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EXISTENCE OF CRITICAL POINTS IN A PRESCRIBED SET

BY

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0. Introduction. The critical point theory is a powerful tool in studying the topological properties of the space (see, e.g., C h a n g [1], P a l a i s and T e r n g [10]) as well as for proving the solvability of various nonlinear problems (see, e.g., C h a n g [1], P a l a i s and T e r n g [10], R a b i n o w i t z [12], S t r u w e [13]). A basic method to establish the existence of critical points is the use of deformation theorems (cf. C h a n g [1], D u [2], P a l a i s and T e r n g [10], R a b i n o w i t z [12], S t r u- w e [13], S z u l k i n [14]). A natural question which is related to the existence results is to locate the critical points. In this respect we refer to the papers of D u [2], G h o u s s o u b [5], G h o u s s o u b and P r e i s s [6]. The aim of this paper is two-fold: (a) to obtain a deformation theorem by means of a given flow; (b) to provide critical points in a prescribed set. Our main results are stated in Theorems 1.1, 2.1, 2.2 and 3.1. In addition to the critical point methods the approach relies on flow invariance properties (see M o t r e a n u and P a v e l [9], P a v e l [11], W a n g [15]) and minimization arguments (see E k e l a n d [3], [4], J o n g e n and W e b e r [7], M o t r e a n u [8]).

1. A deformation result. This Section is devoted to the deformation of level sets of a smooth function along the integral curves of a given locally Lipschitz vector field. In particular, the vector field can be the gradient field on a Riemannian manifold or a pseudo-gradient field on a Finsler manifold. In these specific cases one obtains the classical deformation theorems. The advantage of our general approach lies in the fact that one can make use of special properties of a suitable vector field. For instance, in the next Sections we employ the flow-invariance properties.

Let M be a connected $C^{1,1}$ manifold whose model is a Banach space. We suppose that M is endowed with a continuous function $\|\cdot\| : TM \longrightarrow R$ defined on the tangent bundle TM such that the mapping $d : M \times M \longrightarrow R$ expressed by

$$d(x,y) = \inf\{\int_0^1 \|\gamma'(t)\|dt : \gamma \in C^1([0,1],M), \gamma(0)=x, \gamma(1)=y\}, \quad \forall\, x,y \in M \tag{1.1}$$

determines a metric on M compatible with the manifold topology. For example, M can be a Finsler manifold, in particular a Riemannian manifold (see C h a n g [1], P a l a i s and T e r n g [10], S t r u w e [13]).

We consider a function $f \in C^1(M,R)$, a number $c \in R$ and a locally Lipschitz vector field V on M whose global flow is denoted $\eta(t,x)$.

Throughout the paper we use the following standard notations

$$f_a := \{x \in M : f(x) \le a\} \quad \text{for each } a \in R,$$

$$K_c := \{x \in M : f(x) = c \text{ and } Df(x) = 0\},$$

$$N_\delta(K_c) := \{x \in M : d(x, K_c) < \delta\} \quad \text{for any number } \delta > 0.$$

The hypotheses below are imposed

(i) There is an $\varepsilon_0 > 0$ such that $\eta(t,x)$ is defined and $f(\eta(t,x))$ is non-increasing for $t \in [0,1]$ whenever $x \in f_{c+\varepsilon_0}$.

(ii) If $\{x_n\}$ is a sequence in M such that $f(x_n) \to c$ and $< Df(x_n), V(x_n) > \to 0$, then it contains a subsequence which converges to a critical point of f.

Our deformation theorem is formulated in the result below.

Theorem 1.1. *Under assumptions* (i), (ii), *corresponding to each neighborhood U of K_c there exists $\varepsilon > 0$ such that $\eta(1, f_{c+\varepsilon} \setminus U) \subset f_{c-\varepsilon}$.*

Proof. It is straightforward to check that hypothesis (ii) implies that K_c is a compact subset of M. Then one can find constants $0 < \delta < \delta_0$ and $C_0 > 0$ provided

$$K_c \subset N_\delta(K_c) \subset N_{\delta_0}(K_c) \subset U \tag{1.2}$$

and

$$\|V(x)\| \le C_0, \quad x \in N_{\delta_0}(K_c). \tag{1.3}$$

Hypotheses (i) and (ii) ensure the existence of numbers $0 < \varepsilon_1 < \varepsilon_0$ and $C > 0$ for which one has

$$< Df(x), V(x) > \le -C, \quad x \in f^{-1}[c-\varepsilon_1, c+\varepsilon_1] \setminus N_\delta(K_c). \tag{1.4}$$

We show that the conclusion of Theorem 1.1 holds for every ε with

$$0 < 2\varepsilon \le \min\{C, CC_0^{-1}(\delta_0 - \delta), 2\varepsilon_1\}. \tag{1.5}$$

To justify this let us fix $x \in f_{c+\varepsilon} \setminus U$. Then we may assume

$$\eta(t,x) \in f^{-1}(c-\varepsilon, c+\varepsilon] \quad \text{for each } t \in [0,1) \tag{1.6}$$

because otherwise the result follows by (i).Furthermore, if one has

$$\eta(t,x) \notin N_\delta(K_c) \quad \text{for each } t \in [0,1], \tag{1.7}$$

then relations (1.4)-(1.7) allow us to write

$$f(\eta(1,x)) = f(x) + \int_0^1 \frac{d}{dt} f(\eta(t,x))dt \le$$

$$\le c + \varepsilon + \int_0^1 < Df(\eta(t,x)), V(\eta(t,x)) > dt \le c - \varepsilon.$$

This represents just the desired conclusion.

In the case where (1.7) is not true, let us take according to (1.2) the first time $t_1 \in (0,1)$ satisfying

$$d(\eta(t_1,x), K_c) = \delta \tag{1.8}$$

and the last moment $t_0 \in (0, t_1)$ with

$$d(\eta(t_0,x), K_c) = \delta_0. \tag{1.9}$$

From (1.1)-(1.3), (1.8) and (1.9) we see that

$$\delta_0 - \delta \le \int_{t_0}^{t_1} \|\frac{d}{dt}\eta(t,x)\|dt \le C_0(t_1 - t_0). \tag{1.10}$$

By hypothesis (i) and relations (1.4), (1.5), (1.10) we derive

$$f(\eta(1,x)) = f(x) + \int_0^1 < Df(\eta(t,x)), V(\eta(t,x)) > dt \le$$

$$\le c+\varepsilon+\int_{t_0}^{t_1}\frac{d}{dt}f(\eta(t,x))dt \le c+\varepsilon-C(t_1-t_0)\le c-\varepsilon.$$

This completes the proof of Theorem 1.1.

2. Critical points and flow-invariant sets. A usual procedure to prove the existence of critical points is to construct minimax values by means of deformation results (see Chang [1], Du [2], Palais and Terng [10], Rabinowitz [12], Struwe [13], Szulkin [14]). The deformation constructed in Theorem 1.1 yields the following minimax principle.

Theorem 2.1. *Let the manifold M, the function $f: M \to R$ and the vector field V on M with the global flow $\eta(t,x)$ be as in Section 1. Let Γ be a family of compact subsets of M such that*

$$G\in\Gamma \quad\Rightarrow\quad \eta(1,G) \ \textit{exists and}\ \eta(1,G)\in\Gamma. \tag{2.1}$$

Assume that

$$c := \inf_{G\in\Gamma}\max_{x\in G} f(x) > -\infty \tag{2.2}$$

and for this real number c the conditions (i) *and* (ii) *in Section 1 are valid. Then c is a critical value of f, that is, there is a point $x\in K_c$.*

Proof. Arguing by contradiction suppose that c is not a critical value of f. Then one can apply Theorem 1.1 with $U=\emptyset$. It ensures that, for some $\varepsilon>0$, one has

$$\eta(1,f_{c+\varepsilon}) \subset f_{c-\varepsilon}. \tag{2.3}$$

By (2.2) there exists $G\in\Gamma$ with

$$G\subset f_{c+\varepsilon}. \tag{2.4}$$

Combining (2.3) and (2.4) one finds

$$\eta(1,G)\subset f_{c-\varepsilon},$$

which contradicts (2.2). This contradiction completes the proof of Theorem 2.2.

According to Theorem 2.1, in order to obtain critical points of f it suffices to take a family Γ of subsets of M satisfying (2.1). We show that this can be done by using the tangency properties of the vector field V on M.

The vector field V on the manifold M is said to be tangent to a subset S of M if the below equality holds at each point $x\in S$

$$\lim_{t\to 0} t^{-1}dist(\phi(x)+t<D\phi(x),V(x)>;\phi(dom\ \phi\cap S)) = 0,$$

where $\phi: dom\phi \to E$ is a chart of M with $x\in dom\phi$. The notation $dist$ designates the distance in the Banach space E which is the model of the manifold M. The above definition of tangent (to a set) vector field does not depend on the local chart ϕ. It reduces to the ordinary notion of tangent vector field if the set S is a smooth submanifold of M, so $V(x)\in T_xS$ for each $x\in S$. For more details we refer to Motreanu and Pavel [9], Pavel [11], Wang [15].

The next result discusses the relationship between the critical values of a function $f\in C^1(M,R)$ and the flow-invariant sets of a locally Lipschitz vector field V on the manifold M. We recall that a set $S\subset M$ is flow-invariant with respect to V if the global flow $\eta(t,x)$ of V has the property that $\eta(t,x)\in S$ whenever $\eta(0,x)=x\in S$ and $\eta(t,x)$ is defined. A basic property assures that a closed subset S of M is flow-invariant with respect to the vector field V if and only if V is tangent to S at each point $x\in S$ (cf. Motreanu and Pavel [9] or Pavel [11]).

Theorem 2.2. *Assume that S is a closed subset of M such that $f\in C^1(M,R)$ is bounded from below on S and the locally Lipschitz vector field V on M is tangent to S. If hypotheses* (i), (ii) *are verified with*

$$c = \inf_S f, \tag{2.5}$$

then c is a critical value of f.

Proof. Let us choose the family Γ of compact subsets of M entering Theorem 2.1 as follows

$$\Gamma = \{G\subset S\cap f_{c+\varepsilon_0}:\ G \textit{ is a compact set}\}. \tag{2.6}$$

Since, by the tangency of V to S, the set S is flow-invariant with respect to V, it follows that the trajectory $\eta(\cdot,x)$ of V remains in S as long as it exists if $x\in S$. Taking into account hypothesis (i) with c introduced in (2.5) we see that property (2.1) is true.

On the other hand from (2.5) and (2.6) it is clear that

$$c = \inf_{G\in\Gamma} \max_{x\in G} f(x). \tag{2.7}$$

By hypothesis we have that (2.2) is satisfied. Theorem 2.1 allows then to conclude that c in (2.7) (or (2.5)) is a critical value of f.

Corollary 2.3. (Palais Minimization Theorem [10], [12], [13].) *Let the function $f \in C^1(M,R)$ be bounded from below and satisfy the Palais-Smale condition on a complete $C^{1,1}$ Finsler manifold M. Then f attains its infimum on M.*

Proof. It is sufficient to apply Theorem 2.2 with $S = M$ and V a locally Lipshitz vector field on M with $-V$ equal to a bounded pseudo-gradient vector field of f on the set of regular points of f and vanishing otherwise (see,e.g., C h a n g [1], R a b i n o w i t z [12], S t r u w e [13]). With c given by (2.5) for $S = M$, Corollary 2.3 follows from Theorem 2.2.

3. Minimization on flow-invariant sets. This Section is devoted to solving the minimization problem (2.5) on the set S. Notice that Theorem 2.2 ensures only that c in (2.5) is a critical value of the function f. We show that with further hypotheses the minimum value c is attained on S. The argument to establish this is completely different from the preceding one, namely using instead of the deformation in Theorem 1.1 the Ekeland's variational principle [3],[4]. The result below offers also a location of critical points. Results of this type have been obtained by different methods by D u [2], G h o u s s o u b [5], G h o u s s o u b and P r e i s s [6].

Theorem 3.1. *In addition to the assumptions of Theorem 2.2 we suppose that the subset S of M is complete with respect to the metric d on M given by (1.1) and the vector field V on M is bounded on S. Then problem (2.5) admits a solution, i.e., there exists $x \in S$ with*

$$f(x) = \inf_S f. \tag{3.1}$$

Moreover, each such point x is a critical point of f on M.

Proof. The stated hypotheses allow us to apply Ekeland's variational principle (cf. [3], [4]) to the functional f on the complete metric space S. This determines a sequence $x_n \in S$ such that

$$f(x_n) \le \inf_S f + \frac{1}{n} \tag{3.2}$$

and

$$f(y) \ge f(x_n) - (1/n)d(y,x_n), \quad y \in S. \tag{3.3}$$

The invariance of the set S with respect to the global flow $\eta(\cdot,\cdot)$ of V and property (3.3) yield

$$f(\eta(t,x_n)) - f(x_n) \ge -(1/n)d(\eta(t,x_n),x_n) \tag{3.4}$$

for t near 0 and $n \ge 1$. Using (1.1) and (3.4) one obtains

$$f(\eta(t,x_n)) - f(x_n) \ge -\frac{1}{n}\left|\int_0^t \|V(\eta(s,x_n))\|ds\right|. \tag{3.5}$$

If we divide by $t > 0$ and then pass to the limit for $t \to 0$ in (3.5) we derive that

$$< Df(x_n), V(x_n) > \ge -(1/n)\|V(x_n)\|, \quad n \ge 1. \tag{3.6}$$

Proceeding similarly for $t < 0$ we infer that

$$< Df(x_n), V(x_n) > \le (1/n)\|V(x_n)\|, \quad n \ge 1. \tag{3.7}$$

Letting $n \to \infty$ in (3.6), (3.7) and taking into account the boundedness of V on S it results

$$< Df(x_n), V(x_n) > \to 0 \quad as \quad n \to \infty. \tag{3.8}$$

Then (3.2) and (3.8) show that x_n is a sequence in S satisfying the requirements in hypothesis (ii) with c given by (2.5). We deduce that x_n contains a convergent subsequence denoted again by x_n. If we set

$$x = \lim_{n\to\infty} x_n, \tag{3.9}$$

we see from (3.2) that (3.1) is verified. Finally, we note from (3.2), (3.8) and assumption (ii) that $x \in S$ in (3.9) may be chosen to be critical point of f. This completes the proof of Theorem 3.1.

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FRACTIONAL STEP SCHEME APPLIED TO A CONTROL PROBLEM

BY

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In this paper a control problem governed by a multivalued differrential equation is studied.

The basic feature of the approach consists in approximating the corresponding variational equation by means of a fractional step scheme.

Using the above scheme, the errors between the exact and approximation solution, for two examples, are evaluated.

We consider

Problem (P) Minimize

$$\int_0^T (g(y(t)) + h(u(t)))dt + \varphi_0(y(T)),\ 0 < T < \infty,$$

over the set of all functions $(y,u) \in W^{1,2}(0,T;R^n) \times U$ which satisfy the multivalued differential equation

$$(E) \qquad y'(t) + (F+\beta)(y(t)) \ni Bu(t) \ \text{ a.e. } t \in]0,T[,$$

with the initial condition

$$(IC) \qquad y(0) = y_0 \in D(\beta).$$

The elements in the above equation have the following properties:

(i) *$F : R^n \to R^n$ is a local Lipschitz function which satisfies the condition: there exist the constants $D_1, D_2 \in R_+^*$ such that*

$$(1) \qquad < Fy, y > \geq -D_1\|y\|^2 - D_2 \ \ \textit{for every } y \in R^n;$$