Integrals (9), (10) are the integrals of the model studied in [1],[2] and therefore our problem is solved.

Remarks:

- the variational problem (4) is equivalent with

$$\delta\int B\left(H-\frac{\mu}{r}\right)\left[V^2\left(\frac{dt}{ds}\right)^2-\left(\frac{dr}{ds}\right)^2-r^2\left(\frac{d\theta}{ds}+\omega\frac{dt}{ds}\right)^2-\right.$$
$$\left.-r^2\cos^2\theta\left(\frac{d\varphi}{ds}\right)^2\right]ds=0,$$

  where the parametrization is natural.
- By a carefull analysis it may arrive to think about an equivalence between the relativity and the Newtonian theory of gravitation (?!).
- The integrals (9) and (10) hold even for ω function of time.
- There seems to be some difficulty in the explanation of the advance of perihelion for a planet by using Schwarzschild's model: the corresponding non–inertial frame of this model has a rotation depending on the motion of planet (?!).

REFERENCES

1. B o r ş, C.I. – *Le problème des deux corps dans un repère non galiléen. L'avance de périhélie. Le mouvement des comètes.* An.şt. Univ. "Al.I.Cuza", Iaşi, ser. nouă, sect.I, Mat., t.XXXIII, f.2 (1987), p.185–188.
2. B o r ş, C.I. – *Two body problem in a non-inertial frame of reference and the advance of perihelion.* An.şt. Univ. "Al.I.Cuza", Iaşi, ser. nouă, sect.I, Mat., t.XXXVIII, f.1 (1992), p.23–27.
3. B o r ş, C.I. – *Two body problem in a non-inertial frame of reference and the theory of relativity.* Lucrările Seminarului Matematic "Al.Myller", 3/1994.
4. C h a z y, J.– *La théorie de la relativité et la mécanique célesic.* Gauthier-Villars, t.I, 1928.
5. C h a z y, J.– *Mécanique céleste.* Presses universitaires de France, Paris, 1953.

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RELATIVISTIC MODELS FOR TWO BODY PROBLEM

BY

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After Einstein, it is possible to describe gravitational effects either in terms of a flat space with a gravitational field present, or in terms of a curved four dimensional space without a gravitational field. The second alternative brings geometry back to an experimental science for which the problem is to determine the geometry of the physical world [1].

The fundamental assumption of the theory of relativity is that a free particle will move along a geodesic in four–dimensional space–time.

From the multitude of metrics used in order to solve the two body problem we mention the metric of Schwarzschild

$$ds^2=\left(V^2-\frac{2\mu}{r}\right)dt^2-\frac{dr^2}{1-\frac{2\mu}{V^2r}}-r^2(d\theta^2+\cos^2\theta d\varphi^2)$$

and the metric of de Sitter

$$ds^2=\left(V^2-\frac{2\mu}{r}-\frac{r^2}{R^2}\right)dt^2-\frac{dr^2}{1-\frac{2\mu}{V^2r}-\frac{r^2}{R^2}}-r^2(d\theta^2+\sin^2\theta d\varphi^2).$$

In the above formulae V,μ,R are constants which characterize the models, V being the speed of light [4].

In the following we will agree that the selection of allowed Riemannian geometries, from among merely conceivable Riemannian geometries, demands that the motion so obtained will reduce to the Newtonian predictions [6]. For the problem in discussion, we will build up some models.

Let us consider ds^2 defined by

$$ds^2=B\left(H-\frac{\mu}{r}\right)[V^2dt^2-dr^2-r^2(d\theta^2+\cos^2\theta d\varphi^2)],\tag{1}$$

where B, H, μ, V are constants.

The geodesics of this linear element are given by

$$\delta \int \sqrt{\gamma}\, dt = 0, \tag{2}$$

where

$$\gamma = B\left(H - \frac{\mu}{r}\right)\left[V^2 - \dot{r}^2 - r^2(\dot{\theta}^2 + \cos^2\theta\dot{\varphi}^2)\right].$$

Since the function γ does not involve φ, from the Euler–Lagrange equations we have

$$\frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right) r^2 \cos^2\theta\dot{\varphi} = M, \tag{3}$$

where M is a constant.

From (3) it results $\varphi = const.$ and consequently the trajectory remains in the plane determined by the initial conditions [3],[4]. But in this case $\dot{\varphi} = 0$ and the function γ does not involve θ; evidently, the function γ does not depend on t.

Consequently, from (2) we get also the integrals

$$\frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right) r^2\dot{\theta} = N, \tag{4}$$

$$\sqrt{\gamma} + \frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right)(\dot{r}^2 + r^2\dot{\theta}^2) = A, \tag{5}$$

where N and A are constants determined by initial conditions.

Equation (5) can be written as

$$\frac{1}{\sqrt{\gamma}} BV^2\left(H - \frac{\mu}{r}\right) = A \tag{6}$$

and using (4) we obtain the integral of angular momentum

$$r^2\dot{\theta} = C. \tag{7}$$

Equation (6) gives us the integral of energy which may be written in the form

$$\dot{r}^2 + r^2\dot{\theta}^2 = \frac{2\mu}{r} + 2h \tag{8}$$

if we choose $B = 2\frac{A^2}{V^4}$, h being a constant.

Further on, let us also consider the case when ds^2 is of the form

$$ds^2 = B\left(H - \frac{\mu}{r}\right)\left[V^2dt^2 - dr^2 - r^2(d\theta + \omega dt)^2 - r^2\cos^2\theta d\varphi^2\right], \tag{9}$$

where V, B, H, μ, ω are constants defining another relativistic model; the rotation ω could play an important role in connection with Mach's principle [5].

The geodesics of (9) are given by

$$\delta \int \sqrt{\gamma}\, dt = 0, \tag{10}$$

where

$$\gamma = B\left(H - \frac{\mu}{r}\right)\left[V^2 - \dot{r}^2 - r^2(\dot{\theta} + \omega)^2 - r^2\cos^2\theta\dot{\varphi}^2\right].$$

As well as in the case (2) the trajectory is plane and we have the following integrals

$$\frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right) r^2(\dot{\theta} + \omega) = N, \tag{11}$$

$$\frac{1}{\sqrt{\gamma}} B\left(H - \frac{\mu}{r}\right)\left[V^2 - r^2\omega(\dot{\theta} + \omega)\right] = A. \tag{12}$$

From these integrals we get

$$\frac{r^2(\dot{\theta} + \omega)}{V^2 - r^2\omega(\dot{\theta} + \omega)} = \frac{N}{A}$$

and hence we obtain

$$r^2(\dot{\theta} + \omega) = C. \tag{13}$$

The integral (12) is equivalent to

$$\begin{aligned} &\dot{r}^2 + r^2(\dot{\theta} + \omega)^2 = \frac{2\mu}{r} + 2h, \\ &B = \frac{2A^2}{(V^2 - C\omega)^2}. \end{aligned} \tag{14}$$

Integrals (7), (8) are the equations of the theory of motion in a Newtonian field in a frame of reference with fixed directions with respect to an inertial frame of reference.

Integrals (13), (14) are the equations of the theory of motion for a Newtonian field in a non–inertial frame of reference [2].

On the other hand, let us consider the model of the general theory of motion in a central field depending upon r only:

$$ds^2 = B(H - f(r))\left[V^2dt^2 - dr^2 - r^2(d\theta^2 + \cos^2\theta d\varphi^2)\right], \tag{15}$$

and we have

$$r^2\dot\theta = C,$$
$$\dot r^2 + r^2\dot\theta^2 = 2f(r) + 2h,$$
$$F = -f'(r).$$

In a non–inertial frame of reference

$$ds^2 = B(H - f(r))\left[V^2dt^2 - dr^2 - r^2(d\theta + \omega dt)^2 - r^2\cos^2\theta d\varphi^2\right]. \tag{16}$$

The equations of motion for a particle in a conservative field, and in a inertial frame of reference, are

$$m\ddot x_i = \frac{\partial U}{\partial x_i}$$

and they are the equations of the geodesics of

$$ds^2 = B\left(H - \frac{2}{m}U\right)(V^2dt^2 - dx_1^2 - dx_2^2 - dx_3^2).$$

From the variational problem

$$\delta\int\sqrt{B(H-U)(mV^2-2E)}\,dt = 0,\ 2E = g_{ij}\dot q^i\dot q^j,$$

we obtain the equations of Lagrange

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q^h}\right) - \frac{\partial E}{\partial q^h} = \frac{\partial U}{\partial q^h},\quad (h = \overline{1,n}),$$

for a scleronomic conservative system with n degrees of freedom and mass m.

Some remarks:

- One may say, in our problem we have models and we can look to them from the Newtonian world and from a relativistic point of view.
- Usually, in two body problem, the results of the Newtonian theory are considered as the first approximation of the theory of relativity. According to our result, it seems to be a question of comparison between different models.
- A lot of Newtonian models have their correspondents among relativistic models and vice versa. Perhaps this is the way to establish an equivalence between the theory of relativity and the Newtonian mechanics (?!).
- Metrics used in this paper may be useful for an unified theory of gravitation and electrodynamics. The case $f(r) = 0$ has (for (15) and (16)) a specific significance.

REFERENCES

1. A n d e r s o n, J.L. – *Riemannian geometry. Gravitation and relativity*, ed. by H.–Y. Chiu and W.F. Hoffmann, W.A. Benjamin, Inc., New York. Amsterdam, 1964.
2. B o r ş, C.I. – *Le problème des deux corps dans un repère non galiléen. L'avance de périhélie. Le mouvement des comètes.* An.şt. Univ. "Al.I.Cuza", Iaşi, ser. nouă, sect.I, Mat., t.XXXIII, f.2 (1987), p.185–188.
3. B o r ş, C.I. – *Two body problem in a non-inertial frame of reference and the theory of relativity.* (in press)
4. C h a z y, J.– *La théorie de la relativité et la mécanique céleste.* Gauthier-Villars, t.I-1928, t.II-1933.
5. D i c k e, R.H. – *The many faces of Mach. Gravitation and relativity*, ed. by H.–Y. Chiu and W.F. Hoffmann, W.A. Benjamin, Inc., New York. Amsterdam, 1964.
6. W h e e l e r, J.A. – *Gravitation as geometry – II. Gravitation and relativity*, ed. by H.-Y. Chiu and W.F. Hoffmann, W.A. Benjamin, Inc., New York. Amsterdam, 1964.

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