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EXTREME POINTS OF MEROMORPHIC UNIVALENT FUNCTIONS WITH TWO FIXED POINTS

BY

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1. Introduction. In [1] H. S i l v e r m a n studied the class of univalent functions of the form $f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n$, $a_n\geq 0$ where either $f(z_0)=z_0$ $(-1<z_0<1)$ or $f'(z_0)=1$ $(-1<z_0<1)$.

This paper deals with functions of the form

(1) $$f(z)=\frac{a_0}{z}-\sum_{n=1}^{\infty}a_nz^n,\ (a_n\geq 0)\text{ where either}$$

(2) $$z_0f(z_0)=1\ (-1<z_0<1,\ z_0\neq 0)$$

or

(3) $$-z_0^2f'(z_0)=1\ (-1<z_0<1,\ z_0\neq 0).$$

For $0\leq\alpha<1$, a function $f(z)=\frac{a_0}{z}+\sum_{n=1}^{\infty}a_nz^n$ regular in the punctured disk $E=\{z:0<|z|<1\}$ with a simple pole at $z=0$ is said to be meromorphically starlike of order α if $\operatorname{Re} zf'(z)/f(z)<-\alpha$ for $|z|<1$ and is said to be meromorphically convex of order α if $\operatorname{Re}(1+zf''(z)/f'(z))<-\alpha$ for $|z|<1$.

For $1<\beta\leq 2$ denote by $\Sigma_N^{**}(\beta)$ the subclass of functions $f(z)$ given by (1) satisfying $\operatorname{Re} zf'(z)/f(z)>-\beta$ for $|z|<1$. In [2] it is shown that all functions in $\Sigma_N^{**}(\beta)$ are starlike. Hence univalent.

Given β and z_0 fixed, let $\Sigma^{**}_{N_0}(\beta, z_0)$ and $\Sigma^{**}_{N_1}(\beta, z_0)$ be the subclasses of $\Sigma^{**}_N(\beta)$ satisffying (2) and (3) respectively. In this paper we determine coefficient estimates, extreme points and radius of convexity for the classes $\Sigma^{**}_{N_0}(\beta, z_0)$ and $\Sigma^{**}_{N_1}(\beta, z_0)$. Further the necessary and sufficient condition for a subset B of the real interval $(0,1)$ to have the property that $\bigcup_{z_\gamma \in B} \Sigma^{**}_{N_0}(\alpha, z_\gamma)$ forms a convex family is obtained. Most of the results in this paper reduce to those in [2] by taking $z_0 = 0$.

2. Coefficient inequalities. In [2] it is shown that necessary and sufficient condition for $f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$ to be in $\Sigma^{**}_N(\beta)$ is

$$\sum_{n=1}^{\infty}(n+\beta)a_n \leq \beta - 1.$$

Theorem 1. *A function* $f(z) = \frac{a_0}{z} - \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$ *belongs to* $\Sigma^{**}_N(\beta)$ *if and only if* $\sum_{n=1}^{\infty}(n+\beta)a_n \leq a_0(\beta-1)$.

Proof. Suppose $\sum_{n=1}^{\infty}(n+\beta)a_n \leq a_0(\beta-1)$. It suffices to show that

$$\left|\frac{zf'(z)/f(z)+1}{zf'(z)/f(z)-(1-2\beta)}\right| < 1,\ z \in U.$$

We have

$$\left|\frac{zf'(z)/f(z)+1}{zf'(z)/f(z)-(1-2\beta)}\right| = \left|\frac{\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}}{2(\beta-1)a_0 - \sum_{n=1}^{\infty}(n+2\beta-1)a_n z^{n+1}}\right| \leq$$

$$\leq \frac{\sum_{n=1}^{\infty}(n+1)|a_n|}{2(\beta-1)a_0 - \sum_{n=1}^{\infty}(n+2\beta-1)|a_n|}.$$

This expression is bounded by 1 if

$$\sum_{n=1}^{\infty}(n+1)a_n \leq 2(\beta-1)a_0 - \sum_{n=1}^{\infty}(n+2\beta-1)a_n$$

which is equivalent to

$$\sum_{n=1}^{\infty}(n+\beta)a_n \leq a_0(\beta-1).$$

Conversely suppose

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\frac{-a_0 - \sum_{n=1}^{\infty} n a_n z^{n+1}}{a_0 - \sum_{n=1}^{\infty} a_n z^{n+1}} > -\beta,\ z \in U. \tag{4}$$

Choose values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator in (4) and letting $z \to 1$ through real values, we get

$$\sum_{n=1}^{\infty}(n+\beta)a_n \leq a_0(\beta-1).$$

Theorem 2. *A function* $f(z) = \frac{a_0}{z} - \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0$ *is in* $\Sigma^{**}_{N_0}(\beta, z_0)$ *if and only if* $\sum_{n=1}^{\infty} a_n \left(\frac{n+\beta}{\beta-1} - z_0^{n+1}\right) \leq 1$.

Proof. We have $z_0 f(z_0) = 1 = a_0 - \sum_{n=1}^{\infty} a_n z_0^{n+1}$ i.e.,

$$a_0 = 1 + \sum_{n=1}^{\infty} a_n z_0^{n+1}.$$

Substituting this value of a_0 in Theorem 1 we get

$$\sum_{n=1}^{\infty} a_n \left(\frac{n+\beta}{\beta-1} - z_0^{n+1}\right) \leq 1.$$

Corollary. *If* $f(z)=\frac{a_0}{z}-\sum_{n=1}^{\infty}a_nz^n$, *is in* $\Sigma^{**}_{N_0}(\beta,z_0)$ *then* $a_n\le(\beta-1)/(n+\beta-(\beta-1)z_0^{n+1})$, $n=1,2,\ldots$ *with equality for*

$$f(z)=\left(\frac{n+\beta}{z}-(\beta-1)z^n\right)/(n+\beta-(\beta-1)z_0^{n+1}).$$

Theorem 3. *A function* $f(z)=\frac{a_0}{z}-\sum_{n=1}^{\infty}a_nz^n, a_n\ge 0$ *is in* $\Sigma^{**}_{N_1}(\beta,z_0)$ *if and only if* $\sum_{n=1}^{\infty}a_n\left(\frac{n+\beta}{\beta-1}+nz_0^{n+1}\right)\le 1.$

Proof. Since $-z_0^2f'(z_0)=1=a_0+\sum_{n=1}^{\infty}na_nz_0^{n+1}$.

Substituting $a_0=1-\sum_{n=1}^{\infty}na_nz_0^{n+1}$ in Theorem 1 the result follows.

Corollary. *If* $f(z)=\frac{a_0}{z}-\sum_{n=1}^{\infty}a_nz^n$ *is in* $\Sigma^{**}_{N_1}(\beta,z_0)$ *then* $a_n\le(\beta-1)/(n+\beta+n(\beta-1)z_0^{n+1})$, *with equality for*

$$f(z)=\left(\frac{n+\beta}{z}-(\beta-1)z^n\right)/(n+\beta+n(\beta-1)z_0^{n+1}).$$

3. Radius of convexity.

Theorem 4. *If* $f(z)\in\Sigma^{**}_{N_0}(\beta,z_0)$, *then* $\mathrm{Re}(1+zf''(z)/f'(z))<0$ *in* $|z|<r(\beta)=\inf_n\left(\frac{n+\beta}{n(n+2)(\beta-1)}\right)^{1/n+1}$, $n=1,2,\ldots$. *The result is sharp for* $f(z)=\left(\frac{(n+\beta)}{z}-(\beta-1)z^n\right)/(n+\beta-(\beta-1)z_0^{n+1})$.

Proof. It suffices to show that

$$|zf''(z)/f'(z)+2|<1 \quad \text{for} \quad |z|<r=r(\beta).$$

We have

$$|zf''(z)f(z)+2|=\left|\frac{\sum_{n=1}^{\infty}n(n+1)a_nz^{n+1}}{a_0+\sum_{n=1}^{\infty}na_nz^{n+1}}\right|\le\frac{\sum_{n=1}^{\infty}n(n+1)a_n|z|^{n+1}}{a_0-\sum_{n=1}^{\infty}na_n|z|^{n+1}}.$$

Thus $|zf''(z)/f'(z)+2|<1$ provided

$$\sum_{n=1}^{\infty}n(n+1)a_n|z|^{n+1}\le a_0-\sum_{n=1}^{\infty}na_n|z|^{n+1}$$

or

$$\sum_{n=1}^{\infty}(n^2+2n)a_n|z|^{n+1}\le a_0 \tag{5}$$

Putting $a_0=1+\sum_{n=1}^{\infty}a_nz_0^{n+1}$ in the above inequality, (5) reduces to

$$\sum_{n=1}^{\infty}[n(n+2)|z|^{n+1}-z_0^{n+1}]a_n\le 1. \tag{6}$$

But from Theorem 2 we have $\sum_{n=1}^{\infty}a_n\left(\frac{n+\beta}{\beta-1}-z_0^{n+1}\right)\le 1$. Hence (6) will be satisfied if

$$n(n+2)|z|^{n+1}-z_0^{n+1}\le\frac{n+\beta}{\beta-1}-z_0^{n+1}. \tag{7}$$

Solving (7) for $|z|$ we get

$$|z|\le\left(\frac{n+\beta}{n(n+2)(\beta-1)}\right)^{1/n+1}, \quad n=1,2,\ldots$$

The result follows.

Remark. The conclusion in Theorem 4 is independent of the fixed point z_0.

4. Extreme points. $\Sigma^{**}_{N_0}(\beta, z_0)$ is a convex family. If $f_1(z)$ and $f_2(z)$ are in $\Sigma^{**}_{N_0}(\beta, z_0)$ then it can be shown that $f(z) = \lambda f_1(z) + (1-\lambda) f_2(z)$ is a member of $\Sigma^{**}_{N_0}(\beta, z_0)$ for $0 \le \lambda \le 1$. We shall now show that the extreme points of $\Sigma^{**}_{N_0}(\beta, z_0)$ are $\frac{1}{z}$ and $\frac{(n+\beta)/z - (\beta-1)z^n}{n+\beta-(\beta-1)z_0^{n+1}}$, $n = 1, 2, \ldots$.

Theorem 5. *Set* $f_0(z) = \frac{1}{z}$ *and* $f_n(z) = \frac{(n+\beta)/z - (\beta-1)z^n}{n+\beta-(\beta-1)z_0^{n+1}}$, $n = 1, 2, \ldots$ *Then* $f(z) \in \Sigma^{**}_{N_0}(\beta, z_0)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, *where* $\lambda_n \ge 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$. Then

$$f(z) = \left(\lambda_0 + \sum_{n=1}^{\infty} \frac{\lambda_n (n+\beta)}{n+\beta-(\beta-1)z_0^{n+1}}\right)\frac{1}{z} - \sum_{n=1}^{\infty} \frac{\lambda_n(\beta-1)z^n}{n+\beta-(\beta-1)z_0^{n+1}}.$$

Note that $f(z_0) = \sum_{n=1}^{\infty} \lambda_n f_n(z_0) = \left(\sum_{n=0}^{\infty} \lambda_n\right)\frac{1}{z_0} = \frac{1}{z_0}$, i.e. $z_0 f(z_0) = 1$.

Also

$$\sum_{n=1}^{\infty} \frac{\lambda_n(\beta-1)}{n+\beta-(\beta-1)z_0^{n+1}}\left(\frac{n+\beta-(\beta-1)z_0^{n+1}}{\beta-1}\right) = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \le 1.$$

Thus by Theorem 2 it follows that $f(z) \in \Sigma^{**}_{N_0}(\beta, z_0)$. Conversely suppose $f(z) \in \Sigma^{**}_{N_0}(\beta, z_0)$. Then we have $a_n \le (\beta-1)/(n+\beta-(\beta-1)z_0^{n+1})$, $n = 1, 2, \ldots$ We may set

$$\lambda_n = \{(n+\beta-(\beta-1)z_0^{n+1})/(\beta-1)\}a_n, \quad n = 1, 2, \ldots$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$. Then $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$.

Theorem 6. *Set* $f_0(z) = \frac{1}{z}$ *and* $f_n(z) = \frac{(n+\beta)/z - (\beta-1)z^n}{n+\beta+n(\beta-1)z_0^{n+1}}$ *for* $n = 1, 2, \ldots$ *Then* $f(z) \in \Sigma^{**}_{N_1}(\beta, z_0)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$ *where* $\lambda_n \ge 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof is similar to that of Theorem 5.

Corollary. *The extreme points of* $\Sigma^{**}_{N_1}(\beta, z_0)$ *are the functions* $f_n(z)$, $n = 0, 1, 2, \ldots$

5. Convex Families. Let B be a non empty subset of the real interval $(0, 1)$. We define $\Sigma^{**}_{N_0}(\beta, B)$ by $\Sigma^{**}_{N_0}(\beta, B) = \bigcup_{z_\gamma \in B} \Sigma^{**}_{N_0}(\beta, z_\gamma)$.

As mentioned earlier if B consists of a single element then $\Sigma^{**}_{N_0}(\beta, B)$ is a convex family. Next we shall consider this class for other subsets of B. We need the following lemma.

Lemma. *If* $f(z) \in \Sigma^{**}_{N_0}(\alpha, z_0) \cap \Sigma^{**}_{N_0}(\alpha, z_1)$, *where* z_0 *and* z_1 *are distinct positive numbers, then* $f(z) = \frac{1}{z}$.

Proof. Let $f(z) = \frac{a_0}{z} - \sum_{n=1}^{\infty} a_n z^n$, $a_n \ge 0$. Then we have $a_0 = 1 + \sum_{n=1}^{\infty} a_n z_0^{n+1} = 1 + \sum_{n=1}^{\infty} a_n z_1^{n+1}$. This implies that $a_n \equiv 0$ for $n \ge 1$. Hence $f(z) = \frac{1}{z}$.

Theorem 7. *If* B *is contained in the interval* $(0, 1)$ *and* $1 < \beta \le 2$, *then* $\Sigma^{**}_{N_0}(\beta, B)$ *is a convex family if and only if* B *is connected.*

Proof. Let B be connected. Suppose $z_0, z_1 \in B$ with $z_0 \le z_1$. If $f(z) = \frac{a_0}{z} - \sum_{n=1}^{\infty} a_n z^n$, $(a_n \ge 0) \in \Sigma^{**}_{N_0}(\beta, z_0)$, $g(z) = \frac{b_0}{z} - \sum_{n=1}^{\infty} b_n z^n$, $(b_n \ge 0) \in \Sigma^{**}_{N_0}(\beta, z_1)$, and $0 \le \lambda \le 1$ then we shall show that there exists a z_2 $(z_0 \le z_2 \le z_1)$ such that $h(z) = \lambda f(z) + (1-\lambda) g(z)$ is in $\Sigma^{**}_{N_0}(\beta, z_2)$. Set

$$\begin{aligned} t(z) = zh(z) &= \lambda a_0 - \lambda \sum_{n=1}^{\infty} a_n z^{n+1} + (1-\lambda) b_0 - (1-\lambda) \sum_{n=1}^{\infty} b_n z^{n+1} \\ &= 1 + \lambda \sum_{n=1}^{\infty} a_n (z_0^{n+1} - z^{n+1}) + (1-\lambda) \sum_{n=1}^{\infty} b_n (z_1^{n+1} - z^{n+1}). \end{aligned} \tag{8}$$

Observe that $t(z)$ is real when z is real with $t(z_0) \ge 1$ and $t(z_1) \le 1$. Hence $t(z_2) = 1$ for some $z_2 (z_0 \le z_2 \le z_1)$. Since z_1, z_2 and λ are arbitrary, the family $\Sigma^{**}_{N_0}(\beta, B)$ is convex.

Conversely suppose B is not connected, choose $z_0, z_1 \in B$, $z_2 \notin B$ with $z_0 < z_2 < z_1$. Assume that $f(z)$ and $g(z)$ are not both equal to $1/z$. Using the notation of (8) except that we fix $z = z_2$ and allow λ to vary,

$$t(\lambda) = t(z_2, \lambda) = 1 + \lambda \sum_{n=1}^{\infty} a_n(z_0^{n+1} - z_2^{n+1}) + (1-\lambda) \sum_{n=1}^{\infty} b_n(z_1^{n+1} - z_2^{n+1}).$$

Since $t(z_2, 0) > 1$ and $t(z_2, 1) < 1$, there exists a λ_0, $0 < \lambda_0 < 1$, such that $t(z_2, \lambda_0) = 1$. Hence $h(z) \in \Sigma_{N_0}^{**}(\beta, z_2)$ for $\lambda = \lambda_0$. Since $z_2 \notin B$, applying above lemma, it follows that $h(z) \notin \Sigma_{N_0}^{**}(\beta, B)$ which is a contradiction. Hence B is connected.

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A CLASSIFICATION OF ALMOST CONTACT METRIC SUBMERSIONS BASED ON THE FIBRES

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0. Introduction. It is well known that the fibres of an almost contact metric submersion of type I, in the sense of W a t s o n [17], are almost Hermitian manifolds. Looking through this type of fibration, one can obtain a large number of results such as the following:

Let $f : M \to M'$ be an almost contact metric submersion of type I. If the total space is cosymplectic, quasi-Sasakian or a Kenmotsu manifold, then the fibres are Kählerian.

This result shows that, taken as a total space of an almost contact metric submersion of type I, many different almost contact metric manifolds may yield a determined class of the fibres. Indeed, since the fibres are almost Hermitian manifolds, they have to belong to one of the sixteen classes obtained by G r a y and H e r v e l l a in [7].

According to the classification of Gray and Hervella, we attempt to produce a classification of almost contact metric submersions of type I; this problem was brought up in [14, p.716]. Among the defining relations of the total space, we bring out those which are common and emerge in forcing the fibres to belong to the same class.

In Section 1, we recall some fundamental notions of almost Hermitian and almost contact metric structures which will be used in this text.

Section 2 is specifically devoted to the classification in which we generalize many results of [2, Th. 2.1], [14, Section 3] and [17, Th. 3.1].

In this text, arbitrary vector fields of $\mathcal{X}(M)$, the Lie algebra of smooth vector fields, will be denoted by D, E and G while those of the vertical distribution $V(M)$ will be U, V and W.