By our previous considerations $\theta \le T$. If $\theta = T$ then (H3) cannot take place for any $\delta > 0$, so the estimate in the right hand side is not better. When $\theta < T$, by (H3) one has:

$$\delta < \frac{T-\theta}{T\theta} = \frac{1}{\theta} - \frac{1}{T}. \tag{35}$$

That is the estimate is better if δ satisfies (35).

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FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY IN HILBERT SPACE

BY

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0. Introduction. The aim of this paper is to prove the existence of a solution for a certain class of differential equations with delay in Hilbert spaces, using the theory of nonlinear semigroups in Hilbert spaces.

This work is structured in two sections. In the first one the main result is stated and proved. Section 2 presents some examples to partial differential equations. For other recent related results we refer to [1], [2].

1. Main result. Let us consider the following problem:

$$\begin{cases} \dot{x}(t) \in A_0x(t) + A_1x(t-h) + A_2x(t-h) + f(t), \ (\forall)t \ge 0, \\ x(0) = x_0 \in H, \\ x(s) = x^0(s), \ (\forall) - h \le s < 0, x^0 \in L^2(-h,0;H), \end{cases} \tag{1.1}$$

in which $h > 0, H$ denotes a real Hilbert space with the norm $\|\cdot\|$ and the scalar product $(\cdot,\cdot)$ and A_0, A_1, A_2 satisfy the next four conditions:

(1.2) A_0 is a nonlinear, m-dissipative set in $H \times H$ and $0 \in A_0(0)$.

(1.3) $A_1: \mathcal{D}(A_1) \subset H \longrightarrow H$ is a linear closed densely defined operator and $\|A_1y\| \le M\|A_1^*y\| + C\|y\|, \ (\forall)y \in \mathcal{D}(A_1^*)$.

(1.4) $A_2: H \longrightarrow H$ is a Lipschitz continuous function, $\|A_2y - A_2z\| \le L\|y-z\|, \ (\forall)y,z \in H$.

(1.5) $\|A_1^*(x_0 - y_0)\|^2 \le -\mu_1(u_0 - v_0, x_0 - y_0) + \mu_2\|x_0 - y_0\|^2$, $(\forall)[x_0,u_0],[y_0,v_0] \in A_0$, with $0 \le \mu_1 \le 1$ and $\mu_2 \in \boldsymbol{R}$.

We shall use the standard way to represent equation (1.1) as an abstract differential equation in a Hilbert space (see e.g. [4]). We refer to [1] and [3] for standard results on nonlinear subsets and differential equations of dissipative type in Banach spaces.

We will consider $X = H \times L^2(-h, 0; H)$ a Hilbert space endowed with the scalar product $((u_0, u^0), (v_0, v^0))_X = (u_0, v_0) + (u^0, v^0)_{L^2}$. On this space we define the following operator:

$$\mathcal{D}(\mathcal{A}) = \{(x_0, x^0) | x^0 \in W^{1,2}(-h, 0; H) \cap L^2(-h, 0; \mathcal{D}(A_1)), \ x^0(0) = x_0 \in \mathcal{D}(A_0)\} \tag{1.6}$$

$$\mathcal{A}(x_0, x^0) = (A_0 x_0 + A_1 x^0(-h) + A_2 x^0(-h), \frac{dx^0}{ds}) \tag{1.7}$$

With all these assumptions, it follows:

Proposition 1.1. *$\mathcal{A}$ is $m - \omega$ dissipative.*

Proof: $(\mathcal{A}(x_0, x^0) - \mathcal{A}(y_0, y^0), (x_0, x^0) - (y_0, y^0))_X = (A_0 x_0 - A_0 y_0, x_0 - y_0) + (A_1 x^0(-h) - A_1 y^0(-h), x_0 - y_0) + (A_2 x^0(-h) - A_2 y^0(-h), x_0 - y_0) + \int_{-h}^0 \left(\frac{dx^0}{ds}(u) - \frac{dy^0}{ds}(u)\right)(x^0(u) - y^0(u)) du = (A_0 x_0 - A_0 y_0, x_0 - y_0) + (x^0(-h) - y^0(-h), A_1^*(x_0 - y_0)) + (A_2 x^0(-h) - A_2 y^0(-h), x_0 - y_0) + \frac{\|x_0 - y_0\|^2}{2} - \frac{\|x^0(-h) - y^0(-h)\|^2}{2} \leq (A_0 x_0 - A_0 y_0, x_0 - y_0) + \|x^0(-h) - y^0(-h)\| \cdot (\|A_1^*(x_0 - y_0)\| + L\|x_0 - y_0\|) + \frac{\|x_0 - y_0\|^2}{2} - \frac{\|x^0(-h) - y^0(-h)\|^2}{2} \leq (A_0 x_0 - A_0 y_0, x_0 - y_0) + \|A_1^*(x_0 - y_0)\|^2 + (\frac{1}{2} + L^2)\|x_0 - y_0\|^2 \leq (1 - \mu_1)(A_0 x_0 - A_0 y_0, x_0 - y_0) + (\frac{1}{2} + L^2 + \mu_2)\|x_0 - y_0\|^2 \leq \omega \|x_0 - y_0\|^2 \leq \omega \|(x_0 - y_0, x^0 - y^0)\|_X^2$.

Thus $\mathcal{A}$ is ω-dissipative.

Let us show now that $R(\lambda I - \mathcal{A}) = X$, for λ large enough. It is equivalent to prove that the equation $(\lambda I - \mathcal{A})y = z$ has solutions in $\mathcal{D}(\mathcal{A})$, $(\forall) z \in X$, for λ sufficiently large We may write the last relation in the following form:

$$\lambda y^0 - \frac{dy^0}{ds} = z^0, \tag{1.8}$$

$$\lambda y_0 - A_0 y_0 - A_1 y^0(-h) - A_2 y^0(-h) = z_0, \tag{1.9}$$

where $y = (y_0, y^0) \in \mathcal{D}(\mathcal{A}), z = (z_0, z^0) \in X$.

We deduce from (1.8):

$$y^0(t) = e^{\lambda t} y_0 - \int_0^t e^{\lambda(t-s)} z^0(s) ds. \tag{1.10}$$

Making in (1.10), $t = -h$, it follows:

$$y^0(-h) = e^{-\lambda h} y_0 + \int_{-h}^0 e^{\lambda(-s-h)} z^0(s) ds. \tag{1.11}$$

Combining this relation with (1.9), we obtain:

$$\begin{aligned}(\lambda I - A_0) y_0 = {} & A_1(e^{-\lambda h} y_0 + \textstyle\int_{-h}^0 e^{\lambda(-s-h)} z^0(s) ds) + A_2(e^{-\lambda h} y_0 + \\ & + \textstyle\int_{-h}^0 e^{-\lambda(-s-h)} z^0(s) ds) + z_0.\end{aligned} \tag{1.12}$$

Because of (1.2), for λ large enough, if one defines $w = (\lambda I - A_0) y_0$, we can rewrite (1.12) as:

$$\begin{aligned} w = {} & A_1(e^{-\lambda h} \lambda^{-1} (I - \lambda^{-1} A_0)^{-1} w + \textstyle\int_{-h}^0 \lambda^{-1} e^{-\lambda(s+h)} z_0(s) ds) + \\ & + \lambda^{-1} A_2 (e^{-\lambda h} (I - \lambda^{-1} A_0)^{-1} w + \textstyle\int_{-h}^0 e^{-\lambda(s+h)} z_0(s) ds) + \lambda^{-1} z_0.\end{aligned} \tag{1.13}$$

We shall use the Banach fixed point theorem in order to prove the existence of a solution for (1.13) by $F(w)$, we obtain:

$$\begin{aligned}
& \|F(w) - F(v)\| \leq \lambda^{-1} e^{-\lambda h} \|A_1((I - \lambda^{-1} A_0)^{-1} w - \\
& -(I - \lambda^{-1} A_0)^{-1} v)\| + L \lambda^{-1} e^{-\lambda h} \|(I - \lambda^{-1} A_0)^{-1} w - \\
& -(I - \lambda^{-1} A_0)^{-1} v\| \leq (L + C) \lambda^{-1} e^{-\lambda h} \|(I - \lambda^{-1} A_0))^{-1} w - \\
& -(I - \lambda^{-1} A_0)^{-1} v\| + M \lambda^{-1} e^{-\lambda h} \|A_1^*((I - \lambda^{-1} A_0)^{-1} w - \\
& -(I - \lambda^{-1} A_0)^{-1} v\| \leq M \lambda^{-1} e^{-\lambda h} \cdot \\
& \cdot (-\mu_1 (A_0 (I - \lambda^{-1} A_0)^{-1} w - A_0 (I - \lambda^{-1} A_0)^{-1} v, \\
& (I - \lambda^{-1} A_0)^{-1} w - (I - \lambda^{-1} A_0)^{-1} v) + \mu_2 \| \\
& (I - \lambda^{-1} A_0)^{-1} w - (I - \lambda^{-1} A_0)^{-1} v\|^2)^{1/2} + \\
& +(L + C) \lambda^{-1} e^{-\lambda h} \|(I - \lambda^{-1} A_0)^{-1} w - (I - \lambda^{-1} A_0)^{-1} v\| = \\
& = M \lambda^{-1} e^{-\lambda h} [(\mu_2 - \mu_1 \lambda) \|(I - \lambda^{-1} A_0)^{-1} w - \\
& -(I - \lambda^{-1} A_0)^{-1} v\|^2 + \mu_1 \lambda \cdot (w - v, (I - \lambda^{-1} A_0)^{-1} w - \\
& -(I - \lambda^{-1} A_0)^{-1} v)]^{1/2} + (L + C) \lambda^{-1} e^{-\lambda h} \\
& \|(I - \lambda^{-1} A_0)^{-1} w - (I - \lambda^{-1} A_0)^{-1} v\| \leq \\
& \leq M \lambda^{-1} e^{-\lambda h} (\mu_1 \lambda)^{1/2} \|w - v\| + (L + C) \lambda^{-1} e^{-\lambda h} \\
& \|w - v\| = (M (\mu_1 \lambda)^{1/2} + L + C) \lambda^{-1} e^{-\lambda h} \|w - v\|,
\end{aligned} \tag{1.14}$$

for λ large enough.

Therefore, with λ large enough, we obtain:

$$(M(\mu_1\lambda)^{1/2}+L+C)\lambda^{-1}e^{-\lambda h}<1.$$

Thus, the proof of the proposition is completed.

According to standard existence theory for nonlinear equations of dissipative type, for $f\in W^{1,1}([0,T];H)$ and $(x_0,x^0)\in\mathcal{D}(\mathcal{A})$, the Cauchy problem

$$\left\{\begin{array}{l}\frac{d\zeta}{dt}\in\mathcal{A}\zeta+(f,0)\\ \zeta(0)=(x_0,x^0)\end{array}\right. \tag{1.15}$$

has a unique solution $\zeta\in W^{1,\infty}([0,T];X)$.

Generaly, if the initial datum $\zeta_0\in\overline{\mathcal{D}(\mathcal{A})}$, then $\zeta\in C([0,T];X)$ is a generalized solution only. We have therefore:

Theorem 1.2. *Suppose that $f\in W^{1,1}([0,T];H)$, $(x_0,x^0)\in\mathcal{D}(\mathcal{A})$ and $A_0x^0\in L^\infty(-h,0;H)$. Then problem (1.1) has a unique solution x, for which:*

$$x\in W^{1,\infty}([0,T];H) \tag{1.16}$$

$$A_0x\in L^\infty([0,T];H),A_1x\in L^\infty([-h,T-h];H). \tag{1.17}$$

With this, first section is concluded.

2. Examples.

Example 1. Let us consider the following equation:

$$\left\{\begin{array}{l} y_t(t,x)\in\Delta y(t,x)-\beta(y(t,x))+\mu(x)\cdot\nabla y(t-h,x)+\\ \qquad\qquad +\varphi(y(t-h,x))+f(t)(x),\ (\forall)t\ge0,\ (\forall)x\in\Omega\\ y(t,x)=0,(\forall)t\ge0,(\forall)x\in\partial\Omega\\ y(0,x)=y_0(x)\in L^2(\Omega)\\ y(s,x)=y^0(s)(x)\in L^2(\Omega),-h\le s<0\end{array}\right. \tag{2.1}$$

in which Ω is an open and bounded subset of R^n with sufficiently smooth boundary $\partial\Omega$ (for instance, of class C^2) and β,φ,μ satisfy the next conditions:

$$\beta\subset \boldsymbol{R}\times\boldsymbol{R} \text{ is a maximal monotone graph such that } 0\in\beta(0). \tag{2.2}$$

$$\mu\in W^{1,\infty}(\Omega;\boldsymbol{R}^n) \text{ and } \|\mu(x)\|\le 2^{-1/2},\ (\forall)x\in\Omega. \tag{2.3}$$

$$\varphi:\boldsymbol{R}\longrightarrow\boldsymbol{R} \text{ is a Lipschitz continuous function.} \tag{2.4}$$

We shall apply theorem 1.2 in the next context:

$H=L^2(\Omega)$
$\mathcal{D}(A_0)=\{y\in H_0^1(\Omega)\cap H^2(\Omega);\ \beta(y)\in L^2(\Omega)\},\ A_0y=\Delta y+\beta(y)$
$\mathcal{D}(A_1)=H_0^1(\Omega),\ A_1y=\mu\cdot\nabla y$
$A_2(y)=\varphi(y)$.

Condition (1.2) and (1.4) are easy to be verified. We shall prove that (1.3) and (1.5) remain true. So, we have:

$$\begin{aligned}(A_1y,z)&=\int_\Omega(\mu\cdot\nabla y)z\,dx=-\int_\Omega y\,\mathrm{div}(\mu z)dx+\int_\Omega \mathrm{div}(\mu yz)dx=\\ &=-\int_\Omega y\,\mathrm{div}(\mu z)dx+\int_{\partial\Omega}yz(\mu\cdot\nu)d\sigma=\int_\Omega y(-\mathrm{div})(\mu z))dx=(y,A_1^*z).\end{aligned}$$

It follows that:

$$\begin{aligned}\|A_1y\|^2&=\int_\Omega(\mu\cdot\nabla y)^2dx\le\int_\Omega[2(\mu\cdot\nabla y+y\,\mathrm{div}\mu)^2+2(y^2(\mathrm{div}\mu)^2]dx=\\ &=2\int_\Omega(-\mathrm{div}(\mu y))^2dx+2\int_\Omega y^2(\mathrm{div}\mu)^2dx\le2\|A_1^*y\|^2+C^2\|y\|\le\\ &\le(2^{1/2}\|A_1^*y\|+C\|y\|)^2.\end{aligned}$$

Thus (1.3) is verified for M and C large enough.

$$\begin{aligned}\|A_1^*u-A_1^*v\|&=\int_\Omega(\mu\cdot\nabla(u-v)+(u-v)\mathrm{div}\mu)^2dx\le\\ &\le2\int_\Omega(\mu\cdot\nabla(u-v))^2+2\int_\Omega(\mathrm{div}\mu)^2(u-v)^2dx\le2\int_\Omega\|\mu\|^2\|\nabla(u-\\ &-v)\|^2dx+\mu_2\int_\Omega(u-v)^2dx\le\mu_1\int_\Omega\|\nabla(u-v)\|^2dx+\mu_2\|u-v\|^2=\\ &=-\mu_1\int_\Omega(u-v)\Delta(u-v)dx+\mu_2\|u-v\|^2\le\\ &\le-\mu_1\int_\Omega(u-v)[\Delta(u-v)-(\beta(u)-\beta(v))]dx+\mu_2\|u-v\|^2=\\ &=-\mu_1(A_0u-A_0v,u-v)+\mu_2\|u-v\|^2,\ \text{for}0\le\mu_1\le1,\ \mu_2\in\boldsymbol{R}.\end{aligned}$$

Thus, according to theorem 1.2, we obtain:

Corollary 2.1. *Suppose that $f\in W^{1,1}([0,T];\ L^2(\Omega)),y^0\in W^{1,2}(-h,0;\ L^2(\Omega))\cap L^2(-h,0;\ H_0^1(\Omega)),\ \Delta y^0+\beta(y^0)\in L^\infty(-h,0;\ L^2(\Omega))$ and*

$y^0(0) = y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\beta(y_0) \in L^2(\Omega)$. *Then problem (2.1) has a unique solution* y, *for which:*

$$y \in W^{1,\infty}([0,T];\ L^2(\Omega)) \cap L^\infty([0,T];\ H_0^1(\Omega) \cap H^2(\Omega)) \tag{2.5}$$

$$\beta(y) \in L^\infty([0,T];\ L^2(\Omega)). \tag{2.6}$$

Example 2. Let us consider the following equation:

$$\begin{cases} y_t(t,x) = \Delta y(t,x) + \mu(x)\cdot\nabla y(t-h,x) + \\ +\varphi(y(t-h,x)) + f(t)(x),\ (\forall)t \geq 0,\ (\forall)x \in \Omega \\ \frac{\partial y}{\partial \nu}(t,x) + \beta(y(t,x)) \ni 0,\ (\forall)t \geq 0,\ (\forall)x \in \partial\Omega \\ y(0,x) = y_0(x) \in L^2(\Omega) \\ y(s,x) = y^0(s)(x) \in L^2(\Omega),\ -h \leq s < 0 \end{cases} \tag{2.7}$$

in which Ω is exactly like in Example 1 and β, φ, μ satisfy the next conditions:

$$\beta \subset \boldsymbol{R}^n \times \boldsymbol{R} \text{ is a maximal monotone graph such that } 0 \in \beta(0) \tag{2.8}$$

$$\mu \in W^{1,\infty}(\Omega; \boldsymbol{R}^n), \|(\mu(x)\| \leq 2^{-1/2},\ (\forall)x \in \Omega \text{ and } \mu\cdot\nu = 0 \text{ on } \partial\Omega \tag{2.9}$$

$$\varphi\colon \boldsymbol{R} \longrightarrow \boldsymbol{R} \text{ is a Lipschitz continuous function.} \tag{2.10}$$

We shall apply theorem 1.2 in the next context:

$$\begin{aligned} &H = L^2(\Omega) \\ &\mathcal{D}(A_0) = \{y \in H^2(\Omega);\ \beta(y) \in L^2(\partial\Omega) \text{ and } 0 \in \beta(y) + \tfrac{\partial y}{\partial \nu} \text{ on } \partial\Omega\}, \\ &A_0 y = \Delta y \\ &\mathcal{D}(A_1) = H^1(\Omega),\ A_1 y = \mu\cdot\nabla y \\ &A_2 y = \varphi(y). \end{aligned}$$

Because of the last condition of (2.9), satisfied by μ, one can easily prove that A_1^* has the same form as in example 1. So (1.3) is true. (1.4) is trivial. Thus (1.2) and (1.5) remain to be verified.

The fact that A_0 is nonlinear, $0 \in A_0(0)$ are very easy to be confirmed. Let's prove that A_0 is dissipative.

$$\begin{aligned} (A_0 y - A_0 z, y - z) &= \textstyle\int_\Omega \Delta(y-z)(y-z)dx = \\ &= -\textstyle\int_\Omega \|\nabla(y-z)\|^2 dx + \int_{\partial\Omega}(y-z)\left(\frac{\partial y}{\partial \nu} - \frac{\partial z}{\partial \nu}\right) d\sigma = \\ &= -\textstyle\int_\Omega \|\nabla(y-z)\|^2 dx - \int_{\partial\Omega}(\beta(y) - \beta(z))(y-z)d\sigma \leq 0. \end{aligned}$$

Therefore, toghether with the maximality of A_0 which is easily deduced, A_0 satisfies (1.2).

For (1.5), as in example 1, it follows:

$$\begin{aligned} &\|A_1^* u - A_1^* v\|^2 = \textstyle\int_\Omega (\mu\cdot\nabla(u-v) + (u-v)\mathrm{div}\mu)^2 dx \leq \\ &\leq \mu_1 \textstyle\int_\Omega \|\nabla(u-v)\|^2 dx + \mu_2\|u-v\|^2 = -\mu_1 \int_\Omega (u-v)\Delta(u-v)dx + \\ &+\mu_1 \textstyle\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu}\right)(u-v)d\sigma + \mu_2\|u-v\|^2 = \\ &= -\mu_1 \textstyle\int_\Omega (u-v)\Delta(u-v)dx - \mu_1 \int_{\partial\Omega}(\beta(u) - \beta(v))(u-v)d\sigma + \mu_2\|u- \leq \\ &-v\|^2 - \mu_1 \textstyle\int_\Omega (u-v)\Delta(u-v)dx + \mu_2\|u-v\|^2 = \\ &= -\mu_1(A_0(u) - A_0(v), u-v) + \mu_2\|u-v\|^2, \text{ for } 0 \leq \mu_1 \leq 1,\ \mu_2 \in \boldsymbol{R}. \end{aligned}$$

Therefore, according to theorem 1.2, we obtain:

Corollary 2.2. *Suppose that* $f \in W^{1,1}([0,T];\ L^2(\Omega))$, $y^0 \in W^{1,2}(-h,0;\ L^2(\Omega)) \cap L^2(-h,0;\ H^1(\Omega)), \Delta y^0 \in L^\infty(-h,0;\ L^2(\Omega))$ *and* $y^0(0) = y_0 \in \{y \in H^2(\Omega);\ \beta(y) \in L^2(\partial\Omega)$ *and* $\frac{\partial y}{\partial \nu} + \beta(y) \ni 0$ *on* $\partial\Omega\}$. *Then problem (2.7) has a unique solution* y, *for which:*

$$y \in W^{1,\infty}([0,T];\ L^2(\Omega)) \cap L^\infty([0,T]; H^2(\Omega)). \tag{2.11}$$

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