Theorem C. *Let M^{2n+1} be an invariant submanifold of codimension 2 of a cs-manifold M^{2n+3}. Then M^{2n+1} is totally geodesic if and only if $M^(2n+1)$ is of constant ϕ-sectional curvature. In Theorem C, if M^{2n+1} is of constant ϕ-sectional curvature, then M^{2n+1} is η-Einstein (see the proof of Theorem 4.4 in [7]). Thus the curvature of M^{2n+1} is harmonic. So, Theorem 7.2, in the case where M^{2n+1} is a proper acs-manifold, is a theorem which corresponds to Theorem C.*

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3-9-1-403, Kameido,
Kōtō-ku 136, Tokyo
JAPAN

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLII, s.I.a, Matematică, 1996, f1.

SEMI-INVARIANT SUBMANIFOLDS OF A QUASI-K-SASAKIAN MANIFOLD

BY

C. CĂLIN

0. Introduction. The notion of semi-invariant submanifold introduced by A. Bejancu - N. Papaghiuc [3] has been intensively studied by many people (see references in [2]). In [7] Oubiña have obtained a clasification for almost contact structures, and in [5] D. Chinea obtained some results on invariant submanifolds of quasi-K-Sasakian manifolds.

It is the purpose of the present paper to consider and study the concept of semi-invariant submanifolds in case of a quasi-K-Sasakian manifold. First we present some general formulae and basic results from the theory of quasi -K- Sasaki manifolds in order to use them in the next sections. Next we obtain necessary and sufficient conditions for the integrability of distributions defined on a semi-invariant submanifold of a quasi-K-Sasakian manifold. Finally we a dealing with totally contact umbilical semi-invariant submanifolds of a quasi-K-Sasakian manifold. More precisely, we prove that a totally contact umbilical semi-invariant submanifold of $\tilde{M}$ is totally contact geodesic if $dim D^{\perp} > 1$ and $fN_f(X,Y) \in \Gamma(D \oplus \{\xi\})$ for any $X, Y \in \Gamma(D^{\perp})$.

1. Preliminaries. Let $\tilde{M}$ be a real 2n+1-dimensional differentiable manifold with (f, ξ, η, g) an almost contact metric structure. In this case we have

$$(1)\qquad \begin{cases} f^2 = -I + \eta \otimes \xi; & \eta(\xi) = 1; \quad \eta(X) = g(X, \xi) \\ \eta \circ f = 0; & f(\xi) = 0, \quad g(X, fY) + g(fX, Y) = 0, \end{cases}$$

for any vector fields X,Y on $\tilde{M}$, where I is the identity on the tangent bundle $T\tilde{M}$ of $\tilde{M}$.

Throughtout the paper, all manifolds and maps are differentiable of class C^∞. We denote by $F(\tilde{M})$ the algebra of the differentiable functions on $\tilde{M}$ and by $\Gamma(E)$ the $F(\tilde{M})$-module of the sections of a vector bundle E over $\tilde{M}$.

J. O u b i ñ a has introduced in [7] the notion of quasi-K-Sasakian structure as follows: an almost contact metric structure (f, ξ, η, g) is a quasi-K- Sasakian structure if and only if

(2) $$(\tilde{\nabla}_X f)Y + (\tilde{\nabla}_{fX} f)fY = 2g(X,Y)\xi + \eta(Y)\tilde{\nabla}_{fX}\xi - 2\eta(Y)X,$$

$\forall\, X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection with respect to the metric $\tilde{g}$. It follows that on a quasi-K-Sasakian manifold $\tilde{M}$ we have

(3) $$(\tilde{\nabla}_\xi f)X = 0; \quad 2fX = f\tilde{\nabla}_{fX}\xi - \tilde{\nabla}_X\xi; \quad \forall X \in \Gamma(T\tilde{M}).$$

Remark 1. It is easy to see that all integral curves of the vector field ξ are geodesic, that is $\tilde{\nabla}_\xi \xi = 0$.

The fundamental 2-form Φ of $\tilde{M}$ is defined by

(4) $$\Phi(X,Y) = g(X, fY), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

Next we have

Lemma 1.1 *Let $\tilde{M}$ be a quasi-K-Sasaki manifold. Then the structure tensor field f satisfies*

(5) $$(\tilde{\nabla}_{fX} f)Y + f(\tilde{\nabla}_X f)Y = 2g(\tilde{\nabla}_X\xi, Y)\xi - \eta(Y)fX - 2\Phi(X,Y)\xi,$$

$$\forall X, Y \in \Gamma(T\tilde{M}).$$

Proof. By straightforward calculation we obtain

(6)
$$(\tilde{\nabla}_X f)fY = -f(\tilde{\nabla}_X f)Y + g(\tilde{\nabla}_X\xi, Y)\xi + 2\eta(Y)\tilde{\nabla}_X\xi, \quad \forall\, X, Y \in \Gamma(T\tilde{M}).$$

By using (2) and (5) we deduce

(7) $$(\nabla_{fX} f)Y = -f(\tilde{\nabla}_X f)Y + g(\tilde{\nabla}_X\xi, Y)\xi - 2\eta(Y)fX - 2\Phi(X,Y)\xi,$$

$\forall X, Y \in \Gamma(T\tilde{M})$, and our assertion follows.

Now, we can prove

Lemma 1.2 *Let $\tilde{M}$ be a quasi-K-Sasaki manifold. Then we have*

(8) $$d\eta(X,Y) + d\eta(fX, fY) = 2\Phi(X,Y), \quad \forall\, X, Y \in \Gamma(T\tilde{M}).$$

Proof. By straightforward calculations, using [3] we deduce

$$2d\eta(X,Y) = g(\tilde{\nabla}_X\xi, Y) - g(\tilde{\nabla}_Y\xi, X) = g(f\tilde{\nabla}_{fX}\xi - 2fX, Y) -$$

$$-g(f\tilde{\nabla}_{fY}\xi - 2fY, X) = 4\Phi(X,Y) - 2d\eta(fX, fY), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

that is we get the assertion.

The Nijenhuis tensor field of f is given by

(9)
$$N_f(X,Y) = [fX, fY] + f^2[X,Y] - f[X, fY] - f[fX, Y], \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

By using [7] we obtain

Proposition 1.1 *Let $\tilde{M}$ be a quasi-K-Sasaki manifold. The Nijenhuis tensor field of f is given by*

(10) $$N_f(X,Y) = 2f((\tilde{\nabla}_Y f)X) - (\tilde{\nabla}_X f)Y + \eta(X)Y - \eta(Y)X -$$

$$-2\Phi(X,Y)\xi, \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

From [10] we derive

Corollary 1.1 *Let $\tilde{M}$ be a quasi-K-Sasaki manifold. Then we have*

(11) $$N_f(X,\xi) = -2f((\tilde{\nabla}_X)\xi + fX), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

Let $\tilde{M}$ be an almost contact metric manifold and M be an m-dimensional submanifold of $\tilde{M}$. Denote by $TM^\perp$ the normal vector bundle to M and by the same symbol g the induced metric tensor on M. The Gauss and Weingarten formulae are

(12) $$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N,$$

$$\forall\, X, Y \in \Gamma(TM), \ N \in \Gamma(TM^\perp),$$

where ∇ and $\nabla^{\perp}$ are the induced connections on TM and $TM^{\perp}$, respectively, A_N is the shape operator with respect to the section N and h is the fundamental form of M. It is known that

$$g(h(X,Y),N) = g(A_N X,Y), \quad \forall X,Y \in \Gamma(TM), \quad N \in \Gamma(TM^{\perp}). \tag{13}$$

Now we can state

Theorem 1.1 *Let M be a submanifold of a quasi-K-Sasaki manifold $\tilde{M}$. If the structure vector field ξ is normal to M then M is anti-invariant submanifold of $\tilde{M}$, that is $f(TM) \subset TM^{\perp}$.*

Proof. First we denote by tX the tangential part of fX. Then by using [3], [12] and [13], we derive

$$2g(tX,Y) = g(A_\xi X,Y) + g(A_\xi tX,tY), \quad \forall X,Y \in \Gamma(TM). \tag{14}$$

Because t is an antisymetric tensor and A_ξ is a symetric tensor with respect to the metric tensor g, from (14) we infer $g(tX,Y) = 0$ and our assertion follows.

Let $\tilde{M}$ be an almost contact metric manifold and M be an m-dimensional submanifold of M such that ξ is tangent to M. We say that M is a *semi-invariant submanifold* of $\tilde{M}$ if there exist two maximal distributions D and $D^{\perp}$ such that
$$TM = D \oplus D^{\perp} \oplus \{\xi\},$$
and
$$fX \in \Gamma(D); \quad fY \in \Gamma(TM^{\perp}), \quad \forall X \in \Gamma(D), Y \in \Gamma(D^{\perp}),$$
where ξ is the 1-dimensional distribution spanned by ξ. In case $D \neq 0$ and $D^{\perp} \neq 0$ we say that M is a proper submanifold of $\tilde{M}$.

Remark 2. The above definition has been given by A. B e j a n c u and N. P a p a g h i u c [3] for semi-invariant submanifolds of a Sasakian manifold. Several results for this case can be found in [2] and [8].

We denote by P and Q the projection morphisms of TM to D and $D^{\perp}$, respectively, and obtain

$$X = PX + QX + \eta(X)\xi, \quad \forall X \in \Gamma(TM). \tag{15}$$

2. Integrability of Distributions on a Semi-invariant Submanifold of a Quasi-K-Sasakian Manifod. The purpose of this section is to find necessary and sufficient conditions for the integrability of distributions on a semi-invariant submanifold of a quasi-K-Sasakian manifold.

First, by using [2], [8], [13] and [14] we deduce

Proposition 2.1 *Let M be a semi-invariant submanifold of a quasi -K - Sasakian manifold $\tilde{M}$. Then we have*

$$h(X,fY) - h(fX,Y) = \frac{1}{2} f N_f(X,Y) + f([X,Y]) + \eta(X)Y - \eta(Y)X + \tag{16}$$
$$+\nabla_X fY - \nabla_Y fX + (d\eta(X,fY) + d\eta(fX,Y))\xi, \quad \forall X,Y \in \Gamma(D \oplus \{\xi\}).$$

Now we can state

Theorem 2.1 *Let M be a proper semi-invariant submanifold of a quasi -K - Sasakian manifold $\tilde{M}$. Then the distribution D is not integrable.*

Proof. Suppose D is integrable. Then for any $X,Y \in \Gamma(D)$, by using (3) we infer
$$0 = \eta([X,Y]) = g(X, \tilde{\nabla}_Y \xi) - g(Y, \tilde{\nabla}_X \xi) =$$
$$g(Y, 2fX - f\tilde{\nabla}_{fX}\xi) - g(X, 2fY - f\tilde{\nabla}_{fY}\xi) = -4\Phi(X,Y) + 2d\eta(fX,fY).$$

Now, with help of (8), we infer $d\eta(X,Y) = 0$ that is $d\eta(fX,fY) = 0$ and consequently $\Phi(X,Y) = 0$. But, if we take $Y = fX$ then we derive $0 = g(X,X), \forall X,Y \in \Gamma(D)$, and our assertion follows.

Theorem 2.2 *Let M be a semi-invariant submanifold of a quasi -K - Sasakian manifold M. The distribution $D \oplus \{\xi\}$ is integrable if and only if the following conditions are satisfied*

$$h(X,fY) = h(fX,Y), \quad \forall X,Y \in \Gamma(D), \tag{17}$$

$$N_f(X,Y) \in \Gamma(D \oplus \{\xi\}) \quad \forall X,Y \in \Gamma(D), \tag{18}$$

$$\eta(A_{fZ}X) = 0, \quad \forall X \in \Gamma(D), \quad Z \in \Gamma(D^{\perp}). \tag{19}$$

Proof. Suppose that $D \oplus \{\xi\}$ is integrable. Then from (7) and (15) follows assertion (17) and (18). Next by straightforward calculation, we deduce
(20)
$$g([fX,\xi],Z) = g(\tilde{\nabla}_{fX}\xi - g(\tilde{\nabla}_\xi fX, Z) = g(\tilde{\nabla}_\xi X, fZ) + g(2fX + \tilde{\nabla}_X \xi, fZ) =$$

$$2g(h(X,\xi), fZ) = 2\eta(A_{fZ}X), \quad \forall X \in \Gamma(D), \quad Z \in \Gamma(D^{\perp}),$$

which proves (19).

Conversely, suppose that the assertion (16)-(19) are true. Then from (8) we deduce

$$f([X,Y]) = -\frac{1}{2}fN_f(X,Y) + \nabla_X fY - \nabla_Y fX \in \Gamma(TM). \tag{21}$$

Now our assertion follow from (19) and (21). Next we have

Proposition 2.3 *Let M be a semi-invariant submanifold of a Quasi-K-Saskian manifold $\tilde{M}$. Then, we have*

$$g(A_{fZ}Y - A_{fY}Z + \frac{1}{2}fN_f(Y,Z), W) = g([Y,Z], W),$$

and

$$\eta([Y,Z]) = 2d\eta(Z,Y), \quad \forall\, Y, Z, W \;\in\; \Gamma(D^{\perp}).$$

Proof. By straightforward calculation, by using (10), (12), (13) we infer

$$g(A_{fZ}Y - A_{fY}Z, W) = g(\tilde{\nabla}_Y W, fZ) - g(\tilde{\nabla}_Z W, fY) =$$
$$= g(W, \tilde{\nabla}_Z fY - \tilde{\nabla}_Y fZ) = g(W, (\tilde{\nabla}_Z f)Y - (\tilde{\nabla}_Y f)Z) =$$
$$= \frac{1}{2}g(fW, f((\tilde{\nabla}_Z f)Y - (\tilde{\nabla}_Y f)Z) + g(W, [Y,Z]) =$$
$$= \frac{1}{2}g(fW, N_f(Y,Z)) + g(W,[Y,Z]), \;\forall\;\, Y, Z, W \in \Gamma(D),$$

and our assertion follows.

Now we can prove

Theorem 2.3 *Let M be a semi-invariant submanifold of a Quasi-K-Saskian manifold $\tilde{M}$. The distribution $D^{\perp}$ is integrable if and only if*

$$d\eta(Y,Z) = 0. \quad \forall Y, Z \in \Gamma(D^{\perp}) \tag{22}$$

$$\frac{1}{2}(fN_f(Y,Z)) - A_{fY}Z + A_{fZ}Y \in \Gamma(D \oplus \{\xi\}), \quad \forall Y, Z \in \Gamma(D^{\perp}) \tag{23}$$

Proof. Let $Z, Y \in \Gamma(D^{\perp})$. By using (10) and Proposition 2.3 and the fact that ∇ is a Levi-Civita connection we obtain the above assertions.

Now we find the conditions necessary and sufficient for integrability of distribution $D^{\perp} \oplus \xi$.

Theorem 2.4 *Let M be a semi-invariant submanifold of a quasi-K-saskian manifold $\tilde{M}$. The distribution $D^{\perp} \oplus \xi$ is integrable if and only if (23) hold and*

$$d\eta(X,Z) = 0, \quad \forall\, X \in \Gamma(D), \quad Z \in \Gamma(D^{\perp}).$$

Proof. Let $Y, Z \in \Gamma(D^{\perp})$. Then $[Y,Z] \in \Gamma(D^{\perp} \oplus \{\xi\})$ iff (23) hold. Next by using (1), (3) and (13) we deduce

$$g([Z,\xi], X) = g(\tilde{\nabla}_Z \xi, X) - g(\tilde{\nabla}_\xi Z, X) = g(\tilde{\nabla}_Z \xi, X) + g(\tilde{\nabla}_{fX}\xi, fZ) = \tag{24}$$
$$= g(\tilde{\nabla}_Z \xi, X) - g(\tilde{\nabla}_X \xi + 2fX, Z) = 2d\eta(Z,X), \;\forall\, X \in \Gamma(D),$$

and our assertion follows.

3.Totally Contact Umbilical Semi-invariant Submanifolds of a Quasi -K- Sasakian Manifold. The purpose of this paragraph is to establish some properties of totally contact semi-invariant submanifolds of a Quasi-K-Sasakian manifold $\tilde{M}$.

Definition 3.1 *We say that a contact semi-invariant submanifold of a Quasi-K-Sasakian manifold $\tilde{M}$ is totally contact umbilical if there exists a normal vector field H, so that*

$$h(X,Y) = g(fX, fY)H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \quad \forall\, X, Y \in \Gamma(TM).$$

If $\quad H = 0$, then M is called totally contact geodesic. First we have

Proposition 3.1. *Let M be a contact semi-invariant submanifold of a quasi -K - Sasakian manifold $\tilde{M}$. Then we have*

$$QA_{fZ}Y = QA_{fY}Z \quad \forall\, Z, Y \in \Gamma(D^{\perp}).$$

if and only if $N_f(Y,Z) \in \Gamma(D \oplus \{\xi\})$.

Proof. Let $Y, Z, W \in \Gamma(D^{\perp})$ and by using (1), (7) (12) and (13) we derive

$$g(A_{fZ}Y - A_{fY}Z, W) = g(\tilde{\nabla}_Z fY - \tilde{\nabla}_Y fZ, W) = \tag{25}$$

$$= g((\tilde{\nabla}_Z f)Y - (\tilde{\nabla}_Y f)Z + f[Z,Y], W) =$$

$$= g(f((\tilde{\nabla}_Z f)Y - (\tilde{\nabla}_Y f)Z), fW) = g((\tilde{\nabla}_{fY} f)Z - (\tilde{\nabla}_{fZ} f)Y, W) =$$

$$= g([fY, fZ] + f\tilde{\nabla}_{fZ}Y - f\tilde{\nabla}_{fY}Z, fW) = g(A_{fY}Z - A_{fZ}Y, W),$$

and consequently $g(A_{fZ}Y - A_{fY}Z, W) = \frac{1}{2}g(fN_f(Y,Z), W)$, which proves our assertion.

Next we have

Proposition 3.2. *Let M be a totally contact umbilical semi-invariant submanifold of a quasi-K- Sasakian manifold $\tilde{M}$, such that $dim D^\perp > 1$ and $fN_f(Y,Z) \in \Gamma(D \oplus \{\xi\})$, for any $Y, Z \in \Gamma(D^\perp)$. Then H is orthogonal to $fD^\perp$.*

Proof. As $dim D^\perp > 1$, take $Y, Z \in \Gamma(D^\perp)$ such that $g(Y,Y) = 1$ and $g(Y,Z) = 0$, and by straightforward calculation with help of Proposition 3.1, we infer

$$g(H, fZ) = g(h(Y,Y), fZ) = g(A_{fZ}Y, Y) = g(QA_{fZ}Y, Y) =$$

$$g(QA_{fY}Z, Y) = g(h(Y,Z), fY) = 0,$$

which proves our assertion.

Now, we can prove

Theorem 3.1 *Let M be a totally contact umbilical semi-invariant submanifold of a quasi-K- Sasakian manifold $\tilde{M}$. If $dim D^\perp > 1$, and $fN_f(Y,Z) \in \Gamma(D \oplus \xi$, for any $Y, Z \in \Gamma(D^\perp)$ then M is totally contact geodesic.*

Proof. Let $X \in \Gamma(D \oplus D^\perp)$ be a unit vector field. Because $dim D^\perp > 1$, from Proposition 3.2 we have $fD^\perp \perp H$, and consequently $fH \in \Gamma(TM^\perp)$. Next by using (5) we deduce

$$(\tilde{\nabla}_{fX} f)X = -f((\tilde{\nabla}_X f)X) + g(X, \tilde{\nabla}_X \xi)\xi$$

and consequently

$$g((\tilde{\nabla}_{fX} f)X, H) = g((\tilde{\nabla}_X f)X, fH).$$

But

(26)
$$g((\tilde{\nabla}_{fX} f)X, H) = -g((\tilde{\nabla}_{fX} f)H, X) = g(A_{fH}fX, X) + g(f\tilde{\nabla}_{fX}H, X) =$$

$$= g(A_H fX, fX) = g(H, H),$$

and

$$g((\tilde{\nabla}_X f)X, fH) = g(h(X, fX), fH) - g(h(X,X), H) = -g(H,H). \tag{27}$$

Now our assertion follow from (26) and (27).

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Technical University "Gh.Asachi" Iasi
Department of Mathematics
6600 Iasi,
ROMANIA