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Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta – 700 019
INDIA

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
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AN APPROXIMATION METHOD FOR THE SOLUTION OF A DIFFERENTIAL SYSTEM

BY

OREST IFTIME and GABRIEL TURINICI

0. Introduction. In this paper an approximation method for determining the solution of a differential equation over finite dimensional spaces is proposed. The algorithm in question could be regarded as a refinement of the succesive approximation method (see [2] cap. II, Section 1). Moreover, under some appropriate hypothesis, one can prove the estimated errors are smaller than those obtained by classical means. This method is an analogue – for differential equations – of Seidel's algorithm (see [1] cap. IX, Sections 3-7) concerning the solution of a linear system of equations.

The problem is enounced in Section 1, where a first result (Proposition 1) with a practical importance is given. Starting from here a refinement for it (Proposition 2) is obtained, along with an analogue for the case of another norm (Proposition 3). One can see however that both have mostly a theoretic feature, a lot of calculations being needed. This is the reason why in Section 3 a statement similar to Proposition 3 is given which has, in the cases it is to be used, the advantage to require less calculations.

1. Preliminary results. Let $\Delta=[a,b]\times[\xi_1-r_1,\xi_1+r_1]\times\cdots\times[\xi_n-r_n,\xi_n+r_n]$ be a given $(n+1)$-dimensional interval and $f=(f_1,...,f_n):\Delta\to R^n$ a continuous function, where $a<b$, $\xi_i,r_i\in R$, $r_i>0$, $i=\overline{1,n}$. We consider the Cauchy problem :

$$(CP) \qquad x_i'(t)=f_i(t,x_1(t),...,x_n(t)),\quad x_i(a)=\xi_i,\ i=\overline{1,n},$$

under standard Lipschitz-type assumptions about the functions f_i

(H1) $|f_i(s,x_1,...,x_n) - f_i(s,y_1,...,y_n)| \leq \sum_{j=1}^{n} L_i^j |x_j - y_j|,$

$(\forall)(s,x_1,...,x_n),(s,y_1,...,y_n) \in \Delta$

Since f_i is continuous over Δ (compact) it follows that there exists $M_i > 0$ such that:

(1) $$|f_i(t,x)| \leq M_i, \quad (\forall)(t,x) \in \Delta, \ i = \overline{1,n}.$$

Let us define:

(D1) $\delta = \min\{b-a, r_1/M_1, ..., r_n/M_n\}$.

From (H1) and the continuity of f over Δ one obtains that (CP) has a unique solution $x = (x_1,...,x_n) : [a, a+\delta] \to R^n$, which satisfies the equivalent integral equation :

(E) $x(t) = \xi + \int_a^t f(s,x(s))ds$ (where $\xi = (\xi_1,...,\xi_n)$)

(see [2] cap. II, Section 1).

In the following we will search for an approximation of the solution above. In order to achieve that we define the sequence of functions $(x^p = (x_1^p,...,x_n^p) : [a, a+\delta] \to R^n)_{p\geq 1}$ by:

(D2) $$\begin{cases} x_i^1(t) = \xi_i, \quad i = \overline{1,n}, \quad \text{and for } p > 1 \\ x_i^{p+1}(t) = \xi_i + \int_a^t f_i(s, x_1^{p+1}(s),...,x_{i-1}^{p+1}(s), x_i^p(s),...,x_n^p(s))ds, \\ \qquad\qquad i = \overline{1,n}. \end{cases}$$

(with the convention $x_1^{p+1}(t) = \xi_1 + \int_a^t f_1(s,x_1^p(s),...,x_n^p(s))ds$).

We prove by induction that $(x^p)_{p\geq 1}$ is well defined, i.e.:

(2) $$|x_i^p(t) - \xi_i| \leq r_i, \ i = \overline{1,n}, \ p \geq 1.$$

For $p = 1 : |x_i^1(t) - \xi_i| = |\xi_i - \xi_i| = 0 \leq r_i$. Supposing now that the functions $(x_1^{p-1},...,x_n^{p-1})$ and $(x_1^p,...,x_{i-1}^p)$ are all well defined we prove the same for x_i^p. In fact:

(3) $$\begin{aligned} |x_i^p(t) - \xi_i| &= |\int_a^t f_i(s,x_1^p(s),...,x_{i-1}^p(s),x_i^{p-1}(s),...,x_n^{p-1}(s))ds| \leq \\ &\leq \int_a^t |f_i(s,x_1^p(s),...,x_{i-1}^p(s),x_i^{p-1}(s),...,x_n^{p-1}(s))|ds \\ &\leq (t-a)M_i \leq \delta M_i \leq r_i. \end{aligned}$$

which proves (2).

We now consider the following norms (for each t in $[a, a+\delta]$):

$$\|y\|_t = \sup_{s\in[a,t]} |y(s)|, \quad y \in C([a,t],R),$$

and also

$$\|y\|_m = \max_{i=\overline{1,n}} |y_i|,$$

where $y = (y_1,...,y_n)$ is an arbitrary element in R^n.

Denote, for simplicity, $(y_1,...,y_n)$ by $(y_i)_{i=1}^n$ and $\|(\|x_i^k - x_i\|_t)_{i=1}^n\|_m$ by $\|x^k - x\|_m^t$.

Let us define:

(D3) $M = \max_{i=\overline{1,n}} M_i$,

(D4) $\alpha_i = \sum_{j=1}^{i-1} L_i^j, \quad \alpha_1 = 0,$

(D5) $\beta_i = \sum_{j=i}^{n} L_i^j, \quad i = \overline{1,n},$

(D6) $L = \max_{i=\overline{1,n}} (\alpha_i + \beta_i)$.

In all that follows we suppose :

(H2) $\delta < L^{-1}$ (so $\alpha_i\delta < 1, \ i = \overline{1,n}$)

Denote :

(D7) $\rho = \max_{i=\overline{1,n}} \dfrac{\beta_i}{1-\alpha_i\delta}$.

Now we can state:

Proposition 1. *If (H2) holds then the sequence $(x^p)_{p>0}$ defined as above converge uniformly on $[a, a+\delta]$ to the unique solution x of (CP) with the following error estimate:*

(4) $\|x^p - x\|_m^{a+\delta} \leq M\rho^{p-1}\dfrac{\delta^p}{p!}$, *where ρ, δ are given by (D7) and (D1).*

Proof. Note that the sequence $(x^p)_{p\geq 1}$ verifies:

$$(5)\qquad \begin{aligned} |x_i^{p+1}(t)-x_i(t)| \leq & \int_a^t \sum_{j=1}^{i-1} L_i^j |x_j^{p+1}(s)-x_j(s)| + \\ & + \sum_{j=i}^{n} L_i^j |x_j^p(s)-x_j(s)| ds \ (p>0) \end{aligned}$$

Indeed:

$$|x_i^{p+1}(t)-x_i(t)| =$$
$$= \left|\int_a^t f_i(s,x_1^{p+1}(s),...,x_{i-1}^{p+1}(s),x_i^p(s)...,x_n^p(s))ds - \int_a^t f_i(s,x_1(s),...,x_n(s))ds\right| \leq$$
$$\leq \int_a^t |f_i(s,x_1^{p+1}(s),...,x_{i-1}^{p+1}(s),x_i^p(s)...,x_n^p(s)) - f_i(s,x_1(s),...,x_n(s))|ds,$$

wherefrom, with (H1), (5) follows.
The analogue to (5) evaluation for $p=0$ is:

$$(6)\qquad \begin{aligned} |x_i^1(t)-x_i(t)| = & \left|\int_a^t f_i(s,x_1(s),...,x_n(s))ds\right| \leq \\ \leq & \int_a^t |f_i(s,x_1(s),...,x_n(s))|ds \leq M_i(t-a). \end{aligned}$$

One can see that:

$$(7)\qquad \begin{aligned} |x_i^2(t)-x_i(t)| \leq & \int_a^t \sum_{j=1}^{i-1} L_i^j |x_j^2(s)-x_j(s)| + \sum_{j=i}^{n} L_i^j |x_j^1(s)-x_j(s)| ds \leq \\ \leq & \int_a^t \sum_{j=1}^{i-1} L_i^j \|x_j^2-x_j\|_t + \sum_{j=i}^{n} L_i^j |x_j^1(s)-x_j(s)| ds \leq \\ \leq & \int_a^t \sum_{j=1}^{i-1} L_i^j \|x_j^2-x_j\|_t ds + \sum_{j=i}^{n} L_i^j M_i(s-a) ds. \end{aligned}$$

Inequality (7) can be written as:

$$|x_i^2(t)-x_i(t)| \leq \int_a^t \sum_{j=1}^{i-1} L_i^j \|x_j^2-x_j\|_t ds + M\beta_i \frac{(t-a)^2}{2!},$$
$$(\forall)t\in[a,a+\delta],\ i=\overline{1,n}.$$

Using the inequality $\|x_i^2-x_i\|_t \leq \|x_i^2-x_i\|_m^t$, $i=\overline{1,n}$, we obtain:

$$|x_i^2(t)-x_i(t)| \leq \alpha_i \|x^2-x\|_m^t \int_a^t dt + M\beta_i \frac{(t-a)^2}{2!} =$$
$$= \alpha_i \|x^2-x\|_m^t (t-a) + M\beta_i \frac{(t-a)^2}{2!},\ i=\overline{1,n}.$$

Since $t-a\leq\delta$ it follows: $|x_i^2(t)-x_i(t)| \leq \alpha_i\|x^2-x\|_m^t\delta + M\beta_i \frac{(t-a)^2}{2!}$, $i=\overline{1,n}$. For $s\leq t$ we have:

$$|x_i^2(s)-x_i(s)| \leq \alpha_i\|x^2-x\|_m^s\delta + M\beta_i \frac{(t-a)^2}{2!} \leq$$
$$\leq \alpha_i\|x^2-x\|_m^t\delta + M\beta_i \frac{(t-a)^2}{2!},\ i=\overline{1,n}.$$

Taking the supremum over $s\in[a,t]$ in the left hand side we obtain:

$$\|x_i^2(t)-x_i(t)\|_t \leq \alpha_i\|x^2-x\|_m^t\delta + M\beta_i \frac{(t-a)^2}{2!},\ i=\overline{1,n}.$$

Taking the maximum in the left hand side too, it results:

$$(8)\qquad \|x^2-x\|_m^t \leq \alpha_{s_2}\delta\|x^2-x\|_m^t + \beta_{s_2}M\frac{(t-a)^2}{2!}.$$

(Here s_2 is an element from $\{1,...,n\}$ with $\|x^2-x\|_m^t = \|x_{s_2}^2-x_{s_2}\|_t$. The index that appears at s shows that it depends on the respective set.) We continue by inductively proving the relation:

$$(9)\qquad \|x^{p+1}-x\|_m^t \leq \frac{M\beta_{s_1}...\beta_{s_p}}{(1-\alpha_{s_1}\delta)...(1-\alpha_{s_p}\delta)}\frac{(t-a)^{p+1}}{(p+1)!}.$$

For $p=1$, the conclusion is immediate because of (8). Suppose (9) holds for p; we will check it for $p+1$.

Rewriting (5) and using $|x_j^q(s)-x_j(s)| \leq \|x_j^q-x_i\|_s$ for $s\leq t$ we have:

$$|x_i^{p+1}(t)-x_i(t)| \leq \int_a^t \sum_{j=1}^{i-1} L_i^j\|x_j^{p+1}-x_j\|_t + \sum_{j=i}^{n} L_i^j\|x_j^p-x_j\|_s ds =$$
$$= (t-a)\sum_{j=1}^{i-1} L_i^j\|x_j^{p+1}-x_j\|_t +$$
$$+ \int_a^t \frac{M\beta_{s_1}...\beta_{s_{p-1}}}{(1-\alpha_{s_1}\delta)...(1-\alpha_{s_{p-1}}\delta)}\frac{(s-a)^{p+1}}{p!}\sum_{j=1}^{n} L_i^j \leq$$
$$\leq (t-a)\|x^{p+1}-x\|_m^t \sum_{j=1}^{i-1} L_i^j + \frac{M\beta_{s_1}...\beta_{s_{p-1}}}{(1-\alpha_{s_1}\delta)...(1-\alpha_{s_{p-1}}\delta)}\frac{(t-a)^{p+1}}{(p+1)!}\beta_i$$

Changing t with u one obtains:

$$|x_i^{p+1}(u) - x_i(u)| \le \alpha_i(u-a)\|x^{p+1} - x\|_m^u + \tag{10}$$
$$+\beta_i \frac{(u-a)^{p+1}}{(p+1)!} \frac{M\beta_{s_1}\dots\beta_{s_{p-1}}}{(1-\alpha_{s_1}\delta)\dots(1-\alpha_{s_{p-1}}\delta)},$$
$$(\forall) u \in [a, a+\delta].$$

The right side is an increasing function of u so, for $u \in [a,t]$, we have:

$$|x_i^{p+1}(u) - x_i(u)| \le (t-a)\|x^{p+1} - x\|_m^t \alpha_i + \tag{11}$$
$$+\frac{(t-a)^{p+1}}{(p+1)!} \frac{M\beta_{s_1}\dots\beta_{s_{p-1}}}{(1-\alpha_{s_1}\delta)\dots(1-\alpha_{s_{p-1}}\delta)} \beta_i.$$

Passing to the supremum in the left hand side over $u \in [a,t]$ and then to maximum over $i \in \{1, 2, ..., n\}$ one has (denoting by s_p that index for which it is attained):

$$\|x^{p+1} - x\|_m^t \le (t-a)\|x^{p+1} - x\|_m^t \alpha_{s_p} + \tag{12}$$
$$+\frac{(t-a)^{p+1}}{(p+1)!} \frac{M\beta_{s_1}\dots\beta_{s_{p-1}}}{(1-\alpha_{s_1}\delta)\dots(1-\alpha_{s_{p-1}}\delta)} \beta_{s_p}.$$

It suffices now to replace $(t-a)$ from first term of the right hand side with δ to get the desired relation.
From (9) one obtains:

$$\|x^p - x\|_m^t \le M\rho^{p-1} \frac{\delta^p}{p!}, \ (\forall) p \ge 1, \ t \in [a, a+\delta].$$

For $t = a+\delta$ we obtain (4).
Since

$$M\rho^{p-1} \frac{\delta^p}{p!} = \frac{M}{\rho} \frac{(\rho\delta)^p}{p!} \to 0 \quad \text{as } p \to \infty,$$

the uniform convergence of x^p to x follows by the Weierstrass theorem.

Remark. The standard succesive approximations method gives an error estimate similar to (12) where the second member is replaced by $ML^{p-1}\frac{\delta^p}{p!}$. We will prove in the following that $\rho \le L$ wherefrom we can conclude the method above gives a better theoretical estimation for the error than the standard one. Indeed, (D7) implies $(\exists) i_0 \in \{i, ..., n\}$ such that $\rho = \dfrac{\beta_{i_0}}{1-\alpha_{i_0}\delta}$.

Inequality $\dfrac{\beta_{i_0}}{1-\alpha_{i_0}\delta} \le \alpha_{i_0} + \beta_{i_0}$ can be written equivalently as:

$$\beta_{i_0} \le \alpha_{i_0} + \beta_{i_0} - (\alpha_{i_0} + \beta_{i_0})\alpha_{i_0}\delta \iff (\alpha_{i_0} + \beta_{i_0})\delta \le 1.$$

But this is true because $\alpha_{i_0} + \beta_{i_0} \le L$. It follows that $\rho \le \alpha_{i_0} + \beta_{i_0} \le L$. Note that there are examples where $\rho < L$.

2. Main results. The facts above lead us to an error estimate like:

$$\|x_i^p - x_i\|_t \le \frac{(t-a)^p}{p!} h_i^p(t), \quad i = \overline{1,n}, \ p > 0. \tag{13}$$

where $(h_1^p, ..., h_n^p)$ are positive increasing functions over $[a, a+\delta]$.
Let us denote $\delta_k(t) = \dfrac{(t-a)^k}{k!}$. We will prove by induction that (13) is true and moreover that $(h_1^p, ..., h_n^p)$ are polinomials in $(t-a)$ having positive coeficients. We have already seen that:

$$\|x_i^1 - x_i\|_t \le M_i(t-a), \quad i = \overline{1,n} \quad \text{(see (7))}$$

and also, by (5) and (8),

$$|x_i^{p+1}(t) - x_i(t)| \le \int_a^t \left[\sum_{j=1}^{i-1} L_i^j |x_j^{p+1}(s) - x_j(s)| + \sum_{j=i}^{n} L_i^j |x_j^p(s) - x_j(s)|\right] ds$$
$$(p > 0).$$

Let us now suppose that (13) is true for $p \le p_0$, $i = \overline{1,n}$ and for $p = p_0 + 1$, till $i-1$. Then:
(14)

$$|x_i^{p+1}(t) - x_i(t)| \le \int_a^t \left[\sum_{j=1}^{i-1} L_i^j \|x_j^{p+1} - x_j\|_s + \sum_{j=i}^{n} L_i^j \|x_j^p - x_j\|\right] ds \le$$
$$\le \int_a^t \left[\sum_{j=1}^{i-1} L_i^j \delta_{p+1}(s) h_j^{p+1}(s) + \sum_{j=i}^{n} L_i^j \delta_p(s) h_j^p(s)\right] ds.$$

By hypothesis h_i^p are polinomials in $(t-a)$ with positive coeficients hence they are in particular increasing functions. From (14) we get:

$$
\begin{aligned}
&|x_i^{p+1}(t) - x_i(t)| \le \\
&\le \int_a^t \left[\sum_{j=1}^{i-1} L_i^j \delta_{p+1}(s) h_j^{p+1}(s) + \sum_{j=i}^{n} L_i^j \delta_p(s) h_j^p(s) \right] ds = \\
&= \sum_{j=1}^{i-1} L_i^j \delta_{p+2}(t) h_j^{p+1}(t) + \sum_{j=i}^{n} L_i^j \delta_{p+1}(t) h_j^p(t)
\end{aligned}
\tag{15}
$$

$$(\text{we used } \int_a^t \delta_k(s)ds = \delta_{k+1}(t)).$$

The functions in the second member of (15) are all increasing in t; it then follows that, for each $t \in [a, u]$:

$$|x_i^{p+1}(t) - x_i(t)| \le \sum_{j=1}^{i-1} L_i^j \delta_{p+2}(u) h_j^{p+1}(u) + \sum_{j=i}^{n} L_i^j \delta_{p+1}(u) h_j^p(u),$$
$$i = \overline{1, n},\ p > 0.$$

Passing to the supremum over $t \in [a, u]$ one finds:

$$\|x_i^{p+1} - x_i\|_u \le \delta_{k+1}(u) \left(\sum_{j=1}^{i-1} L_i^j \frac{\delta_{p+2}(u)}{\delta_{p+1}(u)} h_j^{p+1}(u) + \sum_{j=i}^{n} L_i^j h_j^p(u) \right). \tag{16}$$

But $\dfrac{\delta_{p+2}(u)}{\delta_{p+1}(u)} = \dfrac{u-a}{p+2}$ and, putting:

$$h_i^{p+1} = \sum_{j=1}^{i-1} L_i^j \frac{\delta_{p+2}}{\delta_{p+1}} h_j^{p+1} + \sum_{j=i}^{n} L_i^j h_j^p \tag{17}$$

we obtain the desired relation.

We have two major cases to analyse:

A) The first case is concerned with the norm $\|\cdot\|_m^{a+\delta}$. Denote:

(D8) $v^k(t) = \max\limits_{i=\overline{1,n}} h_i^k(t)$.

(D9) $\rho_k = \max\limits_{s=\overline{1,n}} \dfrac{\delta_k(a+\delta)\beta_s}{\delta_k(a+\delta) - \alpha_s \delta_{k+1}(a+\delta)}$.

Note that $\rho_k \le L$ (in fact $\rho_k \le \alpha_r + \beta_r$ where r is that index for which the equality in (D9) is assured).

Let us now construct a sequence $(y^p = (y_1^p, \dots, y_n^p) : [a, a+\delta] \to R^n)_{p \ge 1}$ of "succesive aproximations" of x by

(D10) $y^1(t) = \xi;\ y^p(t) = \xi + \displaystyle\int_a^t f(t, y^{p-1}(s))ds$ for $p < [\delta L] = p_0$,

and for $p \ge p_0$, let y^p be deduced from y^{p-1} by the method above (given by (D2)). (Here $[\cdot]$ is the integer part function).

Remark. For $k \le p_0$, $h_i^k = L^{k-1}M$ and from p_0 one can use (17). The proof is analogue to the case $p_0 = 1$ so we omit it.

Proposition 2. *The sequence $(y^p)_{p>0}$ defined as above converge uniformly over $[a, a+\delta]$ to the unique solution x of (CP). Moreover, for $p > p_0$, the following error estimate holds:*

$$\|y^p - x\|_m^{a+\delta} \le \frac{\delta^p}{p!} \rho_p \rho_{p-1} \dots \rho_{p_0} L^{p_0 - 2} M. \tag{18}$$

Proof. It follows from (17) that:

$$
\begin{aligned}
h_i^{p+1}(t) &\le \sum_{j=1}^{i-1} L_i^j \frac{\delta_{p+2}(t)}{\delta_{p+1}(t)} v_j^{p+1}(t) + \sum_{j=i}^{n} L_i^j v_j^p(t) = \\
&= v^{p+1}(t) \frac{\delta_{p+2}(t)}{\delta_{p+1}(t)} \alpha_i + v^p(t)\beta_i,\ i = \overline{1, n}.
\end{aligned}
\tag{19}
$$

Let $i = i_0$ be that index which satisfies $v^{p+1}(t) = h_{i_0}^{p+1}$. We have:

$$v^{p+1}(t) \le v^{p+1}(t) \frac{\delta_{p+2}(t)}{\delta_{p+1}(t)} \alpha_{i_0} + v^p(t)\beta_{i_0},$$

wherefrom, by elementary transformations,

$$v^{p+1}(t)(\delta_{p+1}(t) - \alpha_{i_0}\delta_{p+2}(t)) \le \delta_{p+1}(t) v^p(t) \beta_{i_0}. \tag{20}$$

If $p + 2 > \delta L$ then (in view of $L \ge \alpha_i + \beta_i$, $i = \overline{1, n}$) $\delta_{p+1}(t) \ge \alpha_i \delta_{p+2}(t)$ and (20) can be written as:

$$v^{p+1}(t) \le \frac{\delta_{p+1}(t)\beta_{i_0}}{\delta_{p+1}(t) - \alpha_{i_0}\delta_{p+2}(t)} v^p(t). \tag{21}$$

Relation (21) shows that for $p \geq [\delta L] - 1$ we can use the formula:

$$v^{p+1}(t) \leq \rho_{p+1} v^p(t), \quad (\forall)\, t \in [a, a+\delta],$$

hence for $t = a + \delta$ we have:

$$\|y^p - x\|_m^{a+\delta} = \delta_p(a+\delta) v^p(a+\delta) \leq \delta_p(a+\delta)\rho_p v^p(a+\delta)$$

and after iteration:

$$\|y^p - x\|_m^{a+\delta} \leq \delta_p(a+\delta)\rho_p\rho_{p-1}\dots\rho_{p_0} v^{p_0-1} \leq \delta_p(a+\delta)\rho_p\rho_{p-1}\dots\rho_{p_0} L^{p_0-2} M$$

which is exactly (18).

Remarks.

1. One can easily see that $\rho_k \leq \rho_{k-1} \leq \dots \leq \rho_{p_0+1} \leq \rho_{p_0} \leq L$. Moreover, if (H2) is satisfied then $\rho_k \leq \rho \ \ (\forall)\ k > 0$;hence the error estimate is sharper than that one of Proposition 1. The only difficulty appears at the calculation of ρ_k for each k.
2. For a given s we have the relation:

$$\text{(22)} \qquad \frac{\delta_k(a+\delta)\beta_s}{\delta_k(a+\delta) - \alpha_s\delta_{k+1}(a+\delta)} \longrightarrow \beta_s \ \text{ as } k \to \infty$$

hence $\{\rho_k \mid k \in N\}$ has all its cluster points smaller than $\beta = \max\limits_{s=\overline{1,n}} \beta_s$.

This shows that it is recommended to carry out the approximation process in such a way that β to be minimum, so it is not necessarily to start with x_1^{p+1}, x_2^{p+1}, etc. but in that order $x_{\sigma(1)}^{p+1}, \dots, x_{\sigma(n)}^{p+1}$ for which $\max\limits_{i=\overline{1,n}} \sum\limits_{j=i}^{n} L_{\sigma(i)}^j$ has the minimum value. For a given matrix $(L_i^j)_{1\leq i,j\leq n}$, the appropriate substitution is obtained as follows: one calculates the sum of elements on each line and choose as a first line the one with the minimum sum; then one calculates the sum of elements from each of the $n-1$ remaining lines and puts as the second line that with the minimum sum... etc.

B) Now we consider the norm:

$$\text{(D11)} \qquad \|(x_1, \dots, x_n)\|_1 = |x_1| + \dots + |x_n|.$$

We are trying to obtain a result similar to Proposition 2. In order to obtain a (25)-type relation, (see below) one may start as follows. We have already obtained (relation (17)):

$$\delta_{p+1}(t)h_i^{p+1} = \delta_{p+2}(t)\sum_{j=1}^{i-1} L_i^j h_j^{p+1} + \delta_{p+1}(t)\sum_{j=i}^{n} L_i^j h_j^p$$

hence, by summing after i,

$$\sum_{j=1}^{n} \delta_{p+1}(t)h_i^{p+1} = \sum_{j=1}^{n}\left\{\delta_{p+2}(t)\sum_{j=1}^{i-1} L_i^j h_j^{p+1} + \delta_{p+1}(t)\sum_{j=i}^{n} L_i^j h_j^p\right\}.$$

Changing the summation order in the second member yields:

$$\text{(23)} \qquad \sum_{i=1}^{n} \delta_{p+1}(t)h_i^{p+1}(t) = \sum_{j=1}^{n-1} h_j^{p+1}(t)\delta_{p+2}(t)\sum_{i=j+1}^{n} L_i^j + \\ + \sum_{j=1}^{n} h_j^p(t)\delta_{p+1}(t)\sum_{i=1}^{j} L_i^j.$$

Denote:

$$\text{(D12)} \quad s_j = \sum_{i=j+1}^{n} L_i^j,\ t_j = \sum_{i=1}^{j} L_i^j,\ \text{for } j < n, \text{ and } s_n = 0,\ t_n = \sum_{i=1}^{n} L_i^n.$$

Relation (23) can be rewritten as:

$$\sum_{i=1}^{n} \delta_{p+1}(t)h_i^{p+1}(t) = \sum_{j=1}^{n-1} h_j^{p+1}(t)\delta_{p+2}(t)s_j + \sum_{j=1}^{n} h_j^p(t)\delta_{p+1}(t)t_j$$

or, since $s_n = 0$,

$$\sum_{i=1}^{n} h_i^{p+1}(t)[\delta_{p+1}(t) - \delta_{p+2}(t)s_i] = \sum_{i=1}^{n} h_i^p(t)\delta_{p+1}(t)t_i.$$

We impose the condition

$$\delta_{p+1}(a+\delta) - \delta_{p+2}(a+\delta)s_i > 0 \ \ (\Longleftrightarrow\ p+2 > \delta s_i)\ i = \overline{1,n}.$$

Let us take $p > t_0 - 1 = [\delta s] - 1$ where

(D13) $s = \max\limits_{i=\overline{1,n}} s_i$.

Under these circumstances:

$$(24) \quad \sum_{i=1}^{n} h_i^{p+1}(t)[\delta_{p+1}(t) - \delta_{p+2}(t)s_i] \geq \sum_{i=1}^{n} h_i^{p+1}(t)[\delta_{p+1}(t) - \delta_{p+2}(t)s].$$

Denote further:

$$(D14) \quad \eta_k = \max_{j=\overline{1,n}} \frac{\delta_k(a+\delta)t_j}{\delta_{k-1}(a+\delta) - s_j\delta_k(a+\delta)},\ T = \max_{j=\overline{1,n}} (s_j + t_j).$$

Relation (24) yields $[\delta_{p+1}(t) - \delta_{p+2}(t)s] \sum\limits_{i=1}^{n} h_i^{p+1}(t) \leq \sum\limits_{i=1}^{n} h_i^{p}(t)\delta_{p+1}(t)t_i$.

Hence

$$(25) \quad \sum_{i=1}^{n} h_i^{p+1}(t) \leq \sum_{i=1}^{n} \frac{\delta_{p+1}(t)t_i}{\delta_{p+1}(t) - s\delta_{p+2}(t)} h_i^{p}(t).$$

If we had passed from step p to step $p+1$ by the classical method we would have obtained:

$$\delta_{p+1}(t) \textstyle\sum_{i=1}^{n} h_i^{p+1}(t) \leq \displaystyle\sum_{i=1}^{n} h_i^{p}(t)\delta_p(t)(t_i + s_i)$$

$$\text{(that is } \sum_{i=1}^{n} h_i^{p+1}(t) \leq \sum_{i=1}^{n} \frac{\delta_p(t)(t_i+s_i)}{\delta_{p+1}(t)} h_i^{p}(t)).$$

We prove that for sufficiently large p:

$$(26) \quad \frac{\delta_{p+1}(a+\delta)t_i}{\delta_{p+1}(a+\delta) - s\delta_{p+2}(a+\delta)} \leq \frac{\delta_p(a+\delta)(t_i+s_i)}{\delta_{p+1}(a+\delta)}.$$

Indeed, let us notice:

$$(27) \quad \frac{1}{1 - s\dfrac{\delta_{p+2}(a+\delta)}{\delta_{p+1}(a+\delta)}} \leq \frac{t_i+s_i}{t_i} \frac{\delta_p(a+\delta)}{\delta_{p+1}(a+\delta)}.$$

But $\dfrac{t_i+s_i}{t_i} \dfrac{\delta_p(a+\delta)}{\delta_{p+1}(a+\delta)} \geq \dfrac{\delta_p(a+\delta)}{\delta_{p+1}(a+\delta)} \cdot 1 = \dfrac{p+1}{\delta}$ and the condition can be rewritten as

$$\frac{p+1}{\delta} \geq \frac{p+2}{p+2-s\delta}.$$

This is true whenever $p \geq \lambda$, λ being the greatest solution of the equation $(X+1)(X+2-s\delta) = (X+2)\delta$.

Now we construct the sequence $(z^p = (z_1^p, \ldots, z_n^p) : [a, a+\delta] \to R^n)_{p\geq 1}$ by:

$$(D15) \quad z^1 = \xi;\ z^p(t) = \xi + \int_a^t f(t, z^{p-1}(s))ds \text{ for } p < q_0 = \max\{[\lambda]+1, t_0\},$$

and for $p \geq q_0$, let z^p be obtained from z^{p-1} by the formulae (D2)).

Because of the particularities of the norm $\|\cdot\|_1$ it is rather difficult to "measure" the improvement brought by the method above. One can only say that (by (26)) there is an improvement (that is z^p converges faster to solution than in the classical situation). We can therefore state:

Proposition 3. *The sequence $(z^p)_{p>0}$ defined as above converges uniformly over $[a, a+\delta]$ to the unique solution x of (CP). Moreover, for $p \geq q_0$, the following error estimate holds:*

$$\|(\|z_i^p - x_i\|_{a+\delta})_{k=1}^n\|_1 \leq$$

$$(28) \quad \leq \frac{1}{1 - s\dfrac{\delta_{p+1}(a+\delta)}{\delta_p(a+\delta)}} \eta_p\eta_{p-1}\ldots\eta_{q_0} T^{q_0-2} M_0 \delta_{q_0}(a+\delta)$$

where $M_0 = \sum\limits_{i=1}^{n} M_i$.

Proof. We have already obtained that for (D2)–type recurrences takes place:

(29)

$$\sum_{i=1}^{n} h_i^k(a+\delta)[\delta_k(a+\delta) - \delta_{k+1}(a+\delta)s_i] \leq$$

$$\leq \sum_{i=1}^{n} \frac{\delta_k(a+\delta)t_i}{\delta_{k-1}(a+\delta) - s_i\delta_k(a+\delta)} h_i^{k-1}(a+\delta)[\delta_{k-1}(a+\delta) - \delta_k(a+\delta)s_i] \leq$$

$$\leq \eta_k \sum_{i=1}^{n} [\delta_{k-1}(a+\delta) - \delta_k(a+\delta)s_i] h_i^{k-1}(a+\delta).$$

Iterating (29) down to q_0 one has:

$$\sum_{i=1}^{n} h_i^k(a+\delta)[\delta_k(a+\delta) - \delta_{k+1}(a+\delta)s_i] \leq$$

$$\leq \eta_k \ldots \eta_{q_0} \sum_{i=1}^{n} [\delta_{q_0-1}(a+\delta) - \delta_{q_0}(a+\delta)s_i] h_i^{q_0-1}(a+\delta) \leq$$

$$\leq \eta_k \ldots \eta_{q_0} \sum_{i=1}^{n} \delta_{q_0-1}(a+\delta) h_i^{q_0-1}(a+\delta) \leq \eta_k \ldots \eta_{q_0} T^{q_0-2} M_0 \delta_{q_0-1}(a+\delta)$$

(because usual estimates, that is the part till q_0, leads us to

$$\delta_{q_0-1}(a+\delta)h_i^{q_0-1}(a+\delta) \le T^{q_0-2}M_0\delta_{q_0-1}(a+\delta)).$$

Hence:

$$\sum_{i=1}^{n} h_i^k(a+\delta)[\delta_k(a+\delta) - \delta_{k+1}(a+\delta)s_i] \le \eta_k \ldots \eta_{q_0} T^{q_0-2} M_0 \delta_{q_0-1}(a+\delta).$$

It suffices now taking (24) into account to end the argument.

3. Other results. We are now in position to give a result connected to the one before. Denote

(D16) $\theta = \max\{s_j, t_j \mid j = \overline{1,n}\}$.

Proposition 4. *If $\delta < T^{-1}$, then the sequence $(x^p)_{p>0}$ converges uniformly over $[a, a+\delta]$ to the unique solution x of (CP). Moreover the following error estimate holds:*

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le \frac{\theta^p}{(1-\theta\delta)^p} M_0 \delta_{p+1}(t). \tag{30}$$

Proof. With the previous norm and using also the evaluation:

$$\|x_i^{p+1}(t) - x_i(t)\|_t \le \int_a^t \sum_{j=1}^{i-1} L_i^j \|x_j^{p+1} - x_j\|_t + \sum_{j=i}^{n} L_i^j \|x_j^p - x_j\|_s ds,$$
$$(\forall)\ t \in [a, a+\delta],\ i = \overline{1,n}$$

one obtains, by summing all the n inequalities of that kind:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le$$
$$\le \int_a^t \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} L_i^j \|x_j^{p+1} - x_j\|_t + \sum_{j=i}^{n} L_i^j \|x_j^p - x_j\|_s\right] ds.$$

Hence, by changing the order of summation in the right hand side:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le$$
$$\le \int_a^t \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} L_i^j \|x_j^{p+1} - x_j\|_t + \sum_{j=1}^{n} \sum_{i=1}^{j} L_i^j \|x_j^p - x_j\|_s ds, \tag{31}$$

which can also be rewritten as:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le$$
$$\le \int_a^t \left[\sum_{j=1}^{n-1} s_j \|x_j^{p+1} - x_j\|_t + \sum_{j=1}^{n} t_j \|x_j^p - x_j\|_s\right] ds,\ (\forall)\ p > 0. \tag{32}$$

For $p = 0$ one obtains:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 = \sum_{i=1}^{n} \|x_i^1 - x_i\|_t \le \sum_{i=1}^{n} M_i \frac{(t-a)^1}{1!}. \tag{33}$$

By (32) and $s_n = 0$ it follows that:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le \theta\left[\sum_{j=1}^{n}(t-a)\|x_j^{p+1} - x_j\|_t + \int_a^t \sum_{j=1}^{n} \|x_j^p - x_j\|_s ds\right].$$

Replacing $t-a$ by δ we get:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le \theta\delta\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 + \int_a^t \|(\|x_i^p - x_i\|_s)_{k=1}^n\|_1 ds$$

hence:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le \theta[1-\theta\delta]^{-1} \int_a^t \|(\|x_i^p - x_i\|_s)_{k=1}^n\|_1 ds. \tag{34}$$

We will prove the conclusion by induction. For $p = 0$ conclusion is clear, via (33). Suppose that it is true for p. Then, by using (34), one gets:

$$\|(\|x_i^{p+1} - x_i\|_t)_{k=1}^n\|_1 \le \theta[1-\theta\delta]^{-1} \int_a^t \theta^{p-1}[1-\theta\delta]^{1-p} M_0 \delta_p(s) ds =$$
$$= \theta^p[1-\theta\delta]^{-p} M_0 \delta_{p+1}(t).$$

Hence (30) holds, and this assures the uniform convergence of x^p to x.

Remark. If we had followed the usual way the right hand side of (30) would have been $M_0 T^p \delta_{p+1}(t)$. To obtain a better estimate one needs $\frac{\theta}{1-\theta\delta} < T$, or equivalently $\theta < T - T\theta\delta$, that is:

(H3) $\theta < \dfrac{T}{1+\delta T}$.

By our previous considerations $\theta \le T$. If $\theta = T$ then (H3) cannot take place for any $\delta > 0$, so the estimate in the right hand side is not better. When $\theta < T$, by (H3) one has:

$$\delta < \frac{T-\theta}{T\theta} = \frac{1}{\theta} - \frac{1}{T}. \tag{35}$$

That is the estimate is better if δ satisfies (35).

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Faculty of Mathematics
"Al.I.Cuza" University
11, Copou Boulevard
6600 Iaşi
Romania

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
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FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY IN HILBERT SPACE

BY

DAN-ANDREI GEBA

0. Introduction. The aim of this paper is to prove the existence of a solution for a certain class of differential equations with delay in Hilbert spaces, using the theory of nonlinear semigroups in Hilbert spaces.

This work is structured in two sections. In the first one the main result is stated and proved. Section 2 presents some examples to partial differential equations. For other recent related results we refer to [1], [2].

1. Main result. Let us consider the following problem:

$$\begin{cases} \dot{x}(t) \in A_0x(t) + A_1x(t-h) + A_2x(t-h) + f(t), \ (\forall)t \ge 0, \\ x(0) = x_0 \in H, \\ x(s) = x^0(s), \ (\forall) - h \le s < 0, x^0 \in L^2(-h,0;H), \end{cases} \tag{1.1}$$

in which $h > 0, H$ denotes a real Hilbert space with the norm $\|\cdot\|$ and the scalar product $(\cdot,\cdot)$ and A_0, A_1, A_2 satisfy the next four conditions:

(1.2) A_0 is a nonlinear, m-dissipative set in $H \times H$ and $0 \in A_0(0)$.

(1.3)
$A_1 : \mathcal{D}(A_1) \subset H \longrightarrow H$ is a linear closed densely defined operator and $\|A_1y\| \le M\|A_1^*y\| + C\|y\|$, $(\forall)y \in \mathcal{D}(A_1^*)$.

(1.4) $A_2 : H \longrightarrow H$ is a Lipschitz continuous function,
$\|A_2y - A_2z\| \le L\|y - z\|$, $(\forall)y, z \in H$.

$$\|A_1^*(x_0 - y_0)\|^2 \le -\mu_1(u_0 - v_0, x_0 - y_0) + \mu_2\|x_0 - y_0\|^2, \tag{1.5}$$
$(\forall)[x_0, u_0], [y_0, v_0] \in A_0$, with $0 \le \mu_1 \le 1$ and $\mu_2 \in \boldsymbol{R}$.