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SEMI–PSEUDO SYMMETRIC MANIFOLD

BY

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Introduction. In a previous paper [1] M.C. C h a k i introduced a pseudo-symmetric manifold. In this paper, we consider a nonflat Riemannian manifold (M^n, g) $(n > 3)$ whose curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = 2\pi(X)R(Y,Z)W + \pi(Y)R(X,Z)W + \pi(Z)R(Y,X)W + \pi(W)R(Y,Z)X \quad (1)$$

where π is a non-zero 1-form

$$g(X,\rho) = \pi(X), \text{ for every vector field X} \quad (2)$$

and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such a manifold shall be called a Semi-Pseudo-Symmetric manifold and the 1-form π shall be called its associated 1-form. An n-dimensional semi-pseudo-symmetric manifold shall be denoted by $(SPS)_n$.

In this paper, it is shown that if its scalar curvature r is constant, then $r = 0$, while if $r \neq 0$ then its associated 1-form is closed. It has been shown that Einstein $(SPS)_n$ does not exist. $(SPS)_n$ with Codazzi type of Ricci tensor and conformally flat $(SPS)_n$ are studied. It has also been shown that, in a conformally flat $(SPS)_n$ $(n \geq 3)$ with non-zero constant scalar curvature, the Ricci tensor has two distinct eigenvalues, namely $\dfrac{r}{n-1}$ and 0 of which the former is of multiplicity $n-1$ and the latter is simple. Finally, a necessary and sufficient condition has been proved for Ricci tensor to be of Codazzi type in a conformally flat $(SPS)_n$.

1. Preliminaries. Let S and r denote respectively the Ricci tensor of type $(0,2)$ and the scalar curvature and L denote the symmetric endomorphism of the tangent space at each point, corresponding to the Ricci tensor S, i.e.

$$g(LX,Y) = S(X,Y) \text{ for any vector fields } X, Y. \tag{1.1}$$

From (1) we get

$$(\nabla_X S)(Y,Z) = 2\pi(X)S(Y,Z) + \pi(Y)S(X,Z) + \\ + \pi(Z)S(Y,X) + \pi(R(X,Y)Z). \tag{1.2}$$

Contracting (1.2) we have

$$dr(X) = 2\pi(X)r + 3\pi(LX). \tag{1.3}$$

Also from (1.2) we find

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \pi(X)S(Y,Z) - \\ -\pi(Z)S(Y,X) + \pi(R(X,Z)Y). \tag{1.4}$$

Contracting (1.4) we get

$$dr(X) = 2\pi(X)r. \tag{1.5}$$

From (1.3) and (1.5) it follows that

$$\pi(LX) = 0. \tag{1.6}$$

In virtue of (1.6), the equation (1.3) takes the form

$$dr(X) = 2\pi(X)r. \tag{1.7}$$

Further we have

$$rd\pi(X,Y) = 0. \tag{1.8}$$

We shall use these formulas later on.

2. Scalar curvature of a $(SPS)_n$. If the scalar curvature r of a $(SPS)_n$ is constant, then from (1.5) it follows that

$$2\pi(X)r = 0.$$

This gives $r = 0$ as $\pi(X) \neq 0$. Again, if $r \neq 0$, then from (1.8) it follows that

$$d\pi(X,Y) = 0$$

which means that the 1-form π is closed. We can therefore state as follows:

Theorem 1. *If the scalar curvature r of $(SPS)_n$ is constant, then $r = 0$, while if $r \neq 0$, then the associated 1-form π of the manifold is closed.*

3. Einstein $(SPS)_n$ $(n > 2)$. It is known [5] that in an Einstein (M^n, g) $(n > 2)$ manifold, the scalar curvature r is constant. By Theorem 1, we can state

Theorem 2. *An Einstein $(SPS)_n$ $(n > 2)$ is of zero scalar curvature.*

If possible, let a $(SPS)_n$ $(n > 2)$ be of constant curvature. Then we have

$$R(X,Y)Z = k\{g(Y,Z)X - g(Z,X)Y\}. \tag{3.1}$$

Since every manifold of constant curvature is an Einstein manifold, it follows from Theorem 2 that in a $(SPS)_n$ $(n > 2)$ of constant curvature the scalar curvature $r = 0$. But $r = 0$ implies $k = 0$. So, from (3.1) it follows that $R(X,Y)Z = 0$, which is inadmissible by definition. This leads to the following corollary of the above theorem

Corollary 1. *A $(SPS)_n$ cannot be of constant curvature.*

Since a 3-dimensional Einstein manifold is of constant curvature [5] it follows from the above corollary that a $(SPS)_3$ cannot be an Einstein manifold. Hence we have another corollary which can be stated as follows:

Corollary 2. *Einstein $(SPS)_n$ does not exist.*

4. $(SPS)_n$ $(n > 2)$ with Codazzi type of Ricci tensor. In this section we consider a $(SPS)_n$ $(n > 2)$ in which the Ricci tensor S is a Codazzi tensor [3] i.e.

$$(\nabla_X S)(Y,Z) = (\nabla_Z S)(Y,X). \tag{4.1}$$

Using (1.2) we get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = \\ = 4\{\pi(X)S(Y,Z) + \pi(Y)S(Z,X) + \pi(Z)S(X,Y)\}. \tag{4.2}$$

In virtue of (4.1), the equation (4.2) can be written as

$$(4.3)\quad 3(\nabla_X S)(Y,Z) = 4\{\pi(X)S(Y,Z)+\pi(Y)S(Z,X)+\pi(Z)S(X,Y)\}.$$

Contracting (4.3) and using (4.6) we get

$$3dr(X) = 4\pi(X)r = 2dr(X) \quad \text{by (1.7)}.$$

Hence

$$(4.4)\quad dr(X) = 0.$$

It is known [2] that in a Riemannian manifold (M^n, g) $(n > 3)$

$$(4.5)\quad \begin{aligned}(\text{div } C)(X,Y)Z = {}& \frac{n-3}{n-2}\{(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)\}+\\ &+\frac{1}{2(n-1)}\{g(X,Y)dr(Z)-g(Y,Z)dr(X)\},\end{aligned}$$

where C denotes the conformal curvature tensor and div denotes divergence with respect to ∇. In virtue of (4.1) and (4.4) it follows from (4.5) that

$$(\text{div } C)(X,Y)Z = 0,$$

which means that the tensor C is conservative [4]. Hence we can state

Theorem 3. *If in a $(SPS)_n$ $(n > 3)$, the Ricci tensor is a Codazzi tensor, then its conformal curvature tensor is conservative.*

5. Conformally flat $(SPS)_n$ $(n \geq 3)$. It is known [2] that in a conformally flat (M^n, g) $(n > 3)$

$$(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X) = \frac{1}{2(n-1)}\{dr(X)g(Y,Z)-dr(Z)g(X,Y)\}$$

Hence in a conformally flat $(SPS)_n$ $(n \geq 3)$ we have

$$(5.1)\quad \begin{aligned}&(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X) =\\ &= \frac{r}{n-1}\{\pi(X)g(Y,Z)-\pi(Z)g(X,Y)\} \text{by (1.7)}\end{aligned}$$

From (5.1) and (1.4) we have

$$(5.2)\quad \begin{aligned}&(n-1)\pi(X)S(Y,Z)-(n-1)\pi(Z)S(Y,X)+(n-1)\pi(R(X,Z)Y)=\\ &=r\{\pi(X)g(Y,Z)-\pi(Z)g(X,Y)\}(n-1)R(X,Z,Y,\rho)=\\ &=-(n-1)\pi(X)S(Y,Z)+(n-1)\pi(Z)S(Y,X)+\\ &+r\pi(X)g(Y,Z)-r\pi(Z)g(X,Y).\end{aligned}$$

Again in a conformally flat (M^n, g) $(n \geq 3)$ we have

$$(5.3)\quad \begin{aligned}R(X,Y,Z,W) = {}& \frac{1}{n-2}\{S(Z,Y)g(X,W)-S(X,Y)g(Z,W)+\\ &+g(Z,Y)S(X,W)-g(X,Y)S(Z,W)\}+\\ &+\frac{r}{(n-1)(n-2)}\{g(X,Y)g(Z,W)-g(Z,Y)g(X,W)\}\end{aligned}$$

Using (1.6) and (5.3) we get

$$(5.4)\quad \begin{aligned}(n-1)R(X,Y,Z,\rho) = {}& \frac{n-1}{n-2}\{S(Y,Z)\pi(X)-\pi(Z)S(X,Y)\}+\\ &+\frac{r}{n-2}\{g(X,Y)\pi(Z)-g(Y,Z)\pi(X)\}\end{aligned}$$

Hence from (5.2) and (5.4) we find

$$\begin{aligned}&-(n-1)\{\pi(X)S(Y,Z)-\pi(Z)S(Y,X)\}-r\{g(X,Y)\pi(Z)-\\ &-\pi(X)g(Y,Z)\} = \frac{n-1}{n-2}\{S(Y,Z)\pi(X)-\pi(Z)S(X,Y)\}+\\ &+\frac{r}{n-2}\{g(X,Y)\pi(Z)-g(Y,Z)\pi(X)\}\end{aligned}$$

or

$$\begin{aligned}&\{S(Y,Z)\pi(X)-\pi(Z)S(X,Y)\}(n-1)(1+\frac{1}{n-2})+\\ &+r\{g(X,Y)\pi(Z)-g(Y,Z)\pi(X)\}(1+\frac{1}{n-2}) = 0\end{aligned}$$

or

$$\{S(Y,Z)\pi(X)-S(X,Y)\pi(Z)\}(n-1)+r\{g(X,Y)\pi(Z)-g(Y,Z)\pi(X)\} = 0$$

or

$$(5.5)\quad \{S(Y,Z)-\frac{r}{n-1}g(Y,Z)\}\pi(X) = \{S(X,Y)-\frac{r}{n-1}g(X,Y)\}\pi(Z)$$

Putting $X = \rho$ and using (1.6) we get

$$\{S(Y,Z) - \frac{r}{n-1}g(Y,Z)\}\pi(\rho) = -\frac{r}{n-1}\pi(Y)\pi(Z)$$

or

$$S(Y,Z) = \frac{r}{n-1}\{g(Y,Z) - \frac{\pi(Y)\pi(Z)}{\pi(\rho)}\}, \ \pi(\rho) \neq 0$$

$$S(Y,Z) = \frac{r}{n-1}\{g(Y,Z) - T(Y)T(Z)\} \quad \text{where} \tag{5.6}$$

$$T(X) = \frac{\pi(X)}{\sqrt{\pi(\rho)}}. \tag{5.7}$$

Since $S(Y,Z) \neq 0$, from (5.6) it follows that $r \neq 0$. Thus in a conformally flat $(SPS)_n$, the scalar curvature is necessarily different from zero.

In virtue of (1.6), it follows from (5.6) that

$$S(LY,Z) - \frac{r}{n-1}g(LY,Z) = 0 \quad \text{as} \quad T(LY) = \frac{\pi(LY)}{\sqrt{\pi(\rho)}} = 0. \tag{5.8}$$

Let λ be an eigenvalue of L and X be an eigenvector corresponding to λ. Then

$$LX = \lambda X. \tag{5.9}$$

Hence from (5.8) we get

$$(\lambda^2 - \frac{r}{n-1}\lambda)g(Y,Z) = 0$$

or

$$\lambda^2 - \frac{r}{n-1}\lambda = 0$$

from which we get

$$\lambda = \frac{r}{n-1}, 0.$$

If m is the multiplicity of $\frac{r}{n-1}$, then

$$m \cdot \frac{r}{n-1} = r$$

or

$$m = n - 1.$$

We can therefore state the following theorem

Theorem 4. *In a conformally flat $(SPS)_n$ ($n \geq 3$) with non-zero scalar curvature, the Ricci tensor L of type $(1,1)$ has two distinct eigenvalues namely $\frac{r}{n-1}$ and 0 of which the former is of multiplicity $n-1$ and the latter is simple.*

Let us now assume that the scalar curvature r of $(SPS)_n$ be constant. Using Theorem 1, we find from (5.1)

$$(\nabla_X S)(Y,Z) = (\nabla_Z S)(Y,X)$$

i.e. the Ricci tensor S is of Codazzi type.

Conversely, let the Ricci tensor be of Codazzi type in a conformally flat $(SPS)_n$. Hence, from (5.1),

$$\frac{r}{n-1}\{\pi(X)g(Y,Z) - \pi(Z)g(X,Y)\} = 0$$

or

$$\frac{r}{n-1}\{n\pi(X) - \pi(X)\} = 0$$

or

$$r\pi(X) = 0$$

$$\therefore \quad r = 0 \quad \text{as} \quad \pi(X) \neq 0.$$

Thus we can state

Theorem 5. *A conformally flat $(SPS)_n$ ($n > 3$) of zero scalar curvature is the necessary and sufficient condition for a Ricci tensor to be of Codazzi type.*

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AN APPROXIMATION METHOD FOR THE SOLUTION OF A DIFFERENTIAL SYSTEM

BY

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0. Introduction. In this paper an approximation method for determining the solution of a differential equation over finite dimensional spaces is proposed. The algorithm in question could be regarded as a refinement of the succesive approximation method (see [2] cap. II, Section 1). Moreover, under some appropriate hypothesis, one can prove the estimated errors are smaller than those obtained by classical means. This method is an analogue – for differential equations – of Seidel's algorithm (see [1] cap. IX, Sections 3-7) concerning the solution of a linear system of equations.

The problem is enounced in Section 1, where a first result (Proposition 1) with a practical importance is given. Starting from here a refinement for it (Proposition 2) is obtained, along with an analogue for the case of another norm (Proposition 3). One can see however that both have mostly a theoretic feature, a lot of calculations being needed. This is the reason why in Section 3 a statement similar to Proposition 3 is given which has, in the cases it is to be used, the advantage to require less calculations.

1. Preliminary results. Let $\Delta=[a,b]\times[\xi_1-r_1,\xi_1+r_1]\times\cdots\times[\xi_n-r_n,\xi_n+r_n]$ be a given $(n+1)$-dimensional interval and $f=(f_1,...,f_n):\Delta\to R^n$ a continuous function, where $a<b$, $\xi_i,r_i\in R$, $r_i>0$, $i=\overline{1,n}$. We consider the Cauchy problem :

$$(CP) \qquad x_i'(t)=f_i(t,x_1(t),...,x_n(t)),\ \ x_i(a)=\xi_i,\ i=\overline{1,n},$$

under standard Lipschitz-type assumptions about the functions f_i