solutions create sets with special interest in the Theory of Languages and Automata.

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Received: 20.IX.1994

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ON ALGEBRAS OF BOL–MOUFANG TYPE

BY

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Introduction. Loops and algebras satisfying the Bol–Moufang type identities have received attention from numerous authors in various directions. These identities include Moufang identities, Extra identities, Bol identities to mention but a few [2], [3], [4], [5], [7], [8], [9], [10].

In this paper we shall study algebras satisfying the following identities

(1a, b, c) $xy\cdot zx=(x\cdot yz)x;\ (xy\cdot z)y=x(y\cdot zy);\ x(y\cdot xz)=(xy\cdot x)z$

(2a, b, c) $(xy\cdot z)x=x(y\cdot zx);\ yx\cdot zx=(y\cdot xz)x;\ xz\cdot xy=x(zx\cdot y)$

(3) $$(xy\cdot z)y=x(yz\cdot y)$$

(4) $$y(z\cdot yx)=(y\cdot zy)x.$$

Identities in (1) are called Moufang identities, and any loop satisfying one of them satisfies the others.

Identities in (2) are called Extra identities and loop satisfying any one of them satisfies the others.

Identities in (3) and (4) are called Bol identity and left Bol identity respectively, there is duality between the two identities. We study the properties of algebras generated by the loops satisfying the above identities.

H. P f l u g f e l d e r [7] calls a loop $Q(\cdot)$ an M–loop provided there is a mapping λ of $Q(\cdot)$ into $Q(\cdot)$ such that

(5) $$xy\cdot zx^{\lambda}=(x\cdot yz)x^{\lambda}$$

for all $x,y,z\in Q(\cdot)$. C h e i n and R o b i n s o n [3], S h a r m a [8] have established the connection between the identity (5) and the following identities

(6) $$(xy\cdot z)x^{\alpha}=x(y\cdot zx^{\alpha})$$

(7) $$(yx\cdot z)x^{\lambda}=y(x\cdot zx^{\lambda}).$$

We shall obtain similar results for certain algebras some of the results in Chein and Robinson [3], Sharma [8] are obtained as consequences of our results on these algebras.

Associated with a (local) Lie loop G is a tangent algebra $L(a)=T_e(a)$ which carries a bilinear anticommutative multiplication $(x,y) \to [x,y]$, called the commuttor bracket and a trilinear operation $(x,y,z) \to (x,y,z)$ called associator bracket vanishing identically in the case of a (local) Lie group. The two operations are linked by the Aktivis identity

$$\Sigma\{\mathrm{sgn}(\sigma)\cdot\sigma(x.y.z) : \sigma \in S_3\} = [[x,y].z] + [[y,z],x] + [[z,x],y].$$

We call an algebra with a binary and a ternary multiplication satisfies the Aktivis identity and Aktivis Algebra. In particular the tangent algebra of a (local) Lie loop is an Aktivis Algebra.
In this paper we define a quadrilinear operation $(x,y,z,w) \to < x,y,z,w >$ called extra bracket and give an identity linking it with the commutator bracket. This identity vanishes identically in case of algebras satisfies the extra identity and other Bol–Moufang identities.

2. Definitions. In what follows $Q(\cdot)$ is a loop whose identity element is denoted by 1 and whose left nucleus, middle nucleus, right nucleus and nucleus are denoted by N_λ, N_μ, N_ρ and N, respectively. For each $x \in Q$ let xJ be that unique element in Q with the property that $x \cdot xJ = 1$. Clearly, J is a one-to-one mapping of Q onto Q so the inverse mapping J^{-1} exists. For each $x \in Q$, xJ^{-1} is that unique element in Q so that $xJ^{-1}\cdot x = 1$. For the most part we have employed notation and terminology consistent with that of R.H. Bruck [2]. A is an algebra (non–associative) over a field F with an identity element 1. It will often be required that A satisfy identities of Bol–Moufang type, that is some generalization of the associativity identity.

3. Main results.

Theorem 1. *Let A be an algebra over F, then*

(i) *A is a Bol algebra if and only if*

(8) $$(xw\cdot z)y + (xy\cdot z)w - x(wz\cdot y) - x(yz\cdot w) = 0$$

(ii) *A is a Moufang algebra if and only if*

(9) $$xy\cdot zw + wy\cdot zx - (x\cdot yz)w - (w\cdot yz)x = 0 \quad or$$
(10) $$(xy\cdot z)w - x(y\cdot zw) + (xw\cdot z)y - x(w\cdot zy) = 0 \quad or$$
(11) $$x(y\cdot wz) - (xy\cdot w)z + w(y\cdot xz) - (wy\cdot x)z = 0$$

(iii) *A is an Extra algebra if and only if*

(12) $$(y\cdot xz)w - yx\cdot zw + (y\cdot wz)x - yw\cdot zx = 0 \quad or$$
(13) $$(xy\cdot z)w + (wy\cdot z)x - x(y\cdot zw) - w(y\cdot zx) = 0 \quad or$$
(14) $$w(zx\cdot y) - wz\cdot xy + x(zw\cdot y) - xz\cdot wy = 0$$

for all $x,y,z,w \in A$.

Proof. (i) Let A be a Bol algebra, that is A satisfies (3) for all $x,y,z \in A$. A satisfies (3) if and only if

(15) $$(x,y,z)y + (x,yz,y) = 0 \quad \text{for all } x,y,z \in A.$$

Linearizing (15) by substituting $y = y+w$, we obtain $(x,w,z)y+(x,y,z)w+ +(x,yz,w) + (x,wz,y) = 0$. This implies that

(16) $$(xw\cdot z)y + (xy\cdot z)w - x(yz\cdot w) - x(wz\cdot y) = 0.$$

Now let A be an algebra satisfies (8), substituting $w = y$ in (8), we obtain

(17) $$(xy\cdot z)y + (xy\cdot z)y - x(yz\cdot y) - x(yz\cdot y) = 0.$$

This implis that $(xy\cdot z)y = x(yz\cdot y)$ for all $x,y,z \in A$.

Following similar procedures as in (i) and linearizing identities in (1a,b,c) and (2a,b,c), we obtain the results in (ii) and (iii), respectively. This completes the proof of the Theorem.

Theorem 2. *Let A be the loop algebra generated by a Bol loop L, then A is Bol algebra. (Respectively, Moufang, Extra, Left Bol Loops).*

Proof. Let A be the loop algebra generated by the Bol loop L. Then $x,y,z \in A$ are of the form

$$x = \sum_{i=1}^{n_1}\alpha_i x_i,\ y = \sum_{j=1}^{n_2}\beta_j y_j,\ z = \sum_{k=1}^{n_3}\gamma_k z_k,\ x_i,y_j,z_k \in L,$$

and $\alpha_i,\beta_j,\gamma_k \in F$,

(18) $$(xy\cdot z)y = \Sigma\alpha_i\beta_j\gamma_k\beta_q(x_iy_jz_k)y_q$$

for all $x_i,y_j,z_k,y_q \in L$ and $\alpha_i,\beta_j,\gamma_k,\beta_q \in F$.

(19) $$x(yz\cdot y) = \Sigma\alpha_i\beta_j\gamma_k\beta_q x_i(y_jz_k\cdot y_q)$$

for all $x_i,y_j,z_k,y_q \in L$ and $\alpha_i,\beta_j,\gamma_k,\beta_q \in F$.

Now (18) can be written as

$$\sum_{j=q} \alpha_i \beta_j^2 \gamma_k (x_i y_j \cdot z_k) y_j + \sum_{j \neq q} \alpha_i \beta_j \gamma_k \beta_q [(x_i y_j \cdot z_k) y_q + (x_i y_q \cdot z_k) y_j].$$

Also (19) can be written as

$$\sum_{j=q} \alpha_i \beta_j^2 \gamma_k x_i (y_j z_k \cdot y_j) + \sum_{j \neq q} \alpha_i \beta_j \gamma_k \beta_q [x_i (y_j z_k \cdot y_q) + x_i (y_q z_k \cdot y_j)].$$

Now by (3) and (8), $(x_i y_j \cdot z_k) y_j = x_i (y_j z_k \cdot y_j)$;

$$(x_i y_j \cdot z_k) y_q + (x_i y_q \cdot z_k) y_j - x_i (y_j z_k \cdot y_q) - x_i (y_q z_k \cdot y_j) = 0.$$

This implies that $(x_i y_j \cdot z_k) y_q + (x_i y_q \cdot z_k) y_j = x_i (y_j z_k \cdot y_q) + x_i (y_q z_k \cdot y_j)$. Therefore (18) = (19). That is $(xy \cdot z)y = x(yz \cdot y)$ for $x, y, z \in A$. Hence A is a Bol algebra. This completes the proof of the theorem.

The above theorem holds for Moufang and Extra identities since they are special types of Bol loops.

Theorem 3. *In an Extra Algebra A the following are true*

(20) (i) $R_y R_z R_x + L_{xy \cdot z} - R_{(y \cdot zx)} - L_z L_y L_x = 0$, *for all* $x, y, z \in A$.
(21) (ii) $R_y R_z R_{y^{-1}} = R_z$ *for all* $y, z \in A^*$.
(22) (iii) $L_z L_y L_{z^{-1}} = L_y$ *for all* $y, z \in A^*$.
(iv) A^* *is a* G*--loop.*
(v) $x^2 \in N$ *for all* $x \in A$.
(vi) A *satisfies the* M_3*--law.*

where A^* is the loop of units in A.

Proof. (i) Let A be an Extra algebra. From (13) we obtain

$xR_y R_z R_w + xL_{wy \cdot z} - xR_{y \cdot zw} - xL_z L_y L_w = 0$

that implies that (20) holds for all $x, y, z \in A$.

(ii) Let $x = y^{-1}$ in (20), we obtain $R_y R_z R_{y^{-1}} - L_{y^{-1}y \cdot z} - R_{y \cdot zy^{-1}} - L_z L_y L_{y^{-1}} = 0$ (we observe that A^* is an extra loop). This implies that $R_y R_z R_{y^{-1}} - L_z - R_z - L_z = 0$ that is (21).

(iii) Similarly, let $x = z^{-1}$ in (20), we obtain $R_y R_z R_{z^{-1}} + L_{z^{-1}y \cdot z} - R_{(y \cdot zz^{-1})} - L_z L_y L_{z^{-1}} = 0$. This implies that $L_z L_y L_{z^{-1}} = L_z$.

(iv) The results in (ii) and (iii) imply that A^* is conjugacy closed. G o o d a i r e and R o b i n s o n [6] has proved that every conjugacy

loop is a G-loop, that is, it is isomorphic to all its loop isotopes. Therefore A^* is a G-loop.

(v) Substituting $w = z$ in (13), we obtain $(xy \cdot z)z + (zy \cdot z)x - x(y \cdot zz) - z(y \cdot zx) = 0$. This implies that $xy \cdot z^2 + (zy \cdot z)x - x \cdot yz^2 - z(y \cdot zx) = 0$. Applying (1), we obtain $xy \cdot z^2 - x \cdot yz^2 = 0$. This implies that $xy \cdot z^2 = x \cdot yz^2$ for all $x, y, z \in A$. Thus $z^2 \in N_\rho = N$.

(vi) Let A be an Extra algebra.

$$\begin{aligned}(x \cdot yz)x^3 &= [(x \cdot yz)x]x^2 = (xy \cdot zx)x^2 = xy(zx \cdot x^2) \quad (x^2 \in N) \\ &= xy \cdot zx^3.\end{aligned}$$

Thus $xy \cdot zx^3 = (x \cdot yz)x^3$ for all $x, y, z \in A$, implies that A satisfies the M_3-law.

Corollary 1. *Let Q be an extra loop, then*

(i) Q is a G-loop; (ii) $x^2 \in N$ for all $x \in Q$; (iii) Q satisfies the M_3-law.

Proof. Let Q be an extra loop. By Theorem 2 $Q \subset A^*$ for some extra algebra A. The result follows from Theorem 3. The results in this corollary are well known F e r y v e s [4].

Theorem 4. *Let A be an extra algebra. The following identities hold in A^*, for all $x, y, z \in A^*$.*

(i)	$(xy \cdot z)x^{-1} = x(y \cdot zx^{-1})$;	(ii)	$(xy \cdot z)y^{-1} = xz$ (WRIP);
(iii)	$(xy \cdot z)z^{-1} = xy$	(iv)	$x^{-1}(xy \cdot z) = yx \cdot x^{-1}z$;
(v)	$y^{-1}(xy \cdot z) = y^{-1}x \cdot yz$	(vi)	$z^{-1}(xy \cdot z) = z^{-1}x \cdot yz = xy$;
(vii)	$(x \cdot yz)x^{-1} = xy \cdot zx^{-1} = yz$	(viii)	$(x \cdot yz)y^{-1} = xy \cdot zy^{-1}$;
(ix)	$(x \cdot yz)z^{-1} = zx^{-1} \cdot zy$	(x)	$x^{-1}(x \cdot yz) = yz$;
(xi)	$y^{-1}(x \cdot yz) = xz$ (WLIP)	(xii)	$z^{-1}(x \cdot yz) = (z^{-1}x \cdot y)z$;

Proof. It should be observed that Extra identities imply the Moufang identities, while the Moufang identities $\Longrightarrow$ Bol identity. (Extra identity $\Longrightarrow$ Moufang identities $\Longrightarrow$ Bol identity)

(i) Substituting $w = x^{-1}$ in (10), we obtain

$$(xy \cdot z)x^{-1} - x(y \cdot zx^{-1}) + (xx^{-1} \cdot z)y - x(x^{-1} \cdot zy) = 0$$

that is $(xy \cdot z)x^{-1} - x(y \cdot zx^{-1}) = 0$. This implies that $(xy \cdot z)x^{-1} = x(y \cdot zx^{-1})$. This result also holds for Moufang algebra.

Similarly, following the above procedure by substituting $w = y^{-1}$ in (13); $w = z^{-1}$ in (14); $w = y^{-1}$ in (14); $w = y^{-1}$ in (12);$w = y^{-1}$ in (9);

$w = z^{-1}$ in (12); $w = z^{-1}$ in (13); $w = z^{-1}$ in (11); we obtain (ii); (iv); (vi); (vii); (viii); (ix); (xi); (xii); respectively for all $x, y, z \in A^*$.

We may call (ii) weak right inverse property (WRIP).

(vii) implies that every extra loop satisfies the M_k–law where $k = -1$.

(iii) Follows from the right inverse property (RIP).

(v) This result is dual to result number (viii).

(x) Follows from the left inverse property (LIP).

S h a r m a [9] studied the algebraic properties of loops satisfying the identical relation

$$(22)' \qquad x^\sigma(y \cdot xz) = (x^\sigma \cdot yx)z$$

called half–Bol loops, the dual of (22)'

$$(23) \qquad (xy \cdot z)y^\sigma = x(yz \cdot y^\sigma)$$

is called generalized Bol identity (or B_σ–law).

Loops satisfying (23) are called generalized Bol loops. Here we show that Extra loops are generalized Bol loops.

Theorem 5. *Let A be an extra algebra. Then (13) holds.*

Proof. Substituting $w = y^\sigma$ in (13) we obtain

$$(xy \cdot z)y^\sigma + (y^\sigma y \cdot z)x - x(y \cdot zy^\sigma) - y^\sigma(y \cdot zx) = 0.$$

This implies that

$$(24) \qquad (xy \cdot z)y^\sigma + (y^\sigma y \cdot z)x - x(yz \cdot y^\sigma) - y^\sigma y \cdot zx = 0.$$

If A satisfies (23) then (24) becomes $(y^\sigma y \cdot z)x - y^\sigma y \cdot zx = 0$ for all $x, y, z \in A$. This implies that $y^\sigma y \in N_\lambda$. Suppose $y^\sigma y \in N_\lambda$ then $(y^\sigma y \cdot z)x = y^\sigma y \cdot zx$. Substituting this in (24) we obtain $(xy \cdot z)y^\sigma = x(yz \cdot y^\sigma)$ for all $x, y, z \in A$. This completes the proof of the theorem.

Corollary 2. *Let A be an extra algebra. Then A satisfies the B_3–Law.*

Proof. Let A be an extra algebra. By Theorem 3 (v), $y^2 \in N$ for all $y \in A$, this implies that $y^4 \in N$ for all $y \in A$. By Theorem 5, A satisfies B_3–Law.

Corollary 3. *Every extra loop satisfies B_3–law.*

Corollary 4. *Every extra loop satisfies B_{-1}–law.*

Proof. Substituting $\sigma = -1$ in (24) gives $(xy \cdot z)y^{-1} = z(yz \cdot y^{-1})$.

Theorem 6. *Let A be a Bol algebra. A satisfies B_σ–law if and only if $(xy^\sigma \cdot z)y = x(y^\sigma z \cdot y)$ for all $x, y, z \in A$.*

Proof. Let A be a Bol algebra. Then (8) holds. Substituting $w = y^\sigma$ in (8) we obtain

$$(25) \qquad (xy^\sigma \cdot z)y + (xy \cdot z)y^\sigma - x(y^\sigma z \cdot y) - x(yz \cdot y^\sigma) = 0.$$

Now if A satisfies the B_σ–law, then substituting (23) in (25) implies that $(xy^\sigma \cdot z)y = x(y^\sigma z \cdot y)$. Also substituting $(xy^\sigma \cdot z)y = x(y^\sigma z \cdot y)$ in (25) gives $(xy \cdot z)y^\sigma = x(yz \cdot y^\sigma)$ for all $x, y, z \in A$. Therefore A satisfies the B_σ–law.

Theorem 7. *Let A be an algebra. The following are equivalent*

(i) *A satisfies the M_λ–law*

(ii) *A satisfies the B_λ–law and $x \cdot yx^\lambda = xy \cdot x^\lambda$ for all $x, y \in A$.*

Proof. (i) $\Longrightarrow$ (ii). Let A be an algebra satisfying the M_λ–law, then

$$(5)' \qquad (yx \cdot z)x^\lambda = y(x \cdot zx^\lambda)$$

Let $y = 1$ in (5'), we obtain

$$(25)' \qquad x \cdot zx^\lambda = xz \cdot x^\lambda \quad \text{for all} \quad x, z \in A.$$

Now, applying (25)' to (5)' we get $(yx \cdot z)x^\lambda = y(xz \cdot x^\lambda)$ for all $x, y, z \in A$, which implies that A satisfies the B_λ–law.

(ii) $\Longrightarrow$ (i). Let A be an algebra satisfying the B_λ–law and $x \cdot yx^\lambda = xy \cdot x^\lambda$. Then $(yx \cdot z)x^\lambda = y(xz \cdot x^\lambda) = y(x \cdot zx^\lambda)$ which implies that A satisfies the M_λ–law.

Theorem 8. *Let A be a Moufang algebra. The following identities hold in A^* for all $x, y, z \in A^*$.*

(i) $(xy \cdot z)x^{-1} = x(y \cdot zx^{-1})$

(ii) (a) $(xy \cdot z)y^{-1} + (xy^{-1} \cdot z)y = 2xz$
(b) $(xy \cdot z)y^{-1} + (zy \cdot z)y^{-1} = xz + zx$

(iii) $(xy \cdot z)z^{-1} = xy$

(iv) $x^{-1}(xy \cdot z) - x^{-1}(z \cdot yx) = yz - zy$

(v) $y^{-1}(xy \cdot z) = y^{-1}x \cdot yz$

(vi) $z^{-1}(xy \cdot z) = xy$

(vii) $(x \cdot yz)x^{-1} = yz$

(viii) $(x \cdot yz)y^{-1} = xy \cdot zy^{-1}$

(ix) $(x \cdot yz)z^{-1} - z^{-1}(yz \cdot x) + yx - xy = 0$

(x) $x^{-1}(x \cdot yz) = yz$

(xi) (a) $y^{-1}(x \cdot yz) + y(x \cdot y^{-1}z) = 2xz$
(b) $y^{-1}(x \cdot yz) + y^{-1}(z \cdot yx) = xz + zx$

(xii) $z^{-1}(x \cdot yz) = (z^{-1}x \cdot y)z$

(xiii) $y^{-1}(x \cdot yx) = y(x \cdot y^{-1}x) = (xy \cdot x)y^{-1} = (xy^{-1} \cdot x)y = x^2$.

Proof. (i) The same as proof of Theorem 4 (i). Similarly, following the procedures in the proof of Theorem 4, by substituting $w = y^{-1}$ in (10); $z = y^{-1}$ in (11); $w = x^{-1}$ in (9); $w = z^{-1}$ in (9); $x = z^{-1}$ in (10); $w = x^{-1}$ in (11); we obtain (ii)(a); (ii)(b); (vii); (ix); (xi)(a) and (xi)(b), respectively, for all $x, y, z \in A^*$.

(iii) Follows from the right inverse property (RIP).

(v) This result is the same as in Theorem 4 (v).

(viii) This result is the same as in Theorem 4 (viii).

(x) Follows from left inverse property (LIP).

(xii) Follows by substituting $x = x^{-1}$ in (i) above.

(iv) Substituting $x = w^{-1}$ in (8), we obtain

$$(w^{-1}w \cdot z)y + (w^{-1}y \cdot z)w - w^{-1}(wz \cdot y) - w^{-1}(yz \cdot w) = 0.$$

This implies that $zy + (w^{-1}y \cdot z)w - w^{-1}(wz \cdot y) - yz = 0$. By applying (1) we obtain $w^{-1}(y \cdot zw) - w^{-1}(wz \cdot y) = yz - zy$. Therefore $x^{-1}(xz \cdot y) - x^{-1}(y \cdot zx) = = zy - yz$ or $x^{-1}(xy \cdot z) - x^{-1}(z \cdot yx) = yz - zy$.

(vi) Substituting $x = y^{-1}$ in (8), gives

$$(y^{-1}w \cdot z)y + (y^{-1}y \cdot z)w - y^{-1}(wz \cdot y) - y^{-1}(yz \cdot w) = 0.$$

This implies that $(y^{-1}w \cdot z)y + zw - y^{-1}(wz \cdot y) - y^{-1}(yz \cdot w) = 0$. Applying Theorem 8 (i) and (v) we obtain $y^{-1}(wz \cdot y) = wz$ for all $x, y, z \in A^*$.

(xiii) Substituting $z = x$ in (xi)(a) and (b) we obtain

$$y^{-1}(x \cdot yx) + y(x \cdot y^{-1}x) = 2x^2,\ y^{-1}(x \cdot yx) + y^{-1}(x \cdot yx) = 2x^2.$$

These imply that $y^{-1}(x \cdot yx) = x^2 = y(x \cdot y^{-1}x)$. Also substituting $z = x$ in (ii)(a) and (b) we obtain

$$(xy \cdot x)y^{-1} + (xy^{-1} \cdot x)y = 2x^2,\ (xy \cdot x)y^{-1} + (xy \cdot x)y^{-1} = 2x^2.$$

These imply that $(xy \cdot x)y^{-1} = x^2 = (xy^{-1} \cdot x)y$.

Theorem 9. *Let A be a Moufang algebra. A satisfies M_λ-law if and only if*

$$x^\lambda y \cdot zx = (x^\lambda \cdot yz)x \quad \text{for all } x, y, z \in A.$$

Proof. Let A be a Moufang algebra. Then (9) holds. Substituting $w = x^\lambda$ in (9) we obtain

(26) $$xy \cdot zx^\lambda + x^\lambda y \cdot zx - (x \cdot yz)x^\lambda - (x^\lambda \cdot yz)x = 0.$$

Now if A satisfy the M_λ-law, then substituting (5) in (26), we get

$$x^\lambda y \cdot zx = (x^\lambda \cdot yz)x \quad \text{for all } x, y, z \in A.$$

Also substituting $x^\lambda y \cdot zx = (x^\lambda \cdot yz)x$ in (26), gives

$$xy \cdot zx^\lambda = (x \cdot yz)x^\lambda \quad \text{for all } x, y, z \in A.$$

Theorem 10. *Let A be a Moufang algebra. The following are equivalent:*

(i) *A^* satisfies B_{-1}-law.*

(ii) $(xy \cdot z)y^{-1} = (xy^{-1} \cdot z)y$ *for all* $x, y, z \in A^*$.

(iii) $(xy \cdot z)y^{-1} = xz$ *for all* $x, y, z \in A^*$.

(iv) $(zy \cdot x)y^{-1} = zx$ *for all* $x, y, z \in A^*$.

Proof. Let A be a Moufang algebra.

(i) $\Longrightarrow$ (ii). Let A^* satisfy B_{-1}-law, then

(27) $$(xy \cdot z)y^{-1} = x(yz \cdot y^{-1}) \quad \text{for all } x, y, z \in A^*.$$

Apply cross-inverse property in (27) gives:

(28) $$(xy \cdot z)y^{-1} = xz.$$

Substituting (28) in Theorem 6 (ii)(a) gives $(xy^{-1} \cdot z)y = xz$.

(ii) $\Longrightarrow$ (iii). Let $(xy \cdot z)y^{-1} = (xy^{-1} \cdot z)y$, substituting in Theorem 6 (ii)(a) gives $(xy \cdot z)y^{-1} = xz$.

(iii) $\Longrightarrow$ (iv). Let $(xy \cdot z)y^{-1} = xz$, substituting in Theorem 6 (ii)(b) gives $(zy \cdot x)y^{-1} = zx$ for all $x, y, z \in A^*$.

(iv) $\Longrightarrow$ (i). $(zy \cdot x)y^{-1} = zx = z(yx \cdot y^{-1})$ therefore $(zy \cdot x)y^{-1} = z(yx \cdot y^{-1})$ for all $x, y, z \in A^*$.

4. In this section, we consider an algebra A with associated tangent algebra which carries a bilinear anti-commutative multiplication $(x, y) \to [x, y]$ called the commutator bracket and a trilinear operation $(x, y, z) \to (x, y, z)$ called the associator bracket. Also a quadrilinear operation (which we call extra bracket) $(x, y, z, w) \to < x, y, z, w >$ are studied.

Theorem 11. *Let A be an algebra such that $xy \cdot z = xz \cdot y$ and $x \cdot yz = y \cdot xz$ for all $x, y, z \in A$, then the tangent algebra of A is an Aktivis Algebra and a Lie algebra.*

Proof. Let A be an algebra such that $xy \cdot z = xz \cdot y$ and $x \cdot yz = y \cdot xz$ for all $x, y, z \in A$. Consider

$$\begin{aligned} T(x,y,z) &= \Sigma\{\operatorname{sgn}(\sigma) \cdot \sigma(x,y,z) : \sigma \in S_3\} = \\ &= (xy \cdot z - xz \cdot y + yz \cdot x - yx \cdot z + zx \cdot y - zy \cdot x) + \\ &+ (-x \cdot yz + x \cdot zy - y \cdot zx + y \cdot xz - z \cdot xy + z \cdot yx) \end{aligned} \tag{29}$$

Since $xy \cdot z = xz \cdot y$ and $x \cdot yz = y \cdot xz$ for all $x, y, z \in A$, then $T(x, y, z) = 0$. Similarly,

$$J(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y] = T(x,y,z).$$

Therefore, $J(x, y, z) = T(x, y, z) = 0$. Hence A is an Aktivis algebra. Since $T(x, y, z) = 0$, then A is also a Lie algebra.

Let $< x, y, z, w > = (xy \cdot z)w - x(y \cdot zw)$. We call $< x, y, z, w >$ extra bracket because when $w = x$, $< x, y, z, x > = (xy \cdot z)x - x(y \cdot zx)$ which vanishes in case of extra algebra. We shall obtain an identity linking this bracket to the commutator bracket.

Theorem 12. *The extra and commutator brackets are linked by the identity:*

$$\begin{aligned} &T(x,y,z,w) \equiv \Sigma\{\operatorname{sgn}(\sigma) \cdot \sigma < x,y,z,w >: \sigma \in S_4\} = \\ &= \Sigma_\gamma\{-\operatorname{sgn}(\gamma) \cdot (w(\operatorname{sgn}(\sigma) \cdot \sigma(x[y,z])) + \\ &+ (\operatorname{sgn}(\alpha) \cdot \alpha([x,y]z))w) : \sigma \in A_3,\ \alpha \in S_3 - A_3\} \equiv J(x,y,z,w) \end{aligned} \tag{30}$$

where $\gamma \in D = \{(w,x,y,z); (x,w,y,z), (y,x,w,z), (z,x,y,w)\}$ and $D \subset S_4$.

Proof. By routine expansion and manipulation, we obtain

$$\begin{aligned} &\Sigma\{\operatorname{sgn}(\sigma) \cdot \sigma < x,y,z,w >: \sigma \in S_4\} = \\ &= \{w(x[y,z] + y[z,x] + z[x.y]) - ([z,y]x + [x,z]y + [y,x]z)w\} - \\ &- \{x(w[y,z] + y[z.w] + z[w,y]) - ([z,y]w + [w,z]y + [y,w]z)x\} - \\ &- \{y(x[w,z] + w[z,x] + z[x,w]) - ([z,w]x + [x,z]w + [w,x]z)y\} - \\ &- \{z(x[y,w] + y[w,x] + w[x,y]) - ([w,y]x + [x,w]y + [y,x]w)z\} = \\ &= \sum_\gamma \{-\operatorname{sgn}(\gamma) \cdot \gamma(w(\operatorname{sgn}(\sigma) \cdot \sigma(x[y,z])) + \\ &+ (\operatorname{sgn}(\alpha) \cdot \alpha([x,y]z))w) : \sigma \in A_3,\ \alpha \in S_3 - A_3\} \end{aligned}$$

where $\gamma \in D = \{(w,x,y,z); (x,w,y,z), (y,x,w,z), (z,x,y,w)\}$. This completes the proof.

Corollary 5. *Let A be an extra algebra, then*

(i) $T(x,y,z,x) = \Sigma\{\operatorname{sgn}(\sigma) \cdot \sigma < x,y,z,x >: \sigma \in S_4\} \equiv 0$

(ii) $J(x,y,z,x) \equiv 0$.

Proof. $T(x,y,z,x) = \Sigma\{\operatorname{sgn}(\sigma) \cdot \sigma < x,y,z,x >: \sigma \in S_4\}$. Each bracket $\sigma < x,y,z,x >$ in the expansion of $T(x,y,z,x)$ is equal to zero (using the identities (1) and (2), alternative property and the fact that $x^2 \in N_\lambda = N_\mu = N_\rho = N$). Therefore $T(x,y,z,x) = 0$. Since $T(x,y,z.w) = J(x.t,z,w)$, then $J(x,y,z,x) = 0$. This completes the proof.

Corollary 6. *Let A be a Moufang algebra. Then*

(i) $T(x,y,z.y) = \Sigma\{\operatorname{sgn}(\sigma) \cdot \sigma < x,y,z,y >: \sigma \in S_4\} = 0$

(ii) $J(x,y,z,y) = 0$.

Proof. Let A be a Moufang algebra. Then in the expansion of $T(x,y,z,y)$. 8 terms are equal to zero using the Moufang identities and the 16 items which are not equal to zero occur in pairs with opposite signs, therefore cancelling each other. Hence $T(x,y.z,y) = 0$. Since $T(x,y.z,w) = J(x,y,z.w)$. then $J(x,y,z,y) = 0$.

Corollary 7. $T(x,y,z.x) = T(x.y,z,y) = T(x,y.z,z) = 0$.

Acknowledgement. This research was supported by Obafemi Awolowo University URC Grant 1423LS.

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Received: 18.II.94
Revised: 20.IV.95

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EXISTENCE OF CRITICAL POINTS IN A PRESCRIBED SET

BY

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0. Introduction. The critical point theory is a powerful tool in studying the topological properties of the space (see, e.g., C h a n g [1], P a l a i s and T e r n g [10]) as well as for proving the solvability of various nonlinear problems (see, e.g., C h a n g [1], P a l a i s and T e r n g [10], R a b i n o w i t z [12], S t r u w e [13]). A basic method to establish the existence of critical points is the use of deformation theorems (cf. C h a n g [1], D u [2], P a l a i s and T e r n g [10], R a b i n o w i t z [12], S t r u w e [13], S z u l k i n [14]). A natural question which is related to the existence results is to locate the critical points. In this respect we refer to the papers of D u [2], G h o u s s o u b [5], G h o u s s o u b and P r e i s s [6]. The aim of this paper is two-fold: (a) to obtain a deformation theorem by means of a given flow; (b) to provide critical points in a prescribed set. Our main results are stated in Theorems 1.1, 2.1, 2.2 and 3.1. In addition to the critical point methods the approach relies on flow invariance properties (see M o t r e a n u and P a v e l [9], P a v e l [11], W a n g [15]) and minimization arguments (see E k e l a n d [3], [4], J o n g e n and W e b e r [7], M o t r e a n u [8]).

1. A deformation result. This Section is devoted to the deformation of level sets of a smooth function along the integral curves of a given locally Lipschitz vector field. In particular, the vector field can be the gradient field on a Riemannian manifold or a pseudo-gradient field on a Finsler manifold. In these specific cases one obtains the classical deformation theorems. The advantage of our general approach lies in the fact that one can make use of special properties of a suitable vector field. For instance, in the