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ITERATED DEFECT CORRECTION METHODS FOR NON–LINEAR DIFFERENTIAL–ALGEBRAIC SYSTEMS OF INDEX 2

BY

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Introduction. We study the behaviour of the iterated defect correction methods based on linearly implicit Euler method for DAE's of the form

$$\begin{aligned} y' &= f(y,z) \quad & y(0) = y_0 \\ 0 &= g(y) & z(0) = z_0 \end{aligned} \tag{1.1}$$

where $t \in [0,T]$, $y \in \mathbb{R}^{\mu}$ and $z \in \mathbb{R}^{\lambda}$.

We assume that f and g are sufficiently differentiable and that

$$g_y(y) \cdot f_z(y,z) \ \text{ is non} - \text{singular} \tag{1.2}$$

in a neighbourhood of the solution, so that the problem has index 2. The initial values y_0, z_0 are required to be consistent, what means that

$$g(y_0) = 0, \ g_y(y_0) \cdot f(y_0, z_0) = 0. \tag{1.3}$$

Whenever an initial value y_0 satisfying $g(y_0) = 0$ has been specified, the second equation of (1.3) determines a locally unique value for z_0. This is a consequence of (1.2) and the implicit function theorem. Then the equation (1.1) possesses a unique solution. Problems of the form (1.1) are frequently encountered in practice. An important class of such problems are multibody systems with constraints on the velocity level (see e.g., [4]). In the last years, much research has been devoted to the development of numerical methods for the differential–algebraic problem (1.1). Convergence results for BDF methods has been given in [9],[1]. Implicit Runge–Kutta methods have been considered in [2],[5]. L u b i c h [8] has recently proposed an extrapolation method based on an half-explicit midpoint rule. The aim of this paper is to search another efficient method for (1.1). In Section 2, the application of IDeC methods on DAE's of insex 2 will be presented. The asymptotic expansions of the global error for the implicit Euler method is given in Section 3. In Section 4, a theorem for DAE's of index 2 is given, concerning the convergence order of IDeC steps, their dependence on the degree of the interpolation polynomial used in defect correction. It turns out that the convergence order of the differential and algebraic components is increased by one at each defect correction step and the maximum convergence order is limited by the degree m of the interpolating polynomial. The last section presents the numerical results which are obtained from the test problem.

2. Formulation of the IDeC method for DAE's of index 2. For the numerical solution of the original problem (OP) (1.1) on the uniform grid $G=\{t_\nu=\nu h,\ \nu=0,1,\dots,n\cdot m,\ n,m\in I\!N\}$ with the stepsize $h=T/n\cdot m$, we consider the linearly implicit Euler method:

$$\begin{bmatrix} I-hA_1 & -hA_2 \\ -hA_3 & 0 \end{bmatrix}\begin{bmatrix} \eta_{\nu+1}-\eta_\nu \\ \xi_{\nu+1}-\xi_\nu \end{bmatrix}=h\begin{bmatrix} f(\eta_\nu,\xi_\nu) \\ g(\eta_\nu) \end{bmatrix} \tag{2.1}$$

where the Jacobian is evaluated at the initial step and held constant in the whole integration interval so that

$$A_1=\frac{\partial f}{\partial y}(y_0,z_0)\quad A_2=\frac{\partial f}{\partial z}(y_0,z_0)\quad A_3=\frac{\partial g}{\partial y}(y_0). \tag{2.2}$$

We get the following numerical solutions

$$\eta:=(\eta_0,\dots,\eta_\nu,\dots,\eta_{n\cdot m})^T\quad \xi:=(\xi_0,\dots,\xi_\nu,\dots,\xi_{n\cdot m})^T.$$

Interpolating of η and ξ by the vector-valued polynomials $P_h^{[0]}(t)$ and $Q_h^{[0]}(t)$, respectively, yields the defect $d_h^{[0]}(t)$

$$\begin{aligned} d_{1,h}^{[0]}(t)&=P_h^{[0]'}(t)-f(P_h^{[0]}(t),Q_h^{[0]}(t)) \\ d_{2,h}^{[0]}(t)&=-g(P_h^{[0]}(t)). \end{aligned} \tag{2.3}$$

By adding the defect $d_{1,h}^{[0]}(t)$ and $d_{2,h}^{[0]}(t)$ to the right hand side of the (OP) (1.1), we construct the neighbouring problem (NP) whose exact solutions are $P_h^{[0]}(t)$ and $Q_h^{[0]}(t)$.

$$\begin{aligned} y'&=f(y,z) &+\quad d_{1,h}^{[0]}(t)\quad y_0(0)=y_0\quad (NP) \\ 0&=g(y) &+\quad d_{2,h}^{[0]}(t)\quad z_0(0)=z_0 \end{aligned} \tag{2.4}$$

Solving the (NP) (2.4) by the implicit Euler method on the same grid leads the following numerical solution

$$\pi^{[0]}:=(\pi_0^{[0]},\dots,\pi_\nu^{[0]},\dots,\pi_{n\cdot m}^{[0]})^T\quad \omega^{[0]}:=(\omega_0^{[0]},\dots,\omega_\nu^{[0]},\dots,\omega_{n\cdot m}^{[0]})^T$$

where $\pi_\nu^{[0]},\omega_\nu^{[0]}$ are the approximations to the exact solutions $P_h^{[0]}(t)$ and $Q_h^{[0]}(t)$ of (2.4) at the point t_ν. Then the improved solutions of (OP) is given by

$$\begin{aligned} \eta_\nu^{[1]}&=\eta_\nu-(\pi_\nu^{[0]}-P_h^{[0]}(t_\nu)) \\ \xi_\nu^{[1]}&=\xi_\nu-(\omega_\nu^{[0]}-Q_h^{[0]}(t_\nu)). \end{aligned}$$

In the next step, $\eta_\nu^{[1]}$ and $\xi_\nu^{[1]}$ are interpolated by the functions $P_h^{[1]}$ and $Q_h^{[1]}$ and a new (NP) will be constructed. This (NP) is solved by the same implicit Euler method on the same grid G, which leads to the solution$\pi^{[1]}$ and $\omega^{[1]}$ from which $\eta^{[2]}$ and $\xi^{[2]}$ are computed, ans so on. The result of the j^{th} IDeC–step is recursively defined by

$$\eta_\nu^{[j]}=\eta_\nu^{[0]}-(\pi_\nu^{[j-1]}-P_h^{[j-1]}(t))) \tag{2.5a}$$

$$\xi_\nu^{[j]}=\xi_\nu^{[0]}-(\omega_\nu^{[j-1]}-Q_h^{[j-1]}(t))) \tag{2.5b}$$

for $j=1,\dots,j_{\max}$, where $\eta_\nu^{[0]}=\eta_\nu$, $\xi_\nu^{[0]}=\xi_\nu$. The interpolants $P_h^{[j]}(t)$ and $Q_h^{[j]}(t)$ are piecewise polynomials of fixed degree m. We denote here by n the number of subintervals and by m the degree of the interpolation polynomials in each subintervals. Then

$$I_i:=[t^{i-1},t^i]\quad i=1,\dots,n,\quad \text{with } t^i=i\cdot m\cdot h$$

and the piecewise interpolating polynomials are

$$\begin{aligned} P_h^{[j]}(t)&=P_i^{[j]}(t) \\ Q_h^{[j]}(t)&=Q_i^{[j]}(t),\ t\in I_i,\ i=1,\dots,n,\ j=0,\dots,j_{\max}-1. \end{aligned}$$

According to the above definitions, the (NP) is defined piecewise as an initial value problem in the subintervals:

$$\begin{aligned} y'&=F_i^{[j]}(y,z,t) & i&=1,\dots,n \\ 0&=G_i^{[j]}(y,t) & j&=0,\dots,j_{\max}-1. \end{aligned} \tag{2.6}$$

$$y(t^{i-1}) = \begin{cases} y_0 & i = 1 \\ P_{i-1}^{[j]}(t^{i-1}) & i = 2, ..., n, \end{cases} \tag{2.7a}$$

$$z(t^{i-1}) = \begin{cases} z_0 & i = 1 \\ Q_{i-1}^{[j]}(t^{i-1}) & i = 2, ..., n, \end{cases} \tag{2.7b}$$

with the exact solutions $P_i^{[j]}(t)$ and $Q_i^{[j]}(t)$. We consider (2.6) as an IVP in a piecewise fashion with the solutions $P_i^{[j]}(t)$ and $Q_i^{[j]}(t)$. These solutions have jumps in the first derivatives at the end points t^i of the subintervals. This is not contradictory to the assumption that the (OP) and (NP) are problems of the same kind [3], since a smooth original problem is of course a special case of (2.6).

3. Assymptotic expansion of the global error. In this section, we will give the asymptotic expansion of the global error of the linearly implicit Euler method developed in [6], because the traditional analysis of IDeC methods depends on the asymptotic expansion of the global error. The asymptotic expansion of the global error of (NP) is derived for constant stepsize h using the same technique in [6]. According to these, we have the following expansions:

$$\eta_\nu^{[0]} = y(t_\nu) + \sum_{l=1}^{m} h^l a_l(t_\nu) + R_\nu \tag{3.1a}$$

$$\xi_\nu^{[0]} = z(t_\nu) + \sum_{l=1}^{m} h^l b_l(t_\nu) + \tilde{R}_\nu \tag{3.1b}$$

where $a_l(t)$ and $b_l(t)$ are smooth functions and all coefficients vanish for $\nu = 0$, i.e. $a_l(0) = 0$, $b_l(0) = 0$ with $\|R_\nu\| \le const.h^{m+1}$, $\|\tilde{R}_\nu\| \le const.h^{m+1}$, $t \in [0, T]$, $\nu = 0, ..., n \cdot m$.

$a_l(t)$ and $b_l(t)$ are defined as follows:

$$\begin{aligned} a_l'(t) = {} & f_y(y,z)a_l(t) + f_z(y,z)b_l(t) + \\ & + H_l(y, y', ..., y^{(l+1)}, z, z', ..., z^{(l)}, \\ & a_1, a_1', ..., a_1^{(s_1)}, ..., a_{l-1}, a_{l-1}', ..., a_{l-1}^{(s_{l-1})}, \\ & b_1, b_1', ..., b_1^{(k_1)}, ..., b_{l-1}, b_{l-1}', ..., b_{l-1}^{(k_{l-1})}), \end{aligned} \tag{3.2a}$$

$$\begin{aligned} 0 = {} & g_y(y)a_l(t) + \\ & + K_l(y, y', ..., y^{(l)}, a_1, a_1', ..., a_1^{(p_1)}, ..., a_{l-1}, a_{l-1}', ..., a_{l-1}^{(p_{l-1})}) \end{aligned} \tag{3.2b}$$

where $s_1, ..., s_{l-1}, k_1, ..., k_{l-1}, p_1, ..., p_{l-1}$ $(\le l-1)$ denote the order of the derivatives with respect to t and H_l. K_l are sufficiently differentiable functions. Since $(g_y f_z)$ is invertible in a neighbourhood of the solution, by the implicit function theorem and using some backward substituting process, we can rewrite the equations (3.2a) and (3.2b) as follows:

$$\begin{aligned} a_l'(t) &= M_l(y, y', z)a_l(t) + A_l(y, y', ..., y^{(l+1)}, z, z', ..., z^{(l)}) \\ &= \Phi_l(a_l(t), t) \end{aligned} \tag{3.3a}$$

$$\begin{aligned} b_l(t) &= N_l(y, y', z)a_l(t) + B_l(y, y', ..., y^{(l+1)}, z, z', ..., z^{(l)}) \\ &= \Psi_l(a_l(t), t) \end{aligned} \tag{3.3b}$$

with the initial conditions $a_l(0) = 0$, $b_l(0) = 0$, $l = 1, ..., m$ where M_l, A_l, N_l and B_l are smooth functions. We consider now the asymptotic expansion for the (NP)

$$\pi_\nu^{[j]} = P_h^{[j]}(t_\nu) + \sum_{l=1}^{m} h^l a_{l,h}^{[j]}(t_\nu) + R_{\nu,h}^{[j]} \tag{3.4a}$$

$$\omega_\nu^{[j]} = Q_h^{[j]}(t_\nu) + \sum_{l=1}^{m} h^l b_{l,h}^{[j]}(t_\nu) + \tilde{R}_{\nu,h}^{[j]} \tag{3.4b}$$

with $\|R_{\nu,h}^{[j]}\| \le const.h^{m+1}$, $\|\tilde{R}_{\nu,h}^{[j]}\| \le const.h^{m+1}$, and the variational equations for

$$a_{l,h}^{[j]}(t) = a_{l,i}^{[j]}(t),\ b_{l,h}^{[j]}(t) = b_{l,i}^{[j]}(t),\ t \in I_i,\ i = 1, ... n$$

are given by

$$a_{l,i}^{[j]\prime}(t) = \Phi_{l,i}^{[j]}(a_{l,i}^{[j]}(t), t) \tag{3.5a}$$

$$b_{l,i}^{[j]}(t) = \Psi_{l,i}^{[j]}(a_{l,i}^{[j]}(t), t) \tag{3.5b}$$

with the initial conditions

$$a^{[j]}_{l,i}(t^{i-1}) = \begin{cases} 0 & i=1 \\ a^{[j]}_{l,i-1}(t^{i-1}) & i=2,\dots,n \end{cases} \tag{3.6a}$$

$$b^{[j]}_{l,i}(t^{i-1}) = \begin{cases} 0 & i=1 \\ b^{[j]}_{l,i-1}(t^{i-1}) & i=2,\dots,n \end{cases} \tag{3.6b}$$

where the smooth functions $\Phi^{[j]}_{l,i}$, $\Psi^{[j]}_{l,i}$ satisfy the same equations (3.3a,b) substituting $P^{[j]}_i(t)$, $Q^{[j]}_i(t)$ instead of $y(t)$ and $z(t)$.

4. Error analysis.

Theorem. *Let f and g be sufficiently smooth. For an IDeC method based on the Linearly Implicit Euler method and on piecewise interpolation with polynomial of degree m, for non-linear index 2 problem, we have:*

$$\eta^{[j]}_\nu - y(t_\nu) = \mathcal{O}(h^{j+1})$$
$$\xi^{[j]}_\nu - y(t_\nu) = \mathcal{O}(h^{j+1}), \quad j=0,1,\dots,m-1.$$

In this situation further iteration steps do not increase the asymptotic order of the approximation.

Proof. Substracting (3.4a,b) from (3.1a,b), we get

$$\begin{aligned}\|\eta^{[j]}_\nu - y(t_\nu)\| &\le \|\eta^{[0]}_\nu - y(t_\nu)\| + \|\eta^{[j-1]}_\nu - P^{[j-1]}_h(t_\nu)\| \\ &\le \sum_{l=1}^{m} h^l \|a_l(t_\nu) - a^{[j-1]}_{l,h}(t_\nu)\| + \|R_\nu - R^{[j-1]}_{\nu,h}\|\end{aligned}$$

and

$$\|\xi^{[j]}_\nu - z(t_\nu)\| \le \sum_{l=1}^{m} h^l \|b_l(t_\nu) - b^{[j-1]}_{l,h}(t_\nu)\| + \|\tilde R_\nu - \tilde R^{[j-1]}_{\nu,h}\|$$

with

$$\|R_\nu - R^{[j-1]}_{\nu,h}\| \le const.h^{m+1}$$
$$\|\tilde R_\nu - \tilde R^{[j-1]}_{\nu,h}\| \le const.h^{m+1}, \quad \nu = 0,1,\dots,n\cdot m$$

and "*const.*" denotes a constant which depends on the (OP) and (NP) respectively, but is independent of h and ν. From the relations (4.9a,b) of Lemma 4.4, it yields

$$\begin{aligned}\|\eta^{[j]}_\nu - y(t_\nu)\| &\le \sum_{l=1}^{m-j} h^l \|a_l(t_\nu) - a^{[j-1]}_{l,h}(t_\nu)\| + \\ &+ \sum_{l=m-j+1}^{m} h^l \|a_l(t_\nu) - a^{[j-1]}_{l,h}(t_\nu)\| + const.h^{m+1} \le \\ &\le \sum_{l=1}^{m-j} const.h^{l+j} + \sum_{l=m-j+1}^{m} const.h^m + const.h^{m+1} \le \\ &\le const.h^{j+1} + \dots + const.h^m + const.h^{m+1}.\end{aligned}$$

It is seen that the maximum attainable correction step j satisfies the relation $j_{max} = m-1$ for $j = 0,1,\dots,j_{max}$. For the algebraic component z, it can be proved similarly.

Lemma 4.1. *For the non-linear index 2 problem (1.1) and $t \in [0,T] = I$ one has*

$$\max_{t\in I} \left\| \frac{d^k}{dt^k} P^{[0]}_h(t) - \frac{d^k}{dt^k} y(t) \right\| \le \begin{cases} const.h & k=0,1,\dots,m \\ const. & k=m+1 \end{cases} \tag{4.1}$$

$$\max_{t\in I} \left\| \frac{d^k}{dt^k} Q^{[0]}_h(t) - \frac{d^k}{dt^k} z(t) \right\| \le \begin{cases} const.h & k=0,1,\dots,m \\ const. & k=m+1 \end{cases} \tag{4.2}$$

where $P^{[0]}_h(t)$ and $Q^{[0]}_h(t)$ are the m degree polynomials which satisfies the properties

$$\begin{array}{lll} P^{[0]}_h(t) = P^{[0]}_i(t), & Q^{[0]}_h(t) = Q^{[0]}_i(t), & t \in I_i \quad \text{and} \\ P^{[0]}_i(t_\nu) = \eta_\nu, & Q^{[0]}_i(t) = \xi_\nu, & i = 1,2,\dots,n. \end{array}$$

Proof. For the proof of this lemma, we follow similar considerations as in [3] and [10]. Let us define the functions

$$R^{[0]}_h(t) : [0,T] \to \mathbb{R}^\mu \quad \text{with} \quad R^{[0]}_h(t) = R^{[0]}_i(t), \quad i = 1,\dots,n$$

where $R_i^{[0]}(t)$ is a polynomial of degree m which interpolates the R_ν values in the equation (3.1a)

$$R_i^{[0]}(t_\nu) = R_\nu \quad \nu = (i-1)m, \dots, i \cdot m, \; i = 1, 2, \dots, n$$

and let us define the piecewise auxiliary function

$$\varphi_i^{[0]}(t) = y(t) + \sum_{l=1}^{m} h^l a_{l,i}(t) + R_i^{[0]}(t) \tag{4.3}$$

with $\varphi_h^{[0]}(t) = \varphi_i^{[0]}(t), \; i = 1, \dots, n$.

From $\|I\!R_\nu\| \le const.\, h^{m+1}$, it follows after some simple considerations, $\forall i$ and $t \in (t^{i-1}, t^i)$

$$\left\| \tfrac{d^k}{dt^k} R_i^{[0]}(t) \right\| \le const. h^{m+1-k}, \quad k = 0, 1, \dots, m+1, \; t \in (t^{i-1}, t^i), \; i = 1, 2, \dots, n. \tag{4.4}$$

As a consequence, we get

$$\left\| \frac{d^k}{dt^k} \varphi_i^{[0]}(t) - \frac{d^k}{dt^k} y(t) \right\| \le \begin{cases} const. h & k = 0, 1, \dots, m \\ const. & k = m+1 \end{cases} \tag{4.5}$$

At the points t^i, where the k^{th} derivative does not exist, (4.4) and (4.5) hold for the left and right k^{th} derivatives. Since $\varphi_h^{[0]}(t) = \eta_\nu^{[0]}$, $P_h^{[0]}(t)$ interpolates $\varphi_h^{[0]}(t)$ at the points t_ν. Therefore we are able to apply a result due to [7]. For $t \in (t^{i-1}, t^i)$ and $k = 0, \dots, m+1$

$$\left\| \frac{d^k}{dt^k} P_i^{[0]}(t) - \frac{d^k}{dt^k} \varphi_i^{[0]}(t) \right\| \le const. h^{m+1-k}$$

which together with (4.5) leads to our assertion (4.1). The inequalities (4.2) can be proved similarly.

Lemma 4.2. *Let $f, g : I\!R^{n+1} \to I\!R^n$ be arbitrary functions defined on a region $[a, b] \times G$ with $G \subset I\!R^n$ and let $v, w : I\!R^1 \to I\!R^n$ be continuous functions on $[a, b]$, where $(t, v(t)), (t, w(t)) \in ([a, b] \times G)$ for $t \in [a, b]$ with $v(t)$ being the solution of $v' = f(t, v), v(a) = v_a$ and $w(t)$ is the solution of following piecewise problem defined on the partition of $[a, b]$ in subintervals $I_i = [t^{i-1}, t^i]$,*

$$w_i' = g_i(t, w_i), \; t \in I_i, \; w_i(t^{i-1}) = \begin{cases} w_a & i = 1 \\ w_{i-1}(t^{i-1}) & i \ge 2 \end{cases}$$

g_i is defined on $I_i \times G$ and w_i is defined on I_i. Let us further assume that f, g_i are continuous and bounded

$$\|f(t, y_1) - f(t, y_2)\| \le L\|y_1 - y_2\|$$
$$\|f(t, y) - g_i(t, y)\| \le \sigma \quad \text{for } (t, y) \in I_i \times G, \; i = 1, \dots, n$$

where σ is independent of i. Then

$$\|v(t) - w(t)\| \le \|v_a - w_a\| e^{L(t-a)} + \frac{\delta}{L}(e^{L(t-a)} - 1) \quad \text{for } t \in [a, b].$$

Proof. (See [3]) The variational equations of the (NP)'s (3.5a) are defined in a piecewise fashion. For $h \to 0$ the number of the points t^i tends to infinity. Since Lemma 4.2 holds for an arbitrary number of intervals I_i, it may be applied immediately to the variational equations for $a_l(t), a_{l,h}^{[j]}(t)$.

Lemma 4.3. *Let $a_l(t), a_{l,i}^{[j]}(t), b_l(t), b_{l,i}^{[j]}(t)$ satisfy the variational equations (3.3a), (3.5a), (3.3b), (3.5b) respectively, with their initial conditions. Then*

$$\|a_l(t) - a_{l,h}^{[0]}(t)\| \le \begin{cases} const. h & l = 1, 2, \dots, m-1 \\ const. & l = m \end{cases} \tag{4.6}$$

$$\|b_l(t) - b_{l,h}^{[0]}(t)\| \le \begin{cases} const. h & l = 1, 2, \dots, m-1 \\ const. & l = m \end{cases} \tag{4.7}$$

for $t \in [0, T]$.

Proof. Since $\|\Phi(a_l, t) - \Phi(\tilde{a}_l, t)\| \le \|M_l(y, y', z)\| \, \|a_l - \tilde{a}_l\|$ and M_l is a smooth function, i.e. $\|M_l(y, y', z)\| \le L$, $t \in [0, T]$ where L is a constant, $\Phi_l(a_l, t)$ is a Lipschitz function on a closed domain. Consider $t \in (t^{i-1}, t^i)$:

$$\|\Phi(a_l, t) - \Phi_{l,i}^{[0]}(a_l, t)\| \le \|M_l(y, y', z) - M_{l,i}^{[0]}(P_i^{[0]}, P_i^{[0]'}, Q^{[0]})\| \, \|a_l\|$$
$$+ \|A_l(y, \dots, y^{(l+1)}, z, \dots, z^{(l)}) - A_{l,i}^{[0]}(P_i^{[0]}, \dots, P_i^{[0](l+1)}, Q_i^{[0]}, \dots, Q_i^{[0](l)}\|$$

Since $M_l, M_{l,i}, A_l$ and $A_{l,i}$ are smooth functions, using the same considerations in [10], we have

$$\|M_l(y, y', z) - M_{l,i}^{[0]}(P_i^{[0]}, P_i^{[0]'}, Q_i^{[0]})\| \le const. \|y(t) - P_i^{[0]}(t)\|$$
$$+ const. \|y'(t) - P_i^{[0]'}(t)\| + const. \|z(t) - Q_i^{[0]}(t)\|$$

and

$$\|A_l(y,...,y^{(l+1)},z,...,z^{(l)}) - A^{[0]}_{l,i}(P^{[0]}_i,...,P^{[0](l+1)}_i,Q^{[0]}_i,...,Q^{[0](l)}_i)\| \leq$$
$$\leq const.\|y(t) - P^{[0]}_i(t)\| + \cdots + const.\|y^{(l+1)}(t) - P^{[0](l+1)}_i(t)\|$$
$$\leq const.\|z(t) - Q^{[0]}_i(t)\| + \cdots + const.\|z^{(l)}(t) - Q^{[0](l)}_i(t)\|$$

from the relations (4.1),(4.2) in Lemma 4.1, it follows that

$$\|\Phi(a_l,t) - \Phi^{[0]}_{l,i}(a_l,t)\| \leq \begin{cases} const.h\|a_l\| + const.h & l = 1,...,m-1 \\ const.h\|a_l\| + const.h & l = m \end{cases}$$

for $t \in I_i$, $i = 1,...,n$.

Since all solutions $a_i(t)$ are in a certain finite region

$$\|a_l(t)\| \leq S$$

the inequality of the Lemma 4.2 holds and we obtain our assertion (4.6). Since $b_l(t)$ depends on $a_l(t)$ and $N_l, B_l, N^{[0]}_{l,i}, B^{[0]}_{l,i}$ are smooth differentiable, it follows that, $t \in I_i$, $i = 1,...,n$

$$\|b_l(t) - b^{[0]}_{l,i}(t)\| \leq \|N_l(y,y',z) - N^{[0]}_{l,i}(P^{[0]}_i, P^{[0]'}_i, Q^{[0]}_i)\| \; \|a_l\| +$$
$$+\|N^{[0]}_{l,i}(P^{[0]}_i, P^{[0]'}_i, Q^{[0]}_i)\| \; \|a_l - a^{[0]}_{l,i}\| +$$
$$+\|B_l(y,...,y^{(l+1)},z,...,z^{(l+1)}) - B^{[0]}_{l,i}(P^{[0]},...,P^{[0](l+1)}_i,Q^{[0]}_i,...,Q^{[0](l+1)}_i)\|$$
$$\leq const.\|y(t) - P^{[0]}_i(t)\| + \cdots$$
$$+ const.\|y^{(l+1)}(t) - P^{[0](l+1)}_i(t)\|$$
$$+ const.\|z(t) - Q^{[0]}_i(t)\| + \cdots$$
$$+ const.\|z^{(l)}(t) - Q^{[0](l)}_i(t)\|$$
$$+ const.\|a_l - a^{[0]}_{l,i}\|.$$

So the relations (4.6), (4.1) and (4.2) yields (4.7).

Lemma 4.4. *For the same problem of Lemma 4.1, with $j = 0,1,...,m-1$, $I = [0,T]$, one has*

$$\max_{t\in I}\left\|\frac{d^k}{dt^k}P^{[j]}_h(t) - \frac{d^k}{dt^k}y(t)\right\| \leq$$
$$(4.8a) \qquad \leq \begin{cases} const.h^{j+1} & k = 0,1,...,m-j \\ const.h^{m+1-k} & k = m-j+1,...,m+1 \end{cases}$$

$$\max_{t\in I}\left\|\frac{d^k}{dt^k}Q^{[j]}_h(t) - \frac{d^k}{dt^k}z(t)\right\| \leq$$
$$(4.8b) \qquad \leq \begin{cases} const.h^{j+1} & k = 0,1,...,m-j-1 \\ const.h^{m-k} & k = m-j,...,m-1 \\ const.h & k = m \\ const. & k = m+1 \end{cases}$$

$$(4.9a) \qquad \|a_l(t) - a^{[j]}_{l,h}(t)\| \leq \begin{cases} const.h^{j+1} & l = 1,2,...,m-j-1 \\ const.h^{m-l} & k = m-j,...,m \end{cases}$$

$$(4.9b) \qquad \|b_l(t) - b^{[j]}_{l,h}(t)\| \leq \begin{cases} const.h^{j+1} & l = 1,2,...,m-j-1 \\ const.h^{m-l} & k = m-j,...,m \end{cases}$$

where $P^{[j]}_h(t)$ and $Q^{[j]}_h(t)$ are the piecewise polynomials of fixed degree m, $a_l(t), a^{[j]}_{l,h}(t), b_l(t), b^{[j]}_{l,h}(t)$ satisfy the equations (3.3a), (3.5a), (3.3b), (3.5b) *with the associated initial conditions.*

Proof. They will be proved by induction. When $j = 0$, all inequalities are proved in Lemma 4.1 and 4.2, respectively. Suppose these inequalities are true for $j < m-1$. Similar idea will be used, but the auxiliary function defined in Lemma 4.1 is arranged as:

$$\varphi^{[m-1]}_i(t) = y(t) + \sum_{l=1}^{m} h^l(a_l(t) - a^{[m-2]}_{l,i}(t)) + R^{[m-1]}_i(t), \; t \in I_i$$

where

$$R^{[m-1]}_i(t_\nu) = R_\nu - R^{[m-1]}_\nu, \; \nu = (i-1)m,...,m, \; i = 1,2,...,n$$
$$\left\|\frac{d^k}{dt^k}R^{[m-1]}_i(t)\right\| \leq const.h^{m+1-k}, \; k = 0,1,...,m+1.$$

By differentiation with respect to t of the variational equations (3.3a) and (3.5a) for $j = m-2$, we get (the derivatives at the end points of I_i are considered as the left and right derivatives):

$$a^{(k)}_l(t) = \tilde{M}_l(y,y',z)a_l(t) + \tilde{A}_l(y,y',...,y^{(l+k)},z,z',...,z^{(l+k-1)})$$
$$a^{[m-2](k)}_{l,i}(t) = \tilde{M}^{[m-2]}_{l,i}(P^{[m-2]}_i, P^{[m-2]'}_i, Q^{[m-2]}_i)a^{[m-2]}_{l,i}(t) +$$
$$+ \tilde{A}^{[m-2]}_l(P^{[m-2]}_i,...,P^{[m-2](l+k)}_i,Q^{[m-2]}_i,...,Q^{[m-2](l+k-1)}_i)$$

Since the effective term is the functions defined with the maximum order of the derivative, as a consequence we get:

$$\|a_l^{(k)}(t) - a_{l,i}^{[m-2](k)}(t)\| \le const.\|y^{(l+k)}(t) - P_i^{[m-2](l+k)}(t)\|$$

so from the inequalities (4.8a) for $j = m-2$, we write

$$\begin{aligned}
&\left\| \frac{d^k}{dt^k}\varphi_i^{[m-1]}(t) - \frac{d^k}{dt^k}y(t)\right\| \le \\
&\le \sum_{l=1}^{m} const.h^l\|y^{(l+k)}(t) - P_i^{[m-2](l+k)}(t)\| \\
&+const.h^{m+1-k} \le \sum_{l=1}^{2-k} const.h^{l+m-1} + \sum_{l=3-k}^{m} const.h^{m+1+l-k} \\
&+const.h^{m+1-k} \le const.h^m + \cdots + const.h^{m+1-k} \le \\
&\le \begin{cases} const.h^m & k = 0,1 \\ const.h^{m+1-k} & k = 2,..,m+1. \end{cases}
\end{aligned} \tag{4.10}$$

It is obvious that $\varphi_i^{[m-1]}(t_\nu) = \eta_\nu^{[m-1]} = P_i^{[m-1]}(t_\nu)$, $\nu = (i-1)m, \ldots, im$, $i = 1, \ldots, n$. Then from [7], we have again, $t \in I_i$

$$\left\| \frac{d^k}{dt^k}\varphi_i^{[m-1]}(t) - \frac{d^k}{dt^k}P_i^{[m-1]}(t)\right\| \le const.h^{m+1-k}, \quad for\ k = 0, 1, \ldots, m+1. \tag{4.11}$$

(4.10) and (4.11) give (4.8a) for $j = m-1$. The other assertions (4.8b), (4.9a), (4.9b) can be proved similarly using the same techniques in Lemma 4.1 and 4.3 by induction for $j = 0, 1, \ldots, m-1$.

5. Numerical results. We consider the following test example of index 2:

$$\begin{array}{ll}
y_1' = y_2 + 2y_4 + y_3y_6 & y_1(0) = 0 \\
y_2' = y_4 - y_2y_6 & y_2(0) = 1 \\
y_3' = -y_5y_1 & y_3(0) = -1 \\
y_4' = -y_5y_2 & y_4(0) = 0 \\
0 = 1 - y_1^2 - y_2^2 & y_5(0) = -1 \\
0 = y_1y_3 + y_2y_4 & y_6(0) = 0
\end{array}$$

with the exact solution $y_1 = \sin t$, $y_2 = \cos t$, $y_3 = -\cos t$, $y_4 = \sin t$, $y_5 = -1$, $y_6 = 2\tan t$.

The global connection strategy and piecewise polynomials of degree $m = 4$ are used. The predicted orders of convergence for test problem can be derived using Theorem. Here LIE denotes linearly implicit Euler solutions and 1.DC, 2.DC, 3.DC, the first, the second and the third defect correction solutions respectively. The numerical results using constant stepsize and observed orders which are computed by $p = \log(e_n/e_{n+1})/\log(h_n/h_{n+1})$ are listed for each component in the following tables at $t = 0.4$. (e_n, e_{n+1} are the global errors depending on the stepsize h_n, h_{n+1} respectively). The observed orders are in good agreement with the predicted ones. All computations are performed on a 80486 Computer in NDP486 FORTRAN in DOUBLE PRECISION.

Table 1. Global error and observed orders for y_1.

h	LIE	1.DC	2.DC	3.DC
.05	.278D-01	.162D-02	.133D-03	.390D-04
.025	.137D-01	.380D-03	.162D-04	.233D-05
.0125	.683D-02	.924D-04	.199D-05	.145D-06
.00625	.340D-02	.228D-04	.245D-06	.917D-08
	Observed Orders			
	1.020	2.091	3.037	4.065
	1.004	2.040	3.025	4.006
	1.006	2.018	3.021	3.982

Table 2. Global error and observed orders for y_2.

h	LIE	1.DC	2.DC	3.DC
.05	.101D-01	.998D-03	.285D-04	.183D-04
.025	.495D-02	.244D-03	.479D-05	.582D-06
.0125	.245D-02	.604D-04	.645D-06	.304D-07
.00625	.121D-02	.150D-04	.831D-07	.165D-08
	Observed Orders			
	1.028	2.032	2.572	4.974
	1.014	2.014	2.892	4.258
	1.017	2.009	2.956	4.203

Table 3. Global error and observed orders for y_5.

h	LIE	1.DC	2.DC	3.DC
.05	.137D-01	.106D-02	.420D-03	.170D-03
.025	.531D-02	.287D-03	.622D-05	.380D-05
.0125	.232D-02	.788D-04	.102D-05	.215D-06
.00625	.109D-02	.207D-04	.1395D-06	.159D-07
		Observed Orders		
	1.367	1.884	6.077	5.483
	1.194	1.864	2.608	4.143
	1.089	1.928	2.875	3.758

Table 4. Global error and observed orders for y_6.

h	LIE	1.DC	2.DC	3.DC
.05	.428D-01	.187D-02	.751D-04	.386D-04
.025	.221D-01	.373D-03	.127D-04	.357D-05
.0125	.112D-01	.825D-04	.142D-05	.179D-06
.00625	.564D-02	.194D-04	.173D-06	.107D-07
		Observed Orders		
	0.953	2.325	2.563	3.434
	0.980	2.176	3.160	4.317
	0.989	2.088	3.037	4.064

Since y_3 and y_4 are the differential components of DAE, the results are similar to the results for y_1 and y_2.

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