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Universitatea "Al. I. Cuza"
Seminarul Matematic "Al. Myller"
6600 Iasi
ROMANIA

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
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FRACTIONAL STEP SCHEME APPLIED TO A CONTROL PROBLEM

BY

CĂTĂLINA DAVIDEANU

In this paper a control problem governed by a multivalued differrential equation is studied.

The basic feature of the approach consists in approximating the corresponding variational equation by means of a fractional step scheme.

Using the above scheme, the errors between the exact and approximation solution, for two examples, are evaluated.

We consider

Problem (P) Minimize

$$\int_0^T (g(y(t)) + h(u(t)))dt + \varphi_0(y(T)),\ 0 < T < \infty,$$

over the set of all functions $(y,u) \in W^{1,2}(0,T;R^n) \times U$ which satisfy the multivalued differential equation

$$(E) \qquad y'(t) + (F+\beta)(y(t)) \ni Bu(t) \ \text{ a.e. } t \in]0,T[,$$

with the initial condition

$$(IC) \qquad y(0) = y_0 \in D(\beta).$$

The elements in the above equation have the following properties:

(i) *$F : R^n \to R^n$ is a local Lipschitz function which satisfies the condition: there exist the constants $D_1, D_2 \in R_+^*$ such that*

$$(1) \qquad < Fy, y > \geq -D_1\|y\|^2 - D_2 \quad \textit{for every } y \in R^n;$$

(ii) *$\beta : D(\beta) \subset R^n \to R^n$ is a maximal monotone operator, with $0 \in D(\beta)$;*

(iii) *$B : R^m \to R^n$ is a linear continous operator;*

(iv) *$U = \{u \in W^{1,2}(0,T;R^m);\ \|Bu(t)\|^2_{L^2(0,T;R^n)} \leq A\}$ represents the set of admissible controls;*

(v) *the functions $g : R^n \to R_+$ and $\varphi_0 : R^n \to R_+$ satisfy the condition: for every $\delta > 0$ there exists $L_\delta > 0$ such that*

$$|g(y) - g(z)| + |\varphi_0(y) - \varphi_0(z)| \leq L_\delta \|y - z\|_n, \tag{2}$$

for every $y, z \in R^n$ with $\|y\| + \|z\| \leq \delta$;

(vi) *$h : R^m \to \bar{R}$ is a convex, lower semicontinuous function which satisfies the condition: there exist $a_1 \in R^*_+$ and $a_2 \in R$ such that*

$$h(v) \geq a_1 \|v\|^2_m - a_2 \quad \textit{for every } v \in R^m. \tag{3}$$

We make the supplementary, nonrestrictive, hypothesis

$$S(0, C) \subset \operatorname{int} D(\beta), \tag{H}$$

where

$$C = \left((\|y_0\|^2 + 2D_2T + A)\, e^{(2D_1+1)T} \right)^{1/2}.$$

Thus, we take the initial data $y_0 \in D(\beta)$, the constants $D_1, D_2 \in R^*_+$ and the set of admissible controls U such that the relation (H) be fulfiled.

We associate to the initial problem (P) the following partial differential equation of Hamilton–Jacobi type

$$\begin{gathered} \psi_t(t,y) - h^*(-B^*(\psi_y(t,y))) - ((F+\beta)(y), \psi_y(t,y)) + g(y) = 0, \\ \text{for every } t \in [0,T],\ y \in R^n, \end{gathered} \tag{4}$$

with the final condition

$$\psi(T,y) = \varphi_0(y), \quad \text{for every } y \in R^n, \tag{5}$$

where ψ_t and ψ_y represent the partial derivatives of the function ψ with respect to t and y, respectively.

The feedback optimal control is given by the relation

$$u(t) \in \partial h^*(-B^*(\psi_y(t,y))) \tag{6}$$

where h^* is the conjugate function of h, i.e.

$$h^*(p) = \sup\{(p, u) - h(u),\ u \in U\}. \tag{7}$$

If the function ψ is replaced by φ, $\varphi(t,y) = \psi(T-t,y)$, the equation (4) becomes

$$\begin{gathered} \varphi_t(t,y) + h^*(-B^*(\varphi_y(t,y))) - ((F+\beta)(y), \varphi_y(t,y)) = g(y), \\ \text{for every } t \in [0,T],\ y \in R^n, \end{gathered} \tag{8}$$

with the initial condition

$$\varphi(0,y) = \varphi_0(y), \quad \text{for every } y \in R^n. \tag{9}$$

In order to approximate the solution of the variational equation (8) with the initial condition (9), we use the following approximate scheme (see [1], p.127)

$$\begin{cases} \varphi^\varepsilon_t(t,y) + h^*(-B^*(\varphi^\varepsilon_y(t,y))) + (Fy, \varphi^\varepsilon_y(t,y)) = 0 \\ \qquad\qquad \text{for every } (t,y) \in]i\varepsilon, (i+1)\varepsilon[\times R^n, \\ \varphi^\varepsilon_+(i\varepsilon, y) = \varphi^\varepsilon_-(i\varepsilon, e^{-\beta\varepsilon}y) + \varepsilon g(y),\ y \in R^n, \\ \varphi^\varepsilon_+(0,y) = \varphi_0(y),\ y \in R^n, \end{cases} \tag{10}$$

for $i = \overline{0, n-1}$, $n\varepsilon = T$, where φ_+, φ_- represent the right and left–hand side limits of the function $t \to \varphi(t,\cdot)$, respectively.

The function $w(t) = e^{-\beta t}z$ is the solution of the Cauchy problem

$$w'(t) + \beta(w(t)) \ni 0 \quad \text{a.e. } t \in]0,T[\tag{11}$$

with the initial condition

$$w(0) = z \in D(\beta). \tag{12}$$

We recognize in the approximation scheme (10) a product formula of Lie–Trotter type associated to the Hamilton–Jacobi equation (9) with the initial condition (10).

For the beginning, let us construct the approximate scheme

$$(S_i)\qquad \begin{cases} y_\varepsilon'(t) + F(y_\varepsilon(t)) = B(u_\varepsilon(t)) \quad \text{a.e. } t \in]i\varepsilon, (i+1)\varepsilon[, \\ y_\varepsilon^+(i\varepsilon) = e^{-\varepsilon\beta} y_\varepsilon^-(i\varepsilon), \end{cases}$$

for $i = \overline{1, n-1}$, $n\varepsilon = T$, with the particular case $(i = 0)$

$$(S_0)\qquad \begin{cases} y_\varepsilon'(t) + F(y_\varepsilon(t)) = B(u_\varepsilon(t)) \quad \text{a.e. } t \in]0, \varepsilon[, \\ y_\varepsilon^+(0) = y_0 \in D(\beta), \end{cases}$$

where $\{u_\varepsilon\}_{\varepsilon>0} \subset U$.

Theorem 1. *If $\{u_\varepsilon\}_{\varepsilon>0} \subset U$ is a weakly convergent sequence to u^* in $L^1(0,T;R^m)$ then*

$$(13)\qquad y_\varepsilon(t) \to y^*(t), \quad \textit{for every } t \in [0,T],$$

where y^* is the solution to equation (E), corresponding to u^*.

Proof. Multiplying the equation of (S_i) with y_ε and taking into account the properties of the elements which appear, after some calculation, we get

$$(14)\qquad \begin{aligned} \|y_\varepsilon^-(i\varepsilon)\|^2 \le (\|e^{-\beta\varepsilon} y_\varepsilon^-((i-1)\varepsilon)\|^2 + 2D_2\varepsilon + \\ + \int_{(i-1)\varepsilon}^{i\varepsilon} \|B(u_\varepsilon(s))\|^2 ds) e^{D\varepsilon} \le (\|y_\varepsilon^-((i-1)\varepsilon)\|^2 + \\ + 2D_2\varepsilon + \int_{(i-1)\varepsilon}^{i\varepsilon} \|B(u_\varepsilon(s))\|^2 ds) e^{D\varepsilon} \end{aligned}$$

for $i = \overline{2, n-1}$, $D = 2D_1 + 1$.

Analogously, we have

$$(15)\qquad \|y_\varepsilon^-(\varepsilon)\|^2 \le (\|y_0\|^2 + 2D_2\varepsilon + \int_0^\varepsilon \|B(u_\varepsilon(s))\|^2 ds) e^{D\varepsilon}.$$

Adding the inequalities from the relations (14) and (15) we get

$$(16)\qquad \|y_\varepsilon^-(i\varepsilon)\| \le C, \; i = \overline{1,n},$$

where $C^2 = \left(\|y_0\|^2 + 2D_2T + \|Bu_\varepsilon\|_{L^2}^2\right) e^{DT}$. Also, we can show that

$$(17)\qquad \|y_\varepsilon^+(i\varepsilon)\| \le C, \; i = \overline{0, n-1},$$

and the sequence $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $C([0,T];R^n)$. We can calculate the variation of the function y_ε on $[0,T]$ and we obtain

$$(18)\qquad \begin{aligned} \bigvee_0^T y_\varepsilon = \sum_{i=1}^{n-1} \|y_\varepsilon^-(i\varepsilon) - y_\varepsilon^+(i\varepsilon)\| = \sum_{i=1}^{n-1} \|y_\varepsilon^-(i\varepsilon) - e^{-\beta\varepsilon} y_\varepsilon^-(i\varepsilon)\| \le \\ \le \varepsilon \sum_{i=1}^{n-1} \|\beta^0(y_\varepsilon^-(i\varepsilon))\|; \end{aligned}$$

we have used Lemma 2.1, p.121 from [2].

Relation (6) and hypothesis (H) imply $y_\varepsilon^-(i\varepsilon) \in \operatorname{int} D(\beta)$. Since β is bounded on every compact subset of $\operatorname{int} D(\beta)$, it result that

$$(19)\qquad \|\beta^0(y_\varepsilon^-(i\varepsilon))\| \le M.$$

Thus, there exists a constant such that

$$\bigvee_0^T y_\varepsilon \le C_1.$$

Helly's Theorem assures the existence of a subsequence $(\varepsilon_n)_{n\in N}$, with $\varepsilon_n \to 0$, and a function y^*, with bounded variation on $[0,T]$ so that

$$(20)\qquad y_{\varepsilon_n}(t) \to y^*(t), \quad (\forall) t \in [0,T].$$

Using the above assumptions, we can prove that y^* is the solution of equation (E) with (IC) (for the proof see [1]).

Theorem 2. *For $\varepsilon \to 0$ we have the convergence*

$$(21)\qquad \varphi^\varepsilon(t,y) \to \varphi(t,y), \quad (\forall) t \in [0,T], \; y \in R^n,$$

where φ is the solution of the variational equation (8).

Proof is similar with that presented in [1], p.127 and Theorem 1 is used as a partial result.

Remark 1. Theorem 2 permits us to approximate equation (8) by a sequence of more simple equations with solutions easily calculated.

Example 1. Maximize $\int_0^T y(t)dt$ over the set of all the functions $(y,u) \in W^{1,2}(0,T) \times U$ that satisfy the scalar variational inequality

$$y'(t) - y(t) + \partial I_K(y(t)) \ni u(t) \quad \text{a.e. } t \in]0,T[,$$

with the initial condition

$$y(0) = y_0 \in K,$$

where

$$K = \{z \in R;\ |z| \le a,\ a \in R_+^*\},$$
$$U = \{u \in L^2(0,T);\ |u(t)| \le \gamma,\ \gamma \in R_+^*,\ (\forall) t \in [0,T]\}.$$

The above problem is a particular case of the problem (P) for $m = n = 1$, $F = -\mathrm{Id}_R$, $\beta = \partial I_K$, $g(y) = -y$, $h = I_U$, $\varphi_0 = 0$; here Id_R and I_U stands for the identity function and the indicator function, respectively.

We have

$$h^*(u^*) = \gamma|u^*|.$$

The multivalued function $\beta : [-a,a] \subset R \to 2^R$ is defined by the relations

$$\beta(r) = \partial I_K(r) = \begin{cases} R^- & \text{for } r = -a \\ 0 & \text{for } -a < r < a \\ R^+ & \text{for } r = a. \end{cases}$$

Let ψ be the solution of Hamilton–Jacobi equation

$$\psi(t,x) = \sup\{\int_t^T y(s)ds;\ y'(s) - y(s) + \partial I_K(y(s)) \ni u(s)$$
$$\text{a.e. } s \in]t,T[,\ y(t) = x \in K\}.$$

Using the transformation $\varphi(t,x) = \psi(T-t,x)$ the Hamilton–Jacobi equation becomes

$$\begin{cases} \varphi_t(t,x) + \gamma|\varphi_x(t,x)| - x\varphi_x(t,x) = -x, \quad (t,x) \in [0,T] \times K, \\ \varphi(0,x) = 0,\ x \in K. \end{cases}$$

The fractional step scheme leads to the equations:

$$\begin{cases} \varphi_t^\varepsilon(t,x) + \gamma|\varphi_x^\varepsilon(t,x)| - x\varphi_x^\varepsilon(t,x) = 0, \quad (t,x) \in](i-1)\varepsilon, i\varepsilon[\times K;\ i = \overline{1,n}, \\ \varphi_+^\varepsilon((i-1)\varepsilon, x) = \varphi_-^\varepsilon((i-1)\varepsilon, Px) - \varepsilon x, i = \overline{1,n-1}, \\ \varphi_+^\varepsilon(0,x) = 0, \end{cases}$$

where $n\varepsilon = T$ and $P : R \to K$ is the projection function.

The differential equation

$$\varphi_t^\varepsilon(t,x) - (\gamma + x)\varphi_x^\varepsilon(t,x) = 0$$

with the initial condition

$$\varphi_+^\varepsilon((i-1)\varepsilon, x) = \varphi_+^\varepsilon((i-1)\varepsilon, x) - \varepsilon x =$$

$$\begin{cases} \varepsilon((i-2)\gamma - e^{(i-1)\varepsilon}(\gamma - a)(\frac{1}{e^\varepsilon} + \frac{1}{e^{2\varepsilon}} + \cdots + \frac{1}{e^{(i-2)\varepsilon}})) + \varepsilon a;\ x \le -a, \\ \varepsilon((i-2)\gamma - e^{(i-1)\varepsilon}(\gamma + x)(\frac{1}{e^\varepsilon} + \frac{1}{e^{2\varepsilon}} + \cdots + \frac{1}{e^{(i-2)\varepsilon}})) - \varepsilon x;\ -a < x < a, \\ \varepsilon((i-2)\gamma - e^{(i-1)\varepsilon}(\gamma + a)(\frac{1}{e^\varepsilon} + \frac{1}{e^{2\varepsilon}} + \cdots + \frac{1}{e^{(i-2)\varepsilon}})) - \varepsilon a;\ x \ge a, \end{cases}$$

has the solution

$$\varphi^\varepsilon(t,x) = \varepsilon(i-1)\gamma - \varepsilon e^{t-\varepsilon(i-1)}(\gamma + x)\frac{e^{(i-1)\varepsilon} - 1}{e^\varepsilon - 1}.$$

Using Theorem 2 we have

$$\varphi(t,x) = \lim_{\varepsilon \to 0} \varphi^\varepsilon(t,x) = t\gamma - (\gamma + x)(e^t - 1).$$

The exact and the approximate solutions satisfy the following inequality

$$|\varphi^\varepsilon(t,x) - \varphi(t,x)| \le C\varepsilon,$$

where $C = (\gamma + a)(e^T - 1)$.

The above inequality permits us to approximate, with a given error, the exact solution φ.

In this case, the suboptimal feedback control is given by the formula

$$u^\varepsilon(t) = -\gamma\,\mathrm{sgn}(\varphi_x^\varepsilon(t,x)),\quad x \in [(i-1)\varepsilon, i\varepsilon],\ i = \overline{1,n},\ n\varepsilon = T.$$

Example 2. Maximize $\int_0^T y^2(t)dt$ over the set of all the functions $(y,u) \in W^{1,2}(0,T) \times U$ that satisfy the scalar variational inequality

$$y'(t) - y(t) + \partial I_K(y(t)) \ni u(t) \quad \text{a.e. } t \in]0,T[,$$

with the initial condition

$$y(0) = y_0 \in K.$$

The significance of all elements is the same as in Example 1.

Analogously, ψ denote the solution of Hamilton–Jacobi equation

$$\psi(t,x)=\sup\{\int_t^T y^2(s)ds;\ y'(s)-y(s)+\partial I_K(y(s))\ni u(s)$$
$$\text{a.e. } s\in]t,T[,\ y(t)=x\in K\}.$$

Using the transformation $\varphi(t,x)=\psi(T-t,x)$ the Hamilton–Jacobi equation becomes

$$\begin{cases}\varphi_t(t,x)+\gamma|\varphi_x(t,x)|-x\varphi_x(t,x)=-x^2, & (t,x)\in[0,T]\times K,\\ \varphi(0,x)=0, & x\in K.\end{cases}$$

The fractional step scheme leads to the equations:

$$\begin{cases}\varphi_t^\varepsilon(t,x)+\gamma|\varphi_x^\varepsilon(t,x)|-x\varphi_x^\varepsilon(t,x)=0 & (t,x)\in](i-1)\varepsilon,i\varepsilon[\times K;\ i=\overline{1,n},\\ \varphi_+^\varepsilon(i\varepsilon,x)=\varphi_-^\varepsilon(i\varepsilon,Px)-\varepsilon x^2, & i=\overline{1,n-1},\\ \varphi_+^\varepsilon(0,x)=0,\end{cases}$$

where $n\varepsilon=T$ and $P:R\to K$ is the projection function.

Case a. If $\varphi_x^\varepsilon(t,x)<0$ then the differential equation

$$\varphi_t^\varepsilon(t,x)-(\gamma+x)\varphi_x^\varepsilon(t,x)=0,\quad (t,x)\in](i-1)\varepsilon,i\varepsilon[\times K$$

with the initial condition

$$\varphi_+^\varepsilon(i\varepsilon,x)=\varphi_-^\varepsilon(i\varepsilon,Px)-\varepsilon x^2=$$

$$\begin{cases}-\varepsilon\{[(\gamma-a)e^\varepsilon-\gamma]^2+[(\gamma-a)e^{2\varepsilon}-\gamma]^2+\cdots+[(\gamma-a)e^{(i-2)\varepsilon}-\gamma]^2\}-\\ \qquad -\varepsilon a^2, \quad x<-a\\ -\varepsilon\{[(\gamma+x)e^\varepsilon-\gamma]^2+[(\gamma+x)e^{2\varepsilon}-\gamma]^2+\cdots+[(\gamma+x)e^{(i-2)\varepsilon}-\gamma]^2\}-\\ \qquad -\varepsilon x^2, \quad -a\le x\le a\\ -\varepsilon\{[(\gamma+a)e^\varepsilon-\gamma]^2+[(\gamma+a)e^{2\varepsilon}-\gamma]^2+\cdots+[(\gamma+a)e^{(i-2)\varepsilon}-\gamma]^2\}-\\ \qquad -\varepsilon a^2, \quad x>a,\end{cases}$$

has the solution

$$\varphi^\varepsilon(t,x)=-\varepsilon[(\gamma+x)^2e^{2t-2\varepsilon}\frac{1-(e^{-2\varepsilon})^{i-1}}{1-e^{-2\varepsilon}}-$$
$$-2\gamma(\gamma+x)e^{t-\varepsilon}\frac{1-(e^{-\varepsilon})^{i-1}}{1-e^{-\varepsilon}}+(i-1)\gamma^2].$$

Theorem 2 leads to

$$\varphi(t,x)=\lim_{\varepsilon\to0}\varphi^\varepsilon(t,x)=-\frac{1}{2}(\gamma+x)^2(e^{2t}-1)+2\gamma(\gamma+x)(e^t-1)-t\gamma^2.$$

The error between the exact and the approximative solution is

$$|\varphi^\varepsilon(t,x)-\varphi(t,x)|\le C\varepsilon,$$

where $C=(\gamma+a)(e^T-1)(e^T+1+2\gamma)$.

Case b. If $\varphi_x^\varepsilon(t,x)>0$ then the differential equation

$$\varphi_t^\varepsilon(t,x)+(\gamma-x)\varphi_x^\varepsilon(t,x)=0,\quad (t,x)\in](i-1)\varepsilon,i\varepsilon[\times K$$

with the initial condition

$$\varphi_+^\varepsilon(i\varepsilon,x)=\varphi_-^\varepsilon(i\varepsilon,Px)-\varepsilon x^2=$$

$$\begin{cases}-\varepsilon\{[(\gamma-(a+\gamma)e^\varepsilon]^2+[\gamma-(a+\gamma)e^{2\varepsilon}]^2+\cdots+[\gamma-(a+\gamma)e^{(i-2)\varepsilon}]^2\}-\\ \qquad -\varepsilon a^2, \quad x<-a\\ -\varepsilon\{[(\gamma+(x-\gamma)e^\varepsilon]^2+[\gamma+(x-\gamma)e^{2\varepsilon}]^2+\cdots+[\gamma+(x-\gamma)e^{(i-2)\varepsilon}]^2\}-\\ \qquad -\varepsilon x^2, \quad -a\le x\le a\\ -\varepsilon\{[(\gamma+(a-\gamma)e^\varepsilon]^2+[\gamma+(a-\gamma)e^{2\varepsilon}]^2+\cdots+[\gamma+(a-\gamma)e^{(i-2)\varepsilon}]^2\}-\\ \qquad -\varepsilon a^2, \quad x>a,\end{cases}$$

has the solution

$$\varphi^\varepsilon(t,x)=-\varepsilon[(i-1)\gamma^2+2\gamma(x-\gamma)e^{t-\varepsilon}\frac{1-(e^{-\varepsilon})^{i-1}}{1-e^{-\varepsilon}}+$$
$$+(x-\gamma)^2e^{2t-2\varepsilon}\frac{1-(e^{-2\varepsilon})^{i-1}}{1-e^{-2\varepsilon}}.$$

Theorem 2 leads to

$$\varphi(t,x)=\lim_{\varepsilon\to0}\varphi^\varepsilon(t,x)=-t\gamma^2-2\gamma(x-\gamma)(e^t-1)-\frac{1}{2}(x-\gamma)^2(e^{2t}-1).$$

The error between the exact and the approximative solution is

$$|\varphi^\varepsilon(t,x)-\varphi(t,x)|\le C\varepsilon,$$

where $C=(a-\gamma)(e^T-1)(e^T+1-2\gamma)$.

The suboptimal feedback control is

$$u^{\varepsilon}(t) = -\operatorname{sgn}(\varphi_x^{\varepsilon}(t,x)),\ x \in [(i-1)\varepsilon, i\varepsilon],\ i = \overline{1,n},\ n\varepsilon = T.$$

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Technical University "Gh.Asachi" of Iaşi
Department of Mathematics
6600 Iaşi
ROMANIA

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
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ON GENERALIZED PSEUDO PROJECTIVE RICCI SYMMETRIC MANIFOLDS

BY

M.C. CHAKI and S.KOLEY

0. Introduction. In a recent paper [1] the authors introduced and studied a type of non–flat Riemannian manifold (M^n, g) $(n > 2)$ whose Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$(1) \qquad (\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(Y,X)$$

where A, B, C are non–zero–1–forms and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such a manifold was called a generalized pseudo Ricci symmetric manifold, the 1–forms A, B, C were called its associated 1–forms and an n–dimensional manifold of this kind was denoted by $G(PRS)_n$. If, in particular, $B = C = A$, then (1) takes the form

$$(2) \qquad (\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X)$$

which reduces to a pseudo Ricci symmetric manifold introduced by one of the authors (C h a k i) [2] who denoted such an n–dimensional manifold by $(PRS)_n$. This justifies the name generalized pseudo Ricci symmetric manifold for the manifold defined by (1) and the use of the symbol $G(PRS)_n$ for it.

It is known [3] that in a Riemannian manifold the projective curvature tensor W is defined by

$$(3) \qquad W(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]$$

where R and S are respectively the curvature tensor and the Ricci tensor. If

$$(4) \qquad 'W(X,Y,Z,U) = g[W(X,Y,Z),U]$$