$\sum_{n=k+1}^{\infty} (n+1+n\alpha-\alpha) a_{n,j} \leq 2\alpha \left(1 - \sum_{i=1}^{k} p_i\right) \tag{3.5}$$

for $j = 1, 2, \ldots, m$. Thus we obtain

$$\sum_{n=k+1}^{\infty}(n+1+n\alpha-\alpha)\left(\sum_{j=1}^{m} d_j a_{n,j}\right) =$$

$$= \sum_{j=1}^{m} d_j \left(\sum_{n=k+1}^{\infty}(n+1+n\alpha-\alpha)a_{n,j}\right) \le 2\alpha\left(1-\sum_{i=1}^{k} p_i\right)$$

which (in view of Theorem 1) implies that $F \in \Sigma_{p_k}[\alpha]$.

Theorem 3. *Let the functions f_j be defined by (3.1). If $f_j \in \Sigma_{p_k}[\alpha]$ for every $j = 1, 2, \ldots, m$, then the function*

$$g(z) = \frac{1}{z} + \sum_{i=1}^{k} \frac{2\alpha p_i}{i+1+i\alpha-\alpha} z^i + \sum_{n=k+1}^{\infty} b_n z^n \quad (b_n \ge 0) \tag{3.6}$$

is also in the same class $\Sigma_{p_k}[\alpha]$, where

$$b_n = \frac{1}{m}\sum_{j=1}^{m} a_{n,j}. \tag{3.7}$$

Proof. Since $f_j \in \Sigma_{p_k}[\alpha]$ it follows from Theorem 1 that

$$\sum_{n=k+1}^{\infty}(n+1+n\alpha-\alpha)a_{n,j} \le 2\alpha\left(1-\sum_{i=1}^{k} p_i\right). \tag{3.8}$$

Hence

$$\sum_{n=k+1}^{\infty}(n+1+n\alpha-\alpha)b_n =$$

$$= \sum_{n=k+1}^{\infty}(n+1+n\alpha-\alpha)\left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right) = \tag{3.9}$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=k+1}^{\infty}(n+1+n\alpha-\alpha)a_{n,j}\right) \le 2\alpha\left(1-\sum_{i=1}^{k} p_i\right)$$

and the result follows.

Theorem 4. *The class $\Sigma_{p_k}[\alpha]$ is closed under convex linear combination.*

Proof. Let the functions f_j $(j = 1, 2)$ defined by (3.1) be in the class $\Sigma_{p_k}[\alpha]$. it is sufficient to prove that the function H defined by

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \le \lambda \le 1) \tag{3.10}$$

is also in the class $\Sigma_{p_k}[\alpha]$. Since

$$H(z) = \frac{1}{z} + \sum_{i=1}^{k}\frac{2\alpha p_i}{i+1+i\alpha-\alpha}z^i + \sum_{n=k+1}^{\infty}\{\lambda a_{n,1} + (1-\lambda)a_{n,2}\}z^n, \tag{3.11}$$

we observe that

$$\sum_{n=k+1}^{\infty}(n+1+n\alpha-\alpha)\{\lambda a_{n,1} + (1-\lambda)a_{n,2}\} \le 2\alpha\left(1-\sum_{i=1}^{k} p_i\right) \tag{3.12}$$

with the aid of Theorem 1. Hence $H \in \Sigma_{p_k}[\alpha]$.

Theorem 5. *Let*

$$f_k(z) = \frac{1}{z} + \sum_{i=1}^{k}\frac{2\alpha p_i}{i+1+i\alpha-\alpha}z^i \quad \textit{and} \tag{3.13}$$

$$f_n(z) = \frac{1}{z} + \sum_{i=1}^{k}\frac{2\alpha p_i}{i+1+i\alpha-\alpha}z^i + \frac{2\alpha\left(1-\sum_{i=1}^{k}p_i\right)}{n+1+n\alpha-\alpha}z^n \quad (n \ge k+1). \tag{3.14}$$

Then f is in the class $\Sigma_{p_k}[\alpha]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=k}^{\infty}\lambda_n f_n(z), \text{ where } \lambda_n \ge 0 \ (n \ge k) \text{ and} \tag{3.15}$$

$$\sum_{n=k}^{\infty}\lambda_n = 1. \tag{3.16}$$

Proof. We suppose that f can be expressed in the form (3.15). Then it follows from (3.13), (3.14) and (3.16) that

$$(3.17)\quad f(z)=\frac{1}{z}+\sum_{i=1}^{k}\frac{2\alpha p_i}{i+1+i\alpha-\alpha}z^i+\sum_{n=k+1}^{\infty}\frac{2\alpha\left(1-\sum_{i=1}^{k}p_i\right)\lambda_n}{n+1+n\alpha-\alpha}z^n.$$

Note that

$$(3.18)\quad \begin{aligned}&\sum_{n=k+1}^{\infty}2\alpha\left(1-\sum_{i=1}^{k}p_i\right)\lambda_n=\\ &=2\alpha\left(1-\sum_{i=1}^{k}p_i\right)(1-\lambda_k)\le 2\alpha\left(1-\sum_{i=1}^{k}p_i\right),\end{aligned}$$

which implies that $f\in\Sigma_{p_k}[\alpha]$.

For the converse, assume that the function f of the form (1.5) belongs to the class $\Sigma_{p_k}[\alpha]$. Since f satisfies (2.5) for $n\ge k+1$, we may set

$$\lambda_n=(n+1+n\alpha-\alpha)\left[2\alpha\left(1-\sum_{i=1}^{k}p_i\right)\right]^{-1}a_n\le 1\ (n\ge k+1)\text{ and }\lambda_k=1-\sum_{n=k+1}^{\infty}\lambda_n.$$

Hence f has the representation (3.15). This evidently completes the proof of Theorem 5.

4. Radius of Convexity.

Theorem 6. *Let the function f defined by (1.5) be in the class $\Sigma_{p_k}[\alpha]$. Then f is meromorphically convex of order β $(0\le\beta<1)$ in $0<|z|<r=r(\alpha,p_i,\beta)$, where $r(\alpha,p_i,\beta)$ is the largest value for which*

(4.1)

$$\sum_{i=1}^{k}\frac{i(i+2-\beta)2\alpha p_i}{i+1+i\alpha-\alpha}r^{i+1}+\frac{n(n+2-\beta)2\alpha\left(1-\sum_{i=1}^{k}p_i\right)}{n+1+n\alpha-\alpha}r^{n+1}\le 1-\beta.$$

for $n\ge k+1$. The result is sharp for the function f given by (2.4).

Proof. It suffices to show that

$$(4.2)\quad \left|\frac{zf''(z)}{f'(z)}+2\right|\le 1-\beta\ \ (0\le\beta<1)\ \text{ for }\ 0<|z|<r(\alpha,p_i,\beta).$$

Note that

$$(4.3)\quad \left|\frac{zf''(z)}{f'(z)}+2\right|\le\frac{\sum_{i=1}^{k}\frac{i(i+1)2\alpha p_i}{i+1+i\alpha-\alpha}r^{i+1}+\sum_{n=k+1}^{\infty}n(n+1)a_nr^{n+1}}{1-\sum_{i=1}^{k}\frac{2i\alpha p_i}{i+1+i\alpha-\alpha}r^{i+1}-\sum_{n=k+1}^{\infty}na_nr^{n+1}}\le\ \le 1-\beta$$

for $0<|z|\le r$ if and only if

$$(4.4)\quad \sum_{i=1}^{k}\frac{2\alpha i(i+2-\beta)p_i}{i+1+i\alpha-\alpha}r^{i+1}+\sum_{n=k+1}^{\infty}n(n+2-\beta)a_nr^{n+1}\le 1-\beta.$$

Since f is in the class $\Sigma_{p_k}[\alpha]$, from (2.5) we may take

$$(4.5)\quad a_n=\frac{2\alpha\left(1-\sum_{i=1}^{k}p_i\right)\lambda_n}{n+1+n\alpha-\alpha}\quad(n\ge k+1),$$

where $\lambda_n\ge 0$ $(n\ge k+1)$ and

$$(4.6)\quad \sum_{n=k+1}^{\infty}\lambda_n\le 1.$$

For each fixed r, we choose the positive integer $n_0=n_0(r)$ for which $\dfrac{n(n+2-\beta)}{n+1+n\alpha-\alpha}r^{n+1}$ is maximal. Then it follows that

$$(4.7)\quad \sum_{n=k+1}^{\infty}n(n+2-\beta)a_nr^{n+1}\le\frac{n_0(n_0+2-\beta)2\alpha\left(1-\sum_{i=1}^{k}p_i\right)}{n_0+1+n_0\alpha-\alpha}r^{n_0+1}.$$

Hence f is meromorphically convex of order β in $0<|z|<r(\alpha,p_i,\beta)$ provided that

$$(4.8)\quad \sum_{i=1}^{k}\frac{i(i+2-\beta)2\alpha p_i}{i+1+i\alpha-\alpha}r^{i+1}+\frac{n_0(n_0+2-\beta)2\alpha\left(1-\sum_{i=1}^{k}p_i\right)}{n_0+1+n_0\alpha-\alpha}r^{n_0+1}\le\ \le 1-\beta.$$

We find the value $r_0 = r_0(\alpha, p_i, \beta)$ and the corresponding integer $n_0(r_0)$ so that

$$(4.9)\quad \sum_{i=1}^{k} \frac{i(i+2-\beta)2\alpha p_i}{i+1+i\alpha-\alpha} r_0^{i+1} + \frac{n_0(n_0+2-\beta)2\alpha\left(1-\sum_{i=1}^{k} p_i\right)}{n_0+1+n_0\alpha-\alpha} r_0^{n_0+1} = 1-\beta.$$

Then this value r_0 is the radius of meromorphically convex of order β for functions f belonging to the class $\Sigma_{p_k}[\alpha]$.

REFERENCES

1. O w a, S. and S r i v a s t a v a, H. – *A class of analytic functions with fixed finitely many coefficients*, J. Fac. Sci. Tech. Kinki Univ. 23 (1987), 1–10.
2. S i l v e r m a n, H. and S i l v i a, E.M. – *Fixed coefficients for subclasses of starlike functions*, Houston J. Math. 7 (1981), 129–136.
3. U r a l e g a d d i, B.A. – *Meromorphically starlike functions with positive and fixed second coefficients*, Kyungpook Math. J. 29 (1989), no.1, 64–68.

Received: 5.IX.94

Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura,
EGYPT

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f1.

SOME CHARACTERISATIONS OF FINITE π-SOLVABLE AND p-SOLVABLE GROUPS WITH THE HELP OF PRIMITIVE GROUP

BY

A. BHATTACHARYYA

1. Introduction. One of the main goals in finite group theory is to study the maximal subgroups because the knowledge of the maximal subgroups of the finite group often gives us a good information about the group itself. M u k h e r j e e and B h a t t a c h a r y a [14] introduced the concept of θ-pairs for a maximal subgroup of a finite group, which is motivated by the interesting concept of the Index Complex defined in D e s k i n s [6]-[7]. In [11] , we have shown how some conditions imposed on maximal θ-pairs for a maximal subgroup M of a finite group G, characterize the solvability of the group. We introduced a characteristic subgroup $S_P(G)$ in [9] and studied its influences on π-solvable and supersolvable group in [10]. We also introduced another characteristic subgroup $B_p(G)$ in [8]. B a l l e s t e r-B o l i n c h e s [4] studied the structure of a group by using its primitive permutation representation and the normal index (introduced by D e s k i n s in [6]) of its maximal subgroups. The purpose of this paper is to obtain some results about π-solvable and p-solvable group by applying the primitive group theory. All groups considered here are finite and we use standard notation as found in G o r e n s t e i n [12]. Sometimes, we denote $M < G$ to indicate that M is a maximal subgroup of G.

2. Preliminaries. In this section, we collect some well-known definitons and results (whose proofs we omit) that are needed in the sequel.

Definition. *For a group G,* Soc(G), *the socle of G is defined to be the product of all the minimal normal subgroups of G, if $G \neq 1$; and* Soc$(G) = 1$ *if $G = 1$. A primitive group is a group G such that for some*