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MINIMUM PRINCIPLES FOR BINARY MIXTURES OF NON-SIMPLE ELASTIC MATERIALS

BY

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1. Introduction. This paper deals with a continuum theory for binary mixtures of non-simple elastic solids. Precisely, we consider the mixed initial-boundary value problem in the linear isothermal case, and aim to explore minimum properties for related functionals.

A modern approach to thermomechanical theories for mixtures in continuum context is essentially due to T r u e s d e l l and T o u p i n [1] and K e l l y [2]. These authors first stated in a rational form the relevant balance postulates. The subsequent developments carried out by E r i n g e n and I n g r a m [3,4], G r e e n and N a g h d i [5,6], M u l l e r [7], D u n w o o d y and M u l l e r [8] and B o w e n and W i e s e [9]., have provided these theories of many important results from both theoretical applicative standpoints. We refer to the reviewing paper by B o w e n [10], A t k i n and C r a i n e [11] and B e d f o r d and D r u m h e l l e r [12] for an exhaustive description of the suject and of the intended applications.

Ignoring certain still unsatisfactory aspects in the thermodynamic assumptions of the theories for mixtures*, we restrict our attention to the *purelly mechanical* (isothermal) case. We also assume as *non simple* the elastic materials involved in the mixture: this amounts to include second-order displacement gradients in the constitutive variables, as well as to consider higher order stress terms in the balance equations [13–15]. The

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* These aspects regard the particular forms in which an entropy inequality should be imposed to the constituents of the mixture; see the discussion in [12, sect. 1a].

resulting model should be useful to account for some properties of *non locality* in the theory of mixtures that have been proposed by M u l l e r [7], B o w e n and W i e s e [9] and G r e e n and L a w s [16].

As regards the minimizing functionals, we shall follow, with minor modifications, the method conceived by R e i s s in [17] and later cast in general form by R e i s s and H a u g [18]: this method allows to set up such functionals for linear dynamic problems in the usual time domain, instead of the Laplace transform domain as in foregoing approaches (cf.[19]).

Starting from the field equations established for the present context in [20], Section 2 is devoted to introduce the initial–boundary–value problem and some related topics. In Section 3, after recalling the guidelines of Reiss and Haug's method, we shall prove (strict) minimum priciples for homogeneous and nonhomogeneous boundary conditions.

2. Statement of the problem and basic definitions. Throughout the paper, Ω will denote the smooth domain of the physical space ($\equiv R^3$) occupied by the (solid) mixture in a given reference configuration; we shall take as $t=0$ the time on which this happens. Identified R^3 with the associated vector space, an orthonormal frame of reference is introduced, so that vectors and tensors will have components denoted by latin subscripts (ranging over $\{1,2,3\}$, unless otherwise specified). Summation over repeated subscripts and other convetions for differential operations are implied, e.g., superposed dot and ∂_i for partial derivative with respect to time or to i-th coordinate, respectively. In this connection, we shall disregard regularity questions, simply understanding a degree of smoothness sufficient to make sense everywhere.

As discussed in the quoted papers on the argument (cf. also [21,22]), an efficient theory for binary mixtures of elastic bodies can be developed on the basis of a mere *superposition* of two interacting continua C_1 and C_2: this means that each of these follows its own motion in such a way that different particles, one from each constituent body, simultaneously occupy, at any time, the same place in the current domain of the mixture. If X_i and $Y_i \in \Omega$ denote the referential positions of two typical particles of C_1 and C_2, a basic assumption concerned in various approaches [20–22], and here adopted, is that the reference configuration can be chosen so as to get X_i coinciding with Y_i. This implies the motion of the mixture is simply described by two maps

$$x_h = x_h(X_i,t), \quad y_h = y_h(X_i,t) \qquad (X_i,t) \in \Omega \times [0,T).$$

in which of course x_h and y_h are the current (different) positions of those particles occupying the same X_i at $t=0$, and $[0,T)$ is a time interval.

Neglecting the possibility of chemical reactions between the constituent materials, we also assume the mass of each of them is individually conserved, so that the following motion equations can be derived from the classical (mechanical) balance laws [20]:

$$(1) \qquad \begin{cases} \partial_j \tau_{ji}^{(1)} - \partial_j \partial_h \mu_{hji}^{(1)} - p_i + \rho^{(1)} b_i^{(1)} = \rho^{(1)} \ddot{u}_i^{(1)} \\ \partial_j \tau_{ji}^{(2)} - \partial_j \partial_h \mu_{hji}^{(2)} + p_i + \rho^{(2)} b_i^{(2)} = \rho^{(2)} \ddot{u}_i^{(2)}, \quad \text{in } \Omega \times (0,T). \end{cases}$$

(Where concerned, the top label $\alpha = 1,2$ refers the various fields to each constituent C_1 or C_2). In these equations, the difference $\left[\tau_{ji}^{(\alpha)} - \partial_h \mu_{hji}^{(\alpha)}\right]$ represents the usual (non–symmetric) stress tensor, while $\mu_{hji}^{(\alpha)}$ is the so-called *hyperstress* tensor, closely related to the non–simple material structure [13–15]. $\rho^{(\alpha)}$ is the referential mass density and $b_i^{(\alpha)}$ the external body force. Moreover, the vector field p_i characterizes the mechanical interaction between the constituents C_1 and C_2, of which, finally, $u_i^{(1)}$ and $u_i^{(2)}$ are displacement fields, respectively; of course, $u_i^{(1)} = x_i - X_i$, $u_i^{(2)} = = y_i - X_i$.

For the linear theory considered in this paper, the following constitutive equations relate the fields $\tau_{ji}^{(\alpha)}$, $\mu_{hji}^{(\alpha)}$ and p_i to the *strain measures*

$$e_{ij} = 1/2\left(\partial_j u_i^{(1)} + \partial_i u_j^{(1)}\right), \; g_{ij} = \partial_i u_j^{(1)} + \partial_j u_i^{(2)}, \; k_{ijh} = \partial_i \partial_j u_h^{(1)},$$

$$\gamma_{ijh} = \partial_i \partial_j u_h^{(2)}, \; d_i = u_i^{(1)} - u_i^{(2)} \quad [20]:$$

(2)

$$\begin{cases} \tau_{ji}^{(1)} = & (A_{jipq} + B_{pqji}) e_{pq} + (B_{jipq} + C_{jipq}) g_{pq} + (D_{jih} + E_{jih}) d_h + \\ & + (J_{jipqr} + K_{jipqr}) K_{pqr} + (M_{jipqr} + N_{jipqr}) \gamma_{pqr}; \\ \tau_{ji}^{(2)} = & B_{pqij} e_{pq} + C_{ijpq} g_{pq} + E_{ijh} d_h + K_{ijpqr} K_{pqr} + N_{ijpqr} \gamma_{pqr}; \\ \mu_{hji}^{(1)} = & J_{pqhji} e_{pq} + K_{pqhji} g_{pq} + L_{rhji} d_r + G_{hjipqr} K_{pqr} + H_{hjipqr} \gamma_{pqr}; \\ \mu_{hji}^{(2)} = & M_{pqhji} e_{pq} + N_{pqhji} g_{pq} + P_{rhji} d_r + H_{pqrhji} K_{pqr} + I_{hjipqr} \gamma_{pqr}; \\ p_i = & D_{pqi} e_{pq} + E_{pqi} g_{pq} + A_{ij} d_j + L_{ipqr} K_{pqr} + P_{ipqr} \gamma_{pqr}. \end{cases}$$

The material coefficients appearing in these equations can be proved to enjoy the following symmetry properties [20]:

(3)

$$\begin{cases} A_{ijpq} = A_{pqij} = A_{jipq}, \; B_{ijpq} = B_{jipq}, \; C_{ijpq} = C_{pqij}, \; A_{ij} = A_{ji}, \\ D_{ijh} = D_{jih}, G_{ijhpqr} = G_{pqrijh} = G_{jihpqr}, H_{ijhpqr} = H_{ijhqpr} = H_{jihpqr}, \\ I_{ijhpqr} = I_{pqrijh} = I_{jihpqr}, \; J_{ijpqr} = J_{ijqpr} = J_{jipqr}, \; K_{ijpqr} = K_{ijqpr}, \\ L_{ipqr} = L_{iqpr}, M_{ijpqr} = M_{ijqpr} = M_{jipqr}, N_{ijpqr} = N_{ijqpr}, P_{ipqr} = P_{iqpr}. \end{cases}$$

Of interest in the sequel will be the following quadratic form:

(4)
$$\begin{cases} \pi(\xi_i, \xi_{ij}, \eta_{ij}, \xi_{ijh}, \eta_{ijh}) = 1/2[A_{ijpq}\xi_{ij}\xi_{pq} + C_{ijpq}\eta_{ij}\eta_{pq} + \\ +G_{ijhpqr}\xi_{ijh}\xi_{pqr} + I_{ijhpqr}\eta_{ijh}\eta_{pqr} + A_{ij}\xi_i\xi_j] + B_{ijpq}\xi_{ij}\eta_{pq} + D_{ijh}\xi_{ij}\xi_h + \\ +E_{ijh}\xi_h\eta_{ij} + J_{ijpqr}\xi_{ij}\xi_{pqr} + K_{ijpqr}\eta_{ij}\xi_{pqr} + M_{ijpqr}\xi_{ij}\eta_{pqr} + \\ +N_{ijpqr}\eta_{ij}\eta_{pqr} + L_{ipqr}\xi_i\xi_{pqr} + H_{ijhpqr}\xi_{ijh}\eta_{pqr} + P_{ipqr}\xi_i\eta_{pqr}. \end{cases}$$

We note that, for $\xi_i = d_i$, $\xi_{ij} = e_{ij}$, $\eta_{ij} = g_{ij}$, $\xi_{ijh} = K_{ijh}$, $\eta_{ijh} = \gamma_{ijh}$, this form attains the meaning of a *potential energy* density in the mixture associated to some pair of displacement fields $u_i^{(1)}$, $u_i^{(2)}$ [20]. In the case of simple materials, all the above equations reduce to those given in [21,22].

Consider now two pairs of disjoint and complementary subsets of the (smooth) boundary $\Sigma : \{\Sigma_k, \Sigma_{k+1}\}$, $k = 1, 3$. We define $S_k = \Sigma_k \times [0, T)$ and the *normal derivative* $D_f = \partial_i f n_i$, f beeing any scalar field and n_i the outer unit normal to Σ. A typical initial-boundary-value problem for binary mixtures of nonsimple elastic solids can be set up by adding to motion and constitutive equations the following initial and mixed boundary conditions ($\alpha = 1, 2$):

$$u_i^{(\alpha)} = a_i^{(\alpha)}, \quad \dot{u}_i^{(\alpha)} = c_i^{(\alpha)}, \qquad \text{on } \Omega \times \{0\}; \tag{5}$$

$$\begin{cases} u_i^{(\alpha)} = u_{i,\Sigma}^{(\alpha)} \text{ on } S_1; \ P_i^{(1)} + P_i^{(2)} = P_{i,\Sigma}, \ d_i = d_{i,\Sigma} \text{ on } S_2; \\ Du_i^{(\alpha)} = \widetilde{u}_{i,\Sigma}^{(\alpha)} \text{ on } S_3; \ R_i^{(1)} + R_i^{(2)} = R_{i,\Sigma}, \ Dd_i = \widetilde{d}_{i,\Sigma} \text{ on } S_4; \end{cases} \tag{6}$$

where $P_i^{(\alpha)}$ and $R_i^{(\alpha)}$ are suitably defined vector fields such that power of the surface tractions and hypertractions can be written as

$$\int_\Sigma [P_i^{(1)}\dot{u}_i^{(1)} + P_i^{(2)}\dot{u}_i^{(2)} + R_i^{(1)}D\dot{u}_i^{(1)} + R_i^{(2)}D\dot{u}_i^{(2)}]d\Sigma \quad [13-15, 20]$$

We refer to [10,11,12] for comments on these and on other of boundary conditions arising in the applications. Of course, $P_i^{(\alpha)}$ and $R_i^{(\alpha)}$ can be determined by $u_i^{(\alpha)}$ through the constitutive equations. Right-hand terms in (5,6) denote prescribed fields; along with $b_i^{(\alpha)}$, these are the *data* of the mixed problem in concern. An array of fields $\left(u_i^{(1)}, u_i^{(2)}\right)$ meeting equations (1, 2, 5, 6) for some assignment of the data, will be referred to as a (regular) *solution* of this.

For any scalar field $a = a(X_i, t)$, define now

$$\bar{a}(X_i, t) = \int_0^t a(X_i, \tau)d\tau, \qquad t \in [0, T).$$

Useful for the sequel is the following result, essentially due to Ignaczak [23] and Gurtin [24].

Lemma 1. *The fields $(u_i^{(1)}, u_i^{(2)})$ give a solution to problem (1,2,5,6) if and only if they satisfy the boundary conditions (6) and the following field equations*

$$\begin{cases} \partial_j \overset{=(1)}{\tau}_{ji} - \partial_j\partial_h \overset{=(1)}{\mu}_{hji} - \overset{=}{p}_i + f_i^{(1)} = \rho^{(1)}u_i^{(1)} \\ \partial_j \overset{=(2)}{\tau}_{ji} - \partial_j\partial_h \overset{=(2)}{\mu}_{hji} + \overset{=}{p}_i + f_i^{(2)} = \rho^{(2)}u_i^{(2)}, \qquad \text{in } \Omega \times [0, T); \end{cases} \tag{7}$$

where

$$f_i^{(\alpha)} = \rho^{(\alpha)}\left[\overset{=(\alpha)}{b}_i + a_i^{(\alpha)} + tc_i^{(\alpha)}\right], \quad \alpha = 1, 2.$$

In the sequel we shall take $T = +\infty$.

3. Minimum questions. Reiss'approach to minimum principles for linear initial-boundary-value problems* begins by introducting the set $\mathbf{H}$ of *admissible weight functions* [17,18]. A positive function g on $[0, T)$ belongs to $\mathbf{H}$ if

$$g(t) = \int_0^{+\infty} \exp(-st)G(s)ds \tag{8}$$

for some continuous, non-negative $G = G(s)$ having compact support in $(0, +\infty)$. Denote now by $\mathbf{L}$ the set of fields a on $\Omega \times [0, T)$ which admit (real) Laplace transform $\hat{a}$ in Ω:

$$\hat{a}(X_i, s) = \int_0^{+\infty} \exp(-st)a(X_i, t)\,dt.$$

* Minor modifications are here included: cf. [25].

Set also

$$\widehat{\mathbf{L}} = \{a \in \mathbf{L} : \ \partial_i a, \partial_i\partial_j a, \partial_i\partial_j\partial_h a \in \mathbf{L}\}.$$

Fixed any $g \in \mathbf{H}$ we can build the following (symmetric) bilinear form for scalar fields a, b in $\mathbf{L}$:

$$[a,b]_g = \int_\Omega \int_0^{+\infty}\int_0^{+\infty} g(t+\tau)a(X_i,t)\,b(X_i,\tau)\;dt\,d\tau\,d\Omega.$$

Of course, thanks to (8), it also results

$$[a,b]_g = \int_\Omega \int_0^{+\infty} G(s)\hat{a}(X_i,s)\,\hat{b}(X_i,s)\;ds\,d\Omega,$$

what clearly implies the positive definiteness of such a form. When dealing with n–dimensional linear space of vectors $U = (U_i)$, $V = (V_i)$, $i = 1, ..., n$, we shall mean

$$[U,V]_g = \sum_1^n [U_i, V_i]_g\,. \tag{9}$$

Consider now a linear operator

$$A : D_A \subseteq X \to Y$$

where X, Y are linear spaces whose elements belog to $\mathbf{L}$, and D_A is the domain of definition. We say that A is *g-symmetric* in some subspace $\widetilde{D}_A \subseteq D_A$ if

$$[AU,V]_g = [U,AV]_g \qquad \forall\, U,V \in \widetilde{D}_A. \tag{10}$$

We say that A is *g-positive* in $\widetilde{D}_A$ if it is g–symmetric and further

$$[AU,U]_g \geq 0 \quad \forall\, U \in \widetilde{D}_A; \quad [AU,U]_g = 0 \Rightarrow U_i = 0 \;\forall\, i = 1,\dots,n. \tag{11}$$

For such an operator, the solution $U \in \widetilde{D}_A$ of the equation

$$AU = F, \quad F \in Y, \tag{12}$$

gives a strict minimum to the functional on $\widetilde{D}_A$ defined by

$$\widetilde{\Phi}_g(V) = 1/2\,[AV,V]_g - [V,F]_g; \tag{13}$$

this trivially results from the following equation

$$\widetilde{\Phi}_g(V) - \widetilde{\Phi}_g(U) = 1/2\,[AZ,Z]_g + [AU - F, Z]_g, \tag{14}$$

in which $Z = V - U$.

We now aim to cast in the form (12) the field equations (7). To this end, define the six–vectors U, F by

$$U_i = u_i^{(1)},\ U_{i+3} = u_i^{(2)}; \quad F_i = f_i^{(1)},\ F_{i+3} = f_i^{(2)} \tag{15}$$

and, by means of constitutive equations (2),

$$\begin{cases} (AU)_i = \rho^{(1)}u_i^{(1)} - \partial_j\left[\overset{=}{\tau}{}^{(1)}_{ji} - \partial_h\,\overset{=}{\mu}{}^{(1)}_{hji}\right] + \overset{=}{p}_i \\ (AU)_{i+3} = \rho^{(2)}u_i^{(2)} - \partial_j\left[\overset{=}{\tau}{}^{(2)}_{ji} - \partial_h\,\overset{=}{\mu}{}^{(2)}_{hji}\right] - \overset{=}{p}_i \end{cases} \tag{16}$$

Along with equation (12), we firstly consider the following *homogeneous* boundary conditions:

(17)

$$\begin{cases} u_i^{(1)} = u_i^{(2)} = 0 \text{ on } S_1; \quad P_i^{(1)} + P_i^{(2)} = 0,\ \ u_i^{(1)} - u_i^{(2)} = 0 \text{ on } S_2; \\ Du_i^{(1)} = Du_i^{(2)} = 0 \text{ on } S_3; \quad R_i^{(1)} + R_i^{(2)} = 0,\ \ Du_i^{(1)} - Du_i^{(2)} = 0 \text{ on } S_4; \end{cases}$$

and denote by D_A^0 the subspace of fields $\left(u_i^{(1)}, u_i^{(2)}\right)$ in D_A satisfying these conditions (via eqs. (2)). We assume $D_A \subset \widehat{\mathbf{L}}$, and prove the following lemmas.

Lemma 2. *Let the symmetry properties (3) hold for the constitutive coefficients. Then, the operator A defined in (16) is g–symmetric on D_A^0 for any $g \in \mathbf{H}$.*

Proof. Let $U = \left(u_i^{(1)}, u_i^{(2)}\right)$, $V = \left(v_i^{(1)}, v_i^{(2)}\right)$ belong to D_A^0. By eqs. (9,16) and divergence theorem (*)

(18)

$$\begin{cases} [AU,V]_g = \int_\Omega \int_0^\infty \int_0^\infty g(t+\tau)\{[\rho^{(1)}u_i^{(1)}(t)v_i^{(1)}(\tau) + \rho^{(2)}u_i^{(2)}(t)v_i^{(2)}(\tau)] \\ +[\overset{=}{\tau}{}^{(1)}_{ji}(t)\partial_j v_i^{(1)}(\tau) + \overset{=}{\tau}{}^{(2)}_{ji}(t)\partial_j v_i^{(2)}(\tau) + \overset{=}{p}_i(t)(v_i^{(1)} - v_i^{(2)})(\tau) + \\ + \overset{=}{\mu}{}^{(1)}_{hji}(t)\partial_j\partial_h v_i^{(1)}(\tau) + \overset{=}{\mu}{}^{(2)}_{hji}(t)\partial_j\partial_h v_i^{(2)}(\tau)]\}\,dt\,d\tau\,d\Omega - \\ \int_\Sigma \int_0^\infty \int_0^\infty g(t+\tau)[\overset{=}{P}{}^{(1)}_i(t)v_i^{(1)}(\tau) + \overset{=}{R}{}^{(1)}_i(t)Dv_i^{(1)}(\tau) + \overset{=}{P}{}^{(2)}_i(t)v_i^{(2)}(\tau) \\ + \overset{=}{R}{}^{(2)}_i(t)Dv_i^{(2)}(\tau)]\,dt\,d\tau\,d\Sigma. \end{cases}$$

(*) Spatial dependence is omitted for convenience.

Of course, the boundary term above can be written as

$$
(19)\qquad
\begin{cases}
\int_{\Sigma_1}\left[\overset{=}{P}{}_i^{(1)}(t)v_i^{(1)}(\tau)+\overset{=}{P}{}_i^{(2)}(t)v_i^{(2)}(\tau)\right]d\Sigma+\\
+\int_{\Sigma_2}\left[\left(\overset{=}{P}{}_i^{(1)}+\overset{=}{P}{}_i^{(2)}\right)(t)v_i^{(2)}(\tau)+\overset{=}{P}{}_i^{(1)}(t)\left(v_i^{(1)}-v_i^{(2)}\right)(\tau)\right]d\Sigma+\\
+\int_{\Sigma_3}\left[\overset{=}{R}{}_i^{(1)}(t)Dv_i^{(1)}(\tau)+\overset{=}{R}{}_i^{(2)}(t)Dv_i^{(2)}(\tau)\right]d\Sigma+\\
+\int_{\Sigma_4}\left[\left(\overset{=}{R}{}_i^{(1)}+\overset{=}{R}{}_i^{(2)}\right)(t)Dv_i^{(2)}(\tau)+\overset{=}{R}{}_i^{(1)}(t)\left(Dv_i^{(1)}-Dv_i^{(2)}\right)(\tau)\right]d\Sigma,
\end{cases}
$$

and clearly vanishes in D_A^0. Insertion of eq. (8) in eq. (18) makes Laplace transform of the various fields appear; taking into account the constitutive equations, the claimed symmetry properties finally imply that equation (10) does hold for $\widetilde{D}_A \equiv D_A^0$. Q.E.D.

Lemma 3. *Assume the hypotheses of Lemma 2 hold, and let $\rho^{(1)},\rho^{(2)}>0$. Provided the quadratic form π of eq. (4) be positive semi-definite, the operator A is g-positive on D_A^0 for any $g\in\mathbf{H}$.*

Proof. Put $V=U$ in eq. (18), and in what comes out insert eqs. (2,8). We get

$$
[AU,U]_g=\int_\Omega\int_0^{+\infty}G(s)[\rho^{(1)}\hat{u}_i^{(1)}(s)\hat{u}_i^{(1)}(s)+\rho^{(2)}\hat{u}_i^{(2)}(s)\hat{u}_i^{(2)}(s)+
$$

$$
+(2/s^2)\pi(\hat{d}_i(s),\hat{e}_{ij}(s),\hat{g}_{ij}(s),\hat{k}_{ijh}(s),\hat{\gamma}_{ijh}(s))]dsd\Omega,
$$

that of course is non-negative. Futher, $[AU,U]_g=0$ implies $\hat{u}_i^{(1)}=\hat{u}_i^{(2)}=0$ in Ω for all s in the support of G, and the whole of $\Omega\times(0,+\infty)$. We conclude that also $u_i^{(1)}=u_i^{(2)}\equiv 0$ in $\Omega\times[0,T)$ Q.E.D.

The previous lemmas straightly enable us to lay down a strict minimum principle in the case of homogeneous boundary conditions (recall eqs. (13,14)).

Theorem 1. *Let $U=\left(u_i^{(1)},u_i^{(2)}\right)$ be a solution of the homogeneous problem (12,17), and let $f_i^{(\alpha)}$ belong to $\mathbf{L}$. Given any $g\in\mathbf{H}$, consider the functional on D_A^0 : $\Phi_g^0(V)=1/2[AV,V]_g-[V,F]_g$ (V as in the proof of Lemma 2), that is,*

$$
(20)\qquad
\begin{cases}
\Phi_g^0(V)=1/2\int_\Omega\int_0^{+\infty}\int_0^{+\infty}g(t+\tau)\{([\rho^{(1)}v_i^{(1)}(t)v_i^{(1)}(\tau)+\\
+\rho^{(2)}v_i^{(2)}(t)v_i^{(2)}(\tau)]+[A_{ijpq}\overset{=}{F}_{ij}(t)F_{pq}(\tau)+\\
+C_{ijpq}\overset{=}{H}_{ij}(t)H_{pq}(\tau)+G_{ijhpqr}\partial_i\partial_j\overset{=}{v}{}_h^{(1)}(t)\partial_p\partial_q v_r^{(1)}(\tau)+\\
+I_{ijhpqr}\partial_i\partial_j\overset{=}{v}{}_h^{(2)}(t)\partial_p\partial_q v_r^{(2)}(\tau)+A_{ij}\overset{=}{e}_i(t)e_j(\tau)+\\
+B_{ijpq}\overset{=}{F}_{ij}(t)H_{pq}(\tau)+(D_{ijh}\overset{=}{F}_{ij}(t)+E_{ijh}\overset{=}{H}_{ij}(t))e_h(\tau)+\\
+(J_{ijpqr}\overset{=}{F}_{ij}(t)+K_{ijpqr}\overset{=}{H}_{ij}(t)+L_{ipqr}\overset{=}{e}_i(t))\partial_p\partial_q v_r^{(1)}(\tau)+\\
+(M_{ijpqr}\overset{=}{F}_{ij}(t)+N_{ijpqr}\overset{=}{H}_{ij}(t)+H_{ijhpqr}\partial_i\partial_j\overset{=}{v}{}_h^{(1)}(t)+\\
+P_{ipqr}\overset{=}{e}_i(t))\partial_p\partial_q v_r^{(2)}(\tau)]\}dtd\tau d\Omega-\\
-\int_\Omega\int_0^{+\infty}\int_0^{+\infty}g(t+\tau)[f_i^{(1)}(t)v_i^{(1)}(\tau)+\\
+f_i^{(2)}(t)v_i^{(2)}(\tau)]dtd\tau d\Omega,
\end{cases}
$$

where $F_{ij}=1/2[\partial_j v_i^{(1)}+\partial_i v_j^{(1)}]$, $H_{ij}=\partial_i v_j^{(1)}+\partial_j v_i^{(2)}$, $e_i=v_i^{(1)}-v_i^{(2)}$. Then, provided lemmas 2,3 hold, we have

$$
\Phi_g^0(V)\geq\Phi_g^0(U)\qquad\forall\, V\in D_A^0,
$$

and equality above necessarily implies $V=U$.

This result can be used to prove a strict minimum principle in the case of non-homogeneous boundary conditions too. Following [26], let us introduce a *given* six vector $W=\left(w_i^{(1)},w_i^{(2)}\right)\in D_A$ satisfying *all* boundary conditions (6), and let $U=\left(u_i^{(1)},u_i^{(2)}\right)\in D_A$ be a solution of the non-homogeneous mixed problem (12,6); clearly, this is equivalent to require that $U'=U-W$ meets the equation

$$
AU'=F'\equiv F-AW
$$

along with the homogeneous boundary conditions (17) (so that $U'\in D_A^0$). Theorem 1 can then be claimed for U', and leads to consider the minimizing functional:

$$
\Phi_g^0(U')=\frac{1}{2}[AU',U']_g-[U',F']_g=
$$

$$
=\frac{1}{2}[AU,U]_g+\frac{1}{2}([AW,U]_g-[AU,W]_g)-[U,F]_g-\frac{1}{2}[AW,W]_g+[W,F]_g
$$

that can be explicitly evaluated by means of eqs. (9,16,2,3) and divergence theorem (see also (18,19)). Of course, the bracketed term above now does not vanish, as it did for $W, U \in D_A^0$; moreover, some terms behave as stationary with respect to extremum properties. By putting

$$\Phi_g(U) = \Phi_g^0(U'),$$

and neglecting the stationary terms, we are finally in position to state the following

Theorem 2. *Let $U = \left(u_i^{(1)}, u_i^{(2)}\right)$ be a solution of the non-homogeneous problem (12,6), with $f_i^{(\alpha)} \in \mathbf{L}$, and let D_A^* denote the subdomain of all fields $V = \left(v_i^{(1)}, v_i^{(2)}\right) \in D_A$ such that $(\alpha = 1,2)$:*

$$\begin{cases} v_i^{(\alpha)} = u_{i,\Sigma}^{(\alpha)} \text{ on } S_1, & v_i^{(1)} - v_i^{(2)} = d_{i,\Sigma} \text{ on } S_2, \\ Dv_i^{(\alpha)} = \tilde{u}_{i,\Sigma}^{(\alpha)} \text{ on } S_3, & Dv_i^{(1)} - Dv_i^{(2)} = \tilde{d}_{i,\Sigma} \text{ on } S_4. \end{cases} \tag{21}$$

Given any $g \in \mathbf{H}$, consider the following functional on D_A^*

$$\Phi_g(V) = \Phi_g^0(V) - \frac{1}{2}\int_{\Sigma_2}\int_0^\infty\int_0^\infty g(t+\tau)\,\bar{\bar{P}}_{i,\Sigma}(t)\left(v_i^{(1)} + v_i^{(2)}\right)(\tau)\,dt\,d\tau\,d\Sigma -$$

$$-\frac{1}{2}\int_{\Sigma_4}\int_0^\infty\int_0^\infty g(t+\tau)\,\bar{\bar{R}}_{i,\Sigma}(t)\left(Dv_i^{(1)} + Dv_i^{(2)}\right)(\tau)\,dt\,d\tau\,d\Sigma,$$

where $\Phi_g^0(V)$ is as in eq. (20). Then, provided the hypotheses of Lemma 3 hold, we have

$$\Phi_g(V) \geq \Phi_g(U) \qquad \forall\, V \in D_A^*,$$

and equality necessarily implies $V = U$.

Of course, *strict* minimum principles are important since they lead to as many uniqueness theorems; in the connection, see [10,27] for similar results.

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ITERATED DEFECT CORRECTION METHODS FOR NON–LINEAR DIFFERENTIAL–ALGEBRAIC SYSTEMS OF INDEX 2

BY

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Introduction. We study the behaviour of the iterated defect correction methods based on linearly implicit Euler method for DAE's of the form

$$y' = f(y,z) \quad y(0) = y_0$$
$$0 = g(y) \quad z(0) = z_0 \tag{1.1}$$

where $t \in [0,T]$, $y \in I\!R^{\mu}$ and $z \in I\!R^{\lambda}$.

We assume that f and g are sufficiently differentiable and that

$$g_y(y) \cdot f_z(y,z) \quad \text{is non} - \text{singular} \tag{1.2}$$

in a neighbourhood of the solution, so that the problem has index 2. The initial values y_0, z_0 are required to be consistent, what means that

$$g(y_0) = 0, \; g_y(y_0) \cdot f(y_0, z_0) = 0. \tag{1.3}$$

Whenever an initial value y_0 satisfying $g(y_0) = 0$ has been specified, the second equation of (1.3) determines a locally unique value for z_0. This is a consequence of (1.2) and the implicit function theorem. Then the equation (1.1) possesses a unique solution. Problems of the form (1.1) are frequently encountered in practice. An important class of such problems are multibody systems with constraints on the velocity level (see e.g., [4]). In the last years, much research has been devoted to the development of numerical methods for the differential–algebraic problem (1.1). Convergence results for BDF methods has been given in [9],[1]. Implicit Runge–Kutta methods have been