ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f1.

TWO BODY PROBLEM IN A NON–INERTIAL FRAME OF REFERENCE AND THE THEORY OF RELATIVITY

BY

C.I. BORŞ

Long ago, in a short paper [1] the two body problem in a non–inertial frame of reference was studied and some aproach to the theory of relativity was made.

In order to go deeper into this problem let us consider two particles S and P (for example the Sun and a planet) which are free to move under the influence of mutual forces of attraction according to Newton's law.

The center of gravity G moves, with respect to an inertial frame of reference, uniformly in a straight line. There is an inertial frame of reference S_G whose origin is G, and the trajectories of the particles will always remain in a fixed plane of S_G. The astronomic observations point out that the inertiality of a frame of reference defined by *fixed stars* is a fiction.

Let us consider a frame $Sx_1x_2x_3$ with its origin S and the plane Sx_1x_2 in conicidence with the plane of the trajectories of particles.

The rotation of this frame, with respect to S_G, is of the form

$$\overline{w} = \omega \overline{e}_3 \tag{1}$$

where $\overline{e}_3$ is the unit vector of Sx_3, fixed in S_G. The force acting on the particle P in its motion around S, in $Sx_1x_2x_3$, is

$$\begin{aligned}\overline{F} = & -f\,\frac{m_2(m_1+m_2)}{r^2}\,\overline{u} + m_2\left[\frac{C^2}{r^3} - \frac{v^2}{r} + \frac{\dot{r}^2}{r} + \frac{(\dot{\omega} r + 2\omega\dot{r})\dot{r}}{\frac{C}{r} - \omega r}\right]\overline{u} - \\ & - m_2\,\frac{\dot{\omega} r + 2\omega \dot{r}}{\frac{C}{r} - \omega r}\,\overline{v}, \qquad \overline{v} = \dot{r}\overline{u} + r\dot{\theta}\overline{n},\end{aligned} \tag{2}$$

where we used the same notations as in [1]. In the theory of relativity the particle P will move along a geodesic in a four-dimensional space–time.

The trajectory is determined from a variational problem which generalizes the principle of Maupertuis.

If the geometry of the physical world is given by the metric of Schwarzschild

$$ds^2 = \left(V^2 - \frac{2\mu}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2\mu}{V^2 r}} - r^2(d\theta^2 + \cos^2\theta d\varphi^2),$$

then the force acting on the particle P, in Newtonian correspondence, is given by [4]

$$(3)\quad \overline{F} = -\frac{\mu}{r^2}\overline{u} + \frac{\mu}{r^2}\left[\frac{2\mu}{V^2 r} - \frac{2v^2}{V^2} + \frac{2\dot{r}^2}{V^2} + \frac{\dot{r}^2}{V^2 - \frac{2\mu}{r}}\right]\overline{u} - \frac{2\mu\dot{r}}{r^2(V^2 - \frac{2\mu}{r})}\overline{v}.$$

Firstly, we must observe some similarity between (2) and (3). Secondly, in both cases, in addition to the Newtonian force, there are some complementary terms.

Now, our central problem is to find a geometry of physical world so that the force acting on P, in the Newtonian correspondence, coincides with (2). With that end in view, let us consider the metric [3]

$$ds^2 = B\left(H - \frac{\mu}{r}\right)\left[V^2 dt^2 - dr^2 - r^2(d\theta + \omega dt)^2 - r^2\cos^2\theta d\varphi^2)\right],$$

where B, H, μ, ω are constants, ω being in connection with (1).
The geodesics of this linear element are given by the variational problem

$$(4)\quad \delta\int\sqrt{\gamma}\, dt = 0,\ \gamma = B\Big(H - \frac{\mu}{r}\Big)[V^2 - \dot{r}^2 - r^2(\dot{\theta}+\omega)^2 - r^2\cos^2\theta\dot{\varphi}^2)].$$

Since the function γ does not depend on φ, from the Euler–Lagrange equations, we get

$$(5)\quad \frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right) r^2\cos^2\theta\dot{\varphi} = M,$$

where M is a constant.

Following [4], [5], let us consider the relations between the cartesian coordinates x_1, x_2, x_3 and the spherical coordinates r, θ, φ, of the particle P, given by $x_1 = r\cos\theta\cos\varphi$, $x_2 = r\cos\theta\sin\varphi$, $x_3 = r\sin\theta$.

We obtain $\overline{r} = x_i\overline{e}_i = r\overline{\rho}$, $\overline{v} = \dot{x}_i\overline{e}_i = \dot{r}\overline{\rho} + r\dot{\theta}\overline{\eta} + r\cos\theta\dot{\varphi}\overline{\varepsilon}$; where

$$\overline{\rho} = \cos\theta(\cos\varphi\overline{e}_1 + \sin\varphi\overline{e}_2) + \sin\theta\overline{e}_3,$$
$$\overline{\eta} = -\sin\theta(\cos\varphi\overline{e}_1 + \sin\varphi\overline{e}_2) + \cos\theta\overline{e}_3,$$
$$\overline{\varepsilon} = -\sin\varphi\overline{e}_1 + \cos\varphi\overline{e}_2.$$

The angular momentum of particle P is given by

$$\overline{K} = \overline{r}\times m\overline{v} = -mr^2(\cos\dot{\varphi}\overline{\eta} - \dot{\theta}\overline{\varepsilon}) \text{ and } \overline{K}\overline{e}_3 = -mr^2\cos^2\theta\dot{\varphi}.$$

Let $\overline{r}(t_0) = \overline{r}_0$, $\overline{v}(t_0) = \overline{v}_0$ be the initial data of the motion of particle P. One may take the axis of x_3 such that

$$\overline{K}(t_0)\overline{e}_3 = m(\overline{r}_0, \overline{v}_0, \overline{e}_3) = -mr_0^2\cos^2\theta_0\dot{\varphi}_0 = 0.$$

It results $\dot{\varphi}_0 = \dot{\varphi}(t_0) = 0$; then, the constant M of integral (3) is zero and it follows immediately that $\varphi(t) = const.$

Therefore, the trajectory is situated in the plane determined by the initial data $\overline{r}_0$ and $\overline{v}_0$ ($\overline{r}_0\times\overline{v}_0 \neq \overline{0}$). But in this case $\dot{\varphi} = 0$ and the function γ does not involve θ; evidently, the function γ does not depend on t.

Consequently, from (2) we get also the integrals

$$(6)\quad \frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right) r^2(\dot{\theta} + \omega) = N,$$

$$(7)\quad \sqrt{\gamma} + \frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right)[\dot{r}^2 + r^2(\dot{\theta} + \omega)\dot{\theta}] = A,$$

where N, A are constants.

The equation (7) can be written as

$$(8)\quad \frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right)\left[V^2 - r^2\omega(\dot{\theta} + \omega)\right] = A.$$

From (6) and (8) we have

$$\frac{r^2(\dot{\theta} + \omega)}{V^2 - r^2\omega(\dot{\theta} + \omega)} = \frac{N}{A}$$

and it follows

$$(9)\quad r^2(\dot{\theta} + \omega) = C.$$

Integral (8) is equivalent to

$$(10)\quad \dot{r}^2 + r^2(\dot{\theta} + \omega)^2 = \frac{2\mu}{r} + 2h, \quad B = \frac{2A^2}{(V^2 - C\omega)^2}.$$

Integrals (9), (10) are the integrals of the model studied in [1],[2] and therefore our problem is solved.

Remarks:

- the variational problem (4) is equivalent with

$$\delta\int B\left(H-\frac{\mu}{r}\right)\left[V^2\left(\frac{dt}{ds}\right)^2-\left(\frac{dr}{ds}\right)^2-r^2\left(\frac{d\theta}{ds}+\omega\frac{dt}{ds}\right)^2-\right.$$
$$\left.-r^2\cos^2\theta\left(\frac{d\varphi}{ds}\right)^2\right]ds=0,$$

where the parametrization is natural.
- By a carefull analysis it may arrive to think about an equivalence between the relativity and the Newtonian theory of gravitation (?!).
- The integrals (9) and (10) hold even for ω function of time.
- There seems to be some difficulty in the explanation of the advance of perihelion for a planet by using Schwarzschild's model: the corresponding non–inertial frame of this model has a rotation depending on the motion of planet (?!).

REFERENCES

1. B o r ş, C.I. – *Le problème des deux corps dans un repère non galiléen. L'avance de périhélie. Le mouvement des comètes.* An.şt. Univ. "Al.I.Cuza", Iaşi, ser. nouă, sect.I, Mat., t.XXXIII, f.2 (1987), p.185–188.
2. B o r ş, C.I – *Two body problem in a non-inertial frame of reference and the advance of perihelion.* An.şt. Univ. "Al.I.Cuza", Iaşi, ser. nouă, sect.I, Mat., t.XXXVIII, f.1 (1992), p.23–27
3. B o r ş, C.I. – *Two body problem in a non-inertial frame of reference and the theory of relativity.* Lucrările Seminarului Matematic "Al.Myller", 3/1991.
4. C h a z y, J – *La théorie de la relativité et la mécanique célesie.* Gauthier-Villars, t.I, 1928.
5. C h a z y, J.– *Mécanique céleste.* Presses universitaires de France, Paris, 1953

Received: 16.V.1994

Faculty of Mathematics
University of Iaşi
R–6600 Iaşi,
ROMANIA

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f1.

RELATIVISTIC MODELS FOR TWO BODY PROBLEM

BY

C.I. BORŞ

After Einstein, it is possible to describe gravitational effects either in terms of a flat space with a gravitational field present, or in terms of a curved four dimensional space without a gravitational field. The second alternative brings geometry back to an experimental science for which the problem is to determine the geometry of the physical world [1].

The fundamental assumption of the theory of relativity is that a free particle will move along a geodesic in four–dimensional space–time.

From the multitude of metrics used in order to solve the two body problem we mention the metric of Schwarzschild

$$ds^2=\left(V^2-\frac{2\mu}{r}\right)dt^2-\frac{dr^2}{1-\frac{2\mu}{V^2r}}-r^2(d\theta^2+\cos^2\theta d\varphi^2)$$

and the metric of de Sitter

$$ds^2=\left(V^2-\frac{2\mu}{r}-\frac{r^2}{R^2}\right)dt^2-\frac{dr^2}{1-\frac{2\mu}{V^2r}-\frac{r^2}{R^2}}-r^2(d\theta^2+\sin^2\theta d\varphi^2).$$

In the above formulae V,μ,R are constants which characterize the models, V being the speed of light [4].

In the following we will agree that the selection of allowed Riemannian geometries, from among merely conceivable Riemannian geometries, demands that the motion so obtained will reduce to the Newtonian predictions [6]. For the problem in discussion, we will build up some models.

Let us consider ds^2 defined by

$$ds^2=B\left(H-\frac{\mu}{r}\right)[V^2dt^2-dr^2-r^2(d\theta^2+\cos^2\theta d\varphi^2)], \tag{1}$$