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TRANSFORMATIONS OF CONNECTIONS IN THE GENERALIZED HAMILTON SPACES

BY

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Abstract. *In the tangent space of dual vector bundle* $\xi^* = (E^*, \pi^*, M)$, $\dim M = n$, $\dim E^* = n+m$, *using two different nonlinear connections* N_{ai} *and* $\overline{N}_{ai}$, *two adapted bases* $B = \{\delta_i, \partial^a\}$ *and* $\overline{B} = \{\overline{\delta}_i \partial^a\}$ *are introduced. The relations between components of the generalized linear connection* ∇, *the metric tensor and the torsion tensor expressed in these two bases are given. It is proved, that the d-connection in the basis* B *will be d-connection in the basis* $\overline{B}$ *iff* $N_{ai} - \overline{N}_{ai}$ *is h- and v-parallel tensor field. The recurrent connection in* B *will be recurrent in* $\overline{B}$, *but with different vector of recurrency.*

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1. The generalized Hamilton spaces. Let $\xi^* = (E^*, \pi^*, M)$ be a C^∞ 1-form bundle with $\dim M = n, \dim E^* = n+m$. In some local chart the point $u \in E^*$ has coordinates

$$(x^1, \ldots, x^n, p_1, \ldots, p_m) = ((x^i), (p_a)) = (x, p)$$

$$i, j, h, k, l = 1, \ldots, n \qquad a, b, c, d, e, f = 1, \ldots, m.$$

We shall consider the following transformation of coordinate system. If $((x^{i'}), (p_{a'}))$ are the coordinates of the same point u in the new coordinate system, then

$$\text{(1.1)} \qquad \text{(a)}\ x^{i'} = x^{i'}(x^1, \ldots, x^n) \qquad \operatorname{rank}\,[\partial x^{i'}/\partial x^i] = n$$

$$\text{(b)}\ p_{a'} = M^a_{a'}(x^1, \ldots, x^n) p_a \quad \operatorname{rank}\,[\partial p_{a'}/\partial p_a] = m.$$

If (1.1) is valid, then an inverse transformation exists

$$(1.2)\qquad \text{(a)}\ \ x^i = x^i(x^{1'},\dots,x^{n'}) \qquad M_a^{a'} M_{b'}^{a} = \delta_{b'}^{a'}$$

$$\text{(b)}\ \ p_a = M_a^{a'}(x^{1'},\dots,x^{n'})p_{a'}\ \ M_b^{a'} M_{a'}^{a} = \delta_b^a.$$

The tangent space of E^*, the space $T(E^*)$, is spanned by $\{\partial_i,\partial^a\}$, where

$$(1.3)\qquad \partial_i = \partial/\partial x^i \qquad \partial^a = \partial/\partial p_a.$$

They have the following law of transformation

$$(1.4)\qquad \begin{array}{l} (a)\ \ \partial^a = M_{a'}^{a}\partial^{a'} \qquad \partial^{a'} = M_a^{a'}\partial^a \\ (b)\ \ \partial_i = (\partial_i x^{i'})\partial_{i'} + (\partial_i M_{a'}^{a})p_a\partial^{a'}. \end{array}$$

From (1.4b) it is obvious that ∂_i are not transforming as tensors, so we introduce a new, the so called adapted basis $B = \{\delta_i,\partial^a\}$, where

$$(1.5)\qquad \delta_i = \partial_i - N_{ai}(x,p)\partial^a$$

and $N_{ai}(x,p)$ are the coefficients of the nonlinear connection. Under coordinate transformation (1.1) and (1.2) they transform in the following way

$$(1.6)\qquad \begin{array}{l} (a)\ \ N_{a'i'}(x',p') = M_{a'}^{a}(x)(\partial_{i'}x^i)N_{ai}(x,p) - (\partial_i M_{a'}^{a}(x))p_a(\partial_{i'}x^i) \\ (b)\ \ N_{bj}(x,p) = M_b^{b'}(x')(\partial_j x^{j'})N_{b'j'}(x',p') + (\partial_j M_{a'}^{a}(x))p_a M_b^{a'}. \end{array}$$

When M is paracompact, then exists a family of functions N_{ai} of type (1.6). $\delta_i (i = 1,\dots,n)$ are transforming as tensors, i.e.

$$(1.7)\qquad \delta_{i'} = \delta_i(\partial_{i'}x^i).$$

Any field X in $T(E^*)$ in the basis B can be expressed in the form

$$(1.8)\qquad X = X^i\delta_i + X_a\partial^a \quad X^i = X^{i'}(\partial_{i'}x^i) \quad X_a = M_a^{a'}X_{a'}.$$

$X^i\delta_i$ is the horizontal and $X_a\partial^a$ the vertical part of X. The subspace of $T(E^*)$ spanned by $\{\delta_i\}$ we shall denote by $T_H(E^*)$ and the subspace spanned by $\{\partial_a\}$ with $T_V(E^*)$. So we have

$$T(E^*) = T_H(E^*)\oplus T_V(E^*) \quad \dim T_H(E^*) = n \quad \dim T_V(E^*) = m.$$

The dual tangent space of E^*, the space $T^*(E^*)$ is spanned by

$$\{dx^1,\dots,dx^n,dp_1,\dots,dp_m\} = \{dx^i,dp_a\}.$$

From (1.1) follows

$$(1.9)\qquad (a) dx^{i'} = (\partial_i x^{i'})dx^i \quad (b) dp_{a'} = (\partial_i M_{a'}^{a}(x))p_a dx^i + M_{a'}^{a}dp_a.$$

It is obvious that $dp_a (a = 1,\dots,m)$ are not transforming as tensors, so the new basis $B^* = \{dx^i,\delta p_a\}$ of $T^*(E^*)$ is introduced, where

$$(1.10)\qquad \delta p_a = dp_a + N_{ai}dx^i.$$

By the coordinate transformation (1.1) and (1.2) the bases B^* and $B^{*'} = \{dx^{i'},\delta p_{a'}\}$ are related by (1.9a) and

$$(1.11)\qquad (a)\ \ \delta p_a = M_a^{a'}(x')\delta p_{a'} \quad (b)\ \ \delta p_{a'} = M_{a'}^{a}(x)\delta p_a.$$

If B^* and $B^{*'}$ are two bases of $T^*(E^*)$ related by (1.9a) and (1.11), then any field $w \in T^*(E^*)$ satisfies the relations

$$(1.12)\qquad w = w_i dx^i + w^a\delta p_a = w_{i'}dx^{i'} + w^{a'}\delta p_{a'},$$

where

$$(1.13)\qquad w_{i'} = w_i(\partial_{i'}x^i) \qquad w^{a'} = w^a M_a^{a'}.$$

The bases B and B^* are dual to each other. In the space $T^*(E^*)\otimes T^*(E^*)$ with respect to the basis B^* the metric tensor G is given by

$$(1.14)\qquad G = g_{ij}dx^i\otimes dx^j + g_i{}^b dx^i\otimes\delta p_b + g^a{}_j\delta p_a\otimes dx^j + g^{ab}\delta p_a\otimes\delta p_b.$$

Definition 1.1. *The generalized linear connection $\nabla_X : T(E^*)\to T(E^*), X\in T(E^*)$ is a linear connection determined by*

$$(1.15)\qquad \begin{array}{ll} \nabla_{\delta_i}\delta_j = F_j{}^k{}_i\delta_k + F_{jci}\partial^c & \nabla_{\delta_j}\partial^a = F^{ak}{}_j\delta_k + F^a{}_{cj}\partial^c \\ \nabla_{\partial^a}\delta_j = C_j{}^{ka}\delta_k + C_{jc}{}^a\partial^c & \nabla_{\partial^a}\partial^b = C^{bka}\delta_k + C^b{}_c{}^a\partial^c. \end{array}$$

Definition 1.2. *The differentiable manifold E^* (in which the coordinate transformations of type (1.1) and (1.2) are allowed) supplied with arbitrary nonlinear connection N (which satisfies (1.6)) and the metric tensor*

G (given by (1.14)) in which the linear connection ∇ is defined (by (1.15)) is called generalized Hamilton space and is denoted by (E^*, N, G, ∇).

Using the duality of bases $B = \{\delta_i, \partial^a\}$ and $B^* = \{dx^i, \delta p_a\}$ we obtain

$$\nabla_{\delta_i} dx^j = F^{\,j}_{ki} dx^k + F^{jc}{}_i \delta p_c \qquad \nabla_{\delta_i} \delta p_b = F_{bki} dx^k + F_b{}^c{}_i \delta p_c$$
$$\nabla_{\partial^a} dx^j = C^{j}{}_k{}^a dx^k + C^{jc}{}_a \delta p_a \qquad \nabla_{\partial^a} \delta p_b = C_{bk}^{\ a} dx^k + C_b{}^{ca} \delta p_c,$$

where

$$F^{\,j}_{ki} = -F_k{}^j{}_i,\ F^{jc}{}_i = -F^{cj}{}_i,\ F_{bki} = -F_{kbi},\ F_b{}^c{}_i = -F^c{}_{bi}$$

$$C^{j}{}_k{}^a = -C_k{}^{ja},\ C^{jca} = -C^{cja},\ C_{bk}^{\ a} = -C_{kb}^{\ a},\ C_b{}^{ca} = -C^c{}_b{}^a.$$

Proposition 1.1. *If $X, Y \in T(E^*)$, where*

(1.16) $$X = X^i\delta_i + X_a\partial^a \qquad Y = Y^j\delta_j + Y_b\partial^b,$$

then

(1.17) $$\nabla_X Y = (Y^k_{|i} X^i + Y^k|^a X_a)\delta_k + (Y_{c|i} X^i + Y_c|^a X_a)\partial^c$$

(1.18)
$$\begin{aligned}
&(a)\ \ Y^k_{|i} = \delta_i Y^k + F_j{}^k{}_i Y^j + F^{bk}{}_i Y_b\\
&(b)\ \ Y^k|^a = \partial^a Y^k + C_j{}^{ka} Y^j + C^{bka} Y_b\\
&(c)\ \ Y_{c|i} = \delta_i Y_c + F_{jci} Y^j + F^b{}_{ci} Y_b\\
&(d)\ \ Y_c|^a = \partial^a Y_c + C_{jc}^{\ a} Y^j + C^b{}_c{}^a Y_b.
\end{aligned}$$

Proposition 1.2. *If $((x^i),(p_a))$ and $((x^{i'}),(p_{a'}))$ are two coordinate systems connected by (1.1) and (1.2), then*

(1.19) $$\nabla_{X'} Y' = \nabla_X Y$$

iff $Y^k_{|i}, Y^k|^a, Y_{c|i}$ and $Y_c|^a$ are transforming as tensors, i.e.

(1.20)
$$\begin{aligned}
&Y^{k'}_{|i'} = Y^k_{|i}(\partial_{i'} x^i)(\partial_k x^{k'}) \quad Y^{k'}|^{a'} = Y^k|_a(\partial_k x^{k'}) M^a_{a'}\\
&Y_{c'|i'} = Y_{c|i} M^c_{c'} (\partial_{i'} x^i) \qquad Y_{c'}|^{a'} = Y_c|^a M^c_{c'} M^{a'}_a,
\end{aligned}$$

or equivalently, iff the connection coefficients have the following law of transformation

(1.21)
$$\begin{aligned}
&(a)\ \ F_j{}^k{}_i = F_{j'}{}^{k'}{}_{i'} (\partial_j x^{j'})(\partial_{k'} x^k)(\partial_i x^{i'}) + (\partial_i \partial_j x^{k'})(\partial_{k'} x^k)\\
&(b)\ \ F_{jci} = F_{j'c'i'} (\partial_j x^{j'}) M^{c'}_c (\partial_i x^{i'})\\
&(c)\ \ F^{bk}{}_i = F^{b'k'}{}_{i'} M^b_{b'} (\partial_{k'} x^k)(\partial_i x^{i'})\\
&(d)\ \ F^b{}_{ci} = F^{b'}{}_{c'i'} M^b_{b'} M^{c'}_c (\partial_i x^{i'}) + (\partial_i M^b_{c'}) M^{c'}_c\\
&(e)\ \ C_j{}^{ka} = C_{j'}{}^{k'a'} (\partial_j x^{j'})(\partial_{k'} x^k) M^a_{a'}\\
&(f)\ \ C_{jc}^{\ a} = C_{j'c'}^{\ a'} (\partial_j x^{j'}) M^{c'}_c M^a_{a'}\\
&(g)\ \ C^{bka} = C^{b'k'a'} M^b_{b'} (\partial_{k'} x^k) M^a_{a'}\\
&(h)\ \ C^b{}_c{}^a = C^{b'}{}_{c'}{}^{a'} M^b_{b'} M^{c'}_c M^a_{a'}.
\end{aligned}$$

Proposition 1.3. *The torsion tensor*

(1.22) $$T(X, Y) = \nabla_X Y - \nabla_Y X - [XY]$$

for the linear connection ∇ determined by (1.15) has the form

(1.23)
$$\begin{aligned}
T(X,Y) =& (T_j{}^k{}_i Y^j X^i + T^{bk}_{\ \ i} Y_b X^i + T_j{}^{ka} Y^j X_a + T^{bka} Y_b X_a)\delta_k +\\
& (T_{jci} Y^j X^i + T^b{}_{ci} Y_b X^i + T_{jc}{}^a Y^j X_a + T^b{}_c{}^a Y_b X_a)\partial^c,
\end{aligned}$$

where

(1.24)
$$\begin{aligned}
&(a)\ \ T_j{}^k{}_i = F_j{}^k{}_i - F_i{}^k{}_j\\
&(b)\ \ T^{bk}{}_i = F^{bk}{}_i - C_i{}^{kb}\\
&(c)\ \ T_j{}^{ka} = C_j{}^{ka} - F^{ak}{}_j\\
&(d)\ \ T^{bka} = C^{bka} - C^{akb}\\
&(e)\ \ T_{jci} = F_{jci} - F_{icj} + \delta_i N_{cj} - \delta_j N_{ci}\\
&(f)\ \ T^b{}_{ci} = F^b{}_{ci} - C_{ic}{}^b - \partial^b N_{ci}\\
&(g)\ \ T_{jc}{}^a = C_{jc}{}^a - F^a{}_{cj} + \partial^a N_{cj}\\
&(h)\ \ T^b{}_c{}^a = C^b{}_c{}^a - C^a{}_c{}^b.
\end{aligned}$$

Definition 1.3. *The generalized Hamilton space (E^*, N, G, ∇) in which the fixed torsion tensor T is given (which satisfies (1.22)-(1.24)) is denoted by (E^*, N, G, ∇, T).*

2. Transformations of the generalized connection in the generalized Hamilton spaces. The generalized connection ∇ defined by (1.15) is the function of the nonlinear connection N, because the basis vectors of $B = \{\delta_i, \partial^a\}$ are determined by N. We shall consider another nonlinear connection $\overline{N}_{bj}(x,p)$, which has the transformation law of form (1.6). The difference of two nonlinear connections N and $\overline{N}$ is transforming as tensor, namely

$$\overline{N}_{bj} - N_{bj} = M_b^{b'}(x)(\partial_j x^{j'})(\overline{N}_{b'j'} - N_{b'j'}). \tag{2.1}$$

We shall consider the change of components of the generalized connection ∇ expressed in the new basis formed by $\overline{N}$, where

$$\overline{N}_{bj} = N_{bj} + A_{bj}. \tag{2.2}$$

$$(a)\ \ \overline{\delta}_i = \partial_i - \overline{N}_{ai}(x,p)\partial^a, \quad (b)\ \ \overline{\delta}p_a = dp_a + \overline{N}_{ai}(x,p)dx^i. \tag{2.3}$$

The comparation of (2.2), (2.3) with (1.5) and (1.10) gives

$$(a)\ \ \overline{\delta}_i = \delta_i - A_{ai}\partial^a \quad (b)\ \ \delta\overline{p}_a = \delta p_a + A_{ai}dx^i. \tag{2.4}$$

The new adapted bases (formed by $\overline{N}$) of $T(E^*)$ and $T^*(E^*)$ are respectively

$$\overline{B} = \{\overline{\delta}_i, \partial^a\}, \quad \overline{B}^* = \{dx^i, \overline{\delta}p_a\}. \tag{2.5}$$

The vector field $X \in T(E^*)$ and the 1-form $\omega \in T^*(E^*)$ in the bases (2.5) are given by

$$\begin{aligned} &(a)\ \ X = \overline{X}^i\overline{\delta}_i + \overline{X}_a\partial^a = \overline{X}^i(\delta_i - A_{ai}\partial^a) + \overline{X}_a\partial^a \\ &(b)\ \ \omega = \overline{\omega}_i dx^i + \overline{\omega}^a(\delta p_a + A_{ai}dx^i). \end{aligned} \tag{2.6}$$

The comparation of (2.6) with (1.12) and (1.16) results:

$$\begin{aligned} &(a)\ \ X^i = \overline{X}^i \qquad &&(b)\ \ X_a = \overline{X}_a - A_{ai}\overline{X}^i \\ &(c)\ \ \omega_i = \overline{\omega}_i + A_{ai}\overline{\omega}^a \quad &&(d)\ \ \omega^a = \overline{\omega}^a. \end{aligned} \tag{2.7}$$

In [19] the transformation $N \to \overline{N}$ of the connection $H\Gamma(H_j{}^i{}_k, C_i{}^{jk})$ under condition $\nabla X^i = \overline{\nabla} X^i$ was considered. Here we shall examine the

relations between coefficients of the same linear connection ∇ expressed in the different bases B and $\overline{B}$.

Similar to (1.15) we define ∇ in the basis $\overline{B}$ by

$$\begin{aligned} &\nabla_{\overline{\delta}_i}\overline{\delta}_j = \overline{F}_j{}^k{}_i\overline{\delta}_k + \overline{F}_{jci}\partial^c \quad &&\nabla_{\overline{\delta}_j}\partial^a = \overline{F}^{ak}{}_j\overline{\delta}_k + \overline{F}^a{}_{cj}\partial^c \\ &\nabla_{\partial^a}\overline{\delta}_j = \overline{C}_j{}^{ka}\overline{\delta}_k + \overline{C}_{jc}{}^a\partial^c \quad &&\nabla_{\partial^a}\partial^b = \overline{C}^{bka}\overline{\delta}_k + \overline{C}^b{}_c{}^a\partial^c. \end{aligned} \tag{2.8}$$

Proposition 2.1. *The coefficients of the same generalized linear connection $\nabla : T(E^*) \otimes T(E^*) \to T(E^*)$ expressed in the bases B and $\overline{B}$ are connected by*

$$\begin{aligned} &(a)\ \ \overline{F}_j{}^k{}_i = F_j{}^k{}_i - A_{bj}F^{bk}{}_i - A_{ai}C_j{}^{ka} + A_{ai}A_{bj}C^{bka} \\ &(b)\ \ \overline{F}_{jci} = F_{jci} - A_{bj}A_{ck}F^{bk}{}_i - A_{ai}(C_{jc}{}^a - A_{bj}A_{ck}C^{bka}) + \\ &\qquad + A_{ai}A_{cj}|^a - A_{cj|i} \\ &(c)\ \ \overline{F}^{ak}{}_j = F^{ak}{}_j - A_{bj}C^{akb} \\ &(d)\ \ \overline{F}^a{}_{cj} = F^a{}_{cj} + A_{ck}F^{ak}{}_j - A_{bj}(C^a{}_c{}^b + A_{ck}C^{akb}) \\ &(e)\ \ \overline{C}_j{}^{ka} = C_j{}^{ka} - A_{cj}C^{cka} \\ &(f)\ \ \overline{C}_{jc}{}^a = C_{jc}{}^a - A_{bj}A_{ck}C^{bka} - A_{cj}|^a \\ &(g)\ \ \overline{C}^{bka} = C^{bka} \\ &(h)\ \ \overline{C}^{bca} = C^{bca} + A_{ck}C^{bka}, \end{aligned} \tag{2.9}$$

where for $A = A_{cj}\partial^c \otimes dx^j$ we have

$$\begin{aligned} &(a)\ \ A_{cj}|^a = \partial^a A_{cj} + A_{bj}C^b{}_c{}^a - A_{ck}C_j{}^{ka} \\ &(b)\ \ A_{cj|i} = \delta_i A_{cj} + A_{bj}F^b{}_{ci} - A_{ck}F_j{}^k{}_i. \end{aligned} \tag{2.10}$$

Theorem 2.1. *The Miron's d-connection ∇ in the adapted basis $B = \{\delta_i, \partial^a\}$ in general case will not be d-connection in the adapted basis $\overline{B} = \{\overline{\delta}_i, \partial^a\}$.*

Proof. As it is known the d-connection ∇ is the linear connection with the properties:

$$\nabla : T(E^*) \otimes T_H(E^*) \to T_H(E^*)$$
$$\nabla : T(E^*) \otimes T_V(E^*) \to T_V(E^*)$$

or expressed in coordinates:

$$\nabla_{\delta_i}\delta_j = F_j{}^k{}_i\delta_k \quad \nabla_{\delta_j}\partial^a = F^a{}_{cj}\partial^c$$
$$\nabla_{\partial^a}\delta_j = C_j{}^{ka}\delta_k \quad \nabla_{\partial^a}\partial^b = C^b{}_c{}^a\partial^c. \tag{2.11}$$

Comparing (1.15) with (2.11) it is obvious that if we put

$$F_{jci} = 0, \quad F^{ak}{}_j = 0, \quad C_{jc}{}^a = 0, \quad C^{bka} = 0 \tag{2.12}$$

in (1.15), then the d-connection defined by (2.11) is obtained.

By substituting (2.12) into (2.9) we obtain that the coefficients of the d-connection ∇ in the basis $\overline{B}$ have the form:

$$\begin{aligned}
&(a)\ \overline{F}_j{}^k{}_i = F_j{}^k{}_i - A_{ai}C_j{}^{ka}\\
&(b)\ \overline{F}_{jci} = A_{ai}A_{cj}|^a - A_{cj|i}\\
&(c)\ \overline{F}^{ak}{}_j = 0\\
&(d)\ \overline{F}^a{}_{cj} = F^a{}_{cj} - A_{bj}C^a_c{}^b\\
&(e)\ \overline{C}_j{}^{ka} = C_j{}^{ka} \quad (f)\ \overline{C}_{jc}{}^a = -A_{cj}|^a\\
&(g)\ \overline{C}^{bka} = 0 \quad (h)\ \overline{C}^{bca} = C^{bca}.
\end{aligned} \tag{2.13}$$

where $A_{cj}|^a$ and $A_{cj|i}$ are determined by (2.10)(a) and (2.10)(b), but in (2.13) the not overlined connection coefficients are those from (2.11). In [19] the connection only on $T_H(E^*)$ was considered and (6.3) from [19] corresponds to (2.13)(a) and (1.13)(c). From (2.13) follows:

Theorem 2.2. *The necessary and sufficient conditions, that the d-connection ∇ in the basis $B = \{\delta_i, \partial^a\}$, be also d-connection in the basis $\overline{B} = \{\overline{\delta}_i, \partial^a\}$, are*

$$A_{cj}|^a = 0, \quad A_{cj|i} = 0, \tag{2.14}$$

i.e. the tensor field A_{cj} in the basis B, with respect to the d-connection should be h-parallel and v-parallel.

Theorem 2.3. *The components of the same torsion tensor T (defined by (1.22) and (1.23)) of the generalized connection ∇ expressed in the bases*

B and $\overline{B}$ are connected by

$$\begin{aligned}
&(a)\ \overline{T}_j{}^k{}_i = T_j{}^k{}_i - T^{bk}{}_iA_{bj} - T_j{}^{ka}A_{ai} + T^{bka}A_{bj}A_{ai}\\
&(b)\ \overline{T}^{bk}{}_i - T^{bka}A_{ai}\\
&(c)\ \overline{T}_j{}^{ka} = T_j{}^{ka} - T^{bka}A_{bj}\\
&(d)\ \overline{T}^{bka} = T^{bka}\\
&(e)\ \overline{T}_{jci} = T_{jci} + T_j{}^k{}_iA_{ck} - T^{bk}_iA_{bj}A_{ck} + T^{bkj}A_{bi}A_{ck} +\\
&\qquad T^{bka}A_{bj}A_{ck}A_{ai} + T^b_c{}^aA_{ai}A_{bj} + T^a_{cj}A_{ai} - T^a_{ci}A_{aj}\\
&(f)\ \overline{T}^b{}_{ci} = T^b{}_{ci} - T^b_c{}^aA_{ai} + T^{bk}{}_iA_{ck} - T^{bka}A_{ai}A_{ck}\\
&(g)\ \overline{T}_{jc}{}^a = T_{jc}{}^a - T^b_c{}^aA_{bj} + T_j{}^{ka}A_{ck} - T^{bka}A_{bj}A_{ck}\\
&(h)\ \overline{T}^b_c{}^a = T^b_c{}^a + T^{bka}A_{ck}.
\end{aligned} \tag{2.15}$$

The proof follows from (1.24), (2.4) and (2.13).

Theorem 2.4. *The torsion tensor $T(X,Y)$ for the Miron's d-connection defined by (1.22) satisfies (1.24), but their components are given by*

$$\begin{aligned}
&(a)\ T_j{}^k{}_i = F_j{}^k{}_i - F_i{}^k{}_j \quad (e)\ T_{jci} = \delta_iN_{cj} - \delta_jN_{ci}\\
&(b)\ T^{bk}{}_i = -C_i{}^{kb} \quad (f)\ T^b{}_{ci} = F^b{}_{ci} - \partial^bN_{ci}\\
&(c)\ T_j{}^{ka} = C_j{}^{ka} \quad (g)\ T_{jc}{}^a = \partial^aN_{cj} - F^a{}_{cj}\\
&(d)\ T^{akb} = 0 \quad (h)\ T^b_c{}^a = C^b_c{}^a - C^a_c{}^b.
\end{aligned} \tag{2.16}$$

The proof follows from (1.24) and (2.12).

Theorem 2.5. *The components of the same torsion tensor $T(X,Y)$ defined by (1.22) of Miron's d-connection in the bases B and $\overline{B}$ are connected* *by*

$$
\text{(2.17)}\qquad
\begin{aligned}
&(a)\ \overline{T}_{j\ i}^{\ k} = T_{j\ i}^{\ k} - T^{bk}_{\ \ i}A_{bi} - T_j^{\ ka}A_{ai}\\
&(b)\ \overline{T}^{bk}{}_i = T^{bk}{}_i\\
&(c)\ \overline{T}_j^{\ ka} = T_j^{\ ka}\\
&(d)\ \overline{T}^{bka} = T^{bka} = 0\\
&(e)\ \overline{T}_{jci} = T_{jci} + T_{j\ i}^{\ k}A_{ck} - T^{bk}_{\ \ i}A_{bj}A_{ck}+\\
&\qquad T^{bk}{}_jA_{bi}A_{ck} + T^{b\ a}_{\ c}A_{ai}A_{bj} + T^{a}_{cj}A_{ai} - T^{a}_{ci}A_{aj}\\
&(f)\ \overline{T}^{b}_{ci} = T^{b}_{ci} - T^{b\ a}_{\ c}A_{ai} + T^{bk}{}_iA_{ck}\\
&(g)\ \overline{T}_{jc}^{\ \ a} = T_{jc}^{\ \ a} - T^{b\ a}_{\ c}A_{bj} + T_j^{\ ka}A_{ck}\\
&(h)\ \overline{T}^{b\ a}_{\ c} = T^{b\ a}_{\ c}
\end{aligned}
$$

where the components of $T(X,Y)$ in the basis B are given by (2.16).

3. The metric connection in different bases. In $T^*(E^*)\otimes T^*(E^*)$ the metric tensor G expressed in the basis B^* is given by (1.14).

Theorem 3.1. *The components of G in the bases B^* and $\overline{B}^*$ are connected by*

$$
\text{(3.1)}\qquad
\begin{aligned}
\overline{g}_{ij} &= g_{ij} - g_i^{\ b}A_{bj} - g^{a}_{\ j}A_{ai} + g^{ab}A_{ai}A_{bj}\\
\overline{g}_i^{\ b} &= g_i^{\ b} - g^{ab}A_{ai}\\
\overline{g}^{a}_{\ j} &= g^{a}_{\ j} - g^{ab}A_{bj}\\
\overline{g}^{ab} &= g^{ab}.
\end{aligned}
$$

Proof. The metric tensor G given by (1.14), in the basis $\overline{B}^*$ has the form

$$
\text{(3.2)}\qquad \overline{G} = \overline{g}_{ij}dx^i\otimes dx^j + \overline{g}_i^{\ b}dx^i\otimes\overline{\delta}p_b + \overline{g}^{a}_{\ j}\overline{\delta}p_a\otimes dx^j + \overline{g}^{ab}\overline{\delta}p_b.
$$

If we substitute (2.4)(b) into (1.14) and compare the obtained expression with (3.2), we obtain (3.1).

The metrical connection has the property that h- and v-covariant derivatives of all components of the metric tensor are equal to zero. The question is the following: when will the metrical connection in the basis B^* be metrical in the basis $\overline{B}^*$?

To get answer to this question it is necessary to obtain the relations between the covariant derivatives of the metrical tensor in these two bases.

Theorem 3.2. *The relations between the covariant derivatives of the same metric tensor G, with respect to the same generalized connection ∇, but in different bases B^* and $\overline{B}^*$ are given by*

$$
\text{(3.3)}\qquad
\begin{aligned}
&(a)\ \overline{g}_{ij\overline{|}k} = g_{ij|k} - g^{a}_{\ j|k}A_{ai} - g_{i\ |k}^{\ b}A_{bj} + g^{ab}{}_{|k} + g^{ab}{}_{|k}A_{ai}A_{bj}\\
&\qquad - A_{ck}(g_{ij}|^c - g^{a}_{\ j}|^cA_{ai} - g_i^{\ b}|^cA_{bj} + g^{ab}|^cA_{ai}A_{bj})\\
&(b)\ \overline{g}_{i\ \overline{|}k}^{\ b} = g_{i\ |k}^{\ b} - g^{ab}_{\ \ |k}A_{ai} - g_i^{\ b}|^cA_{ck} + g^{ab}|^cA_{ai}A_{ck}\\
&(c)\ \overline{g}^{a}_{\ j\overline{|}k} = g^{a}_{\ j|k} - g^{ab}_{\ \ |k}A_{bj} - g^{a}_{\ j}|^cA_{ck} + g^{ab}|^cA_{bj}A_{ck}\\
&(d)\ \overline{g}^{ab}_{\ \ \overline{|}k} = g^{ab}_{\ \ |k} + g^{ab}|^cA_{ck}\\
&(e)\ \overline{g}_{ij}\overline{|}^c = g_{ij}|^c - g^{a}_{\ j}|^cA_{ai} - g_i^{\ b}|^cA_{bj} + g^{ab}|^cA_{ai}A_{bj}\\
&(f)\ \overline{g}_i^{\ b}\overline{|}^c = g_i^{\ b}|^c - g^{ab}|^cA_{ai}\\
&(g)\ \overline{g}^{a}_{\ j}\overline{|}^c = g^{a}_{\ j}|^c - g^{ab}|^cA_{bj}\\
&(h)\ \overline{g}^{ab}\overline{|}^c = g^{ab}|^c
\end{aligned}
$$

where

$$
\text{(3.4)}\qquad
\begin{aligned}
&(a)\ g_{ij|k} = \delta_kg_{ij} - g_{hj}F_{i\ k}^{\ h} - g^{d}_{\ j}F_{idk} - g_{ih}F_{j\ k}^{\ h} - g_i^{\ d}F_{jdk}\\
&(b)\ g_{ij}|^c = \partial^cg_{ij} - g_{hj}C_i^{\ hc} - g^{d}_{\ j}C_{id}^{\ \ c} - g_{ih}C_j^{\ hc} - g_i^{\ d}C_{jd}^{\ \ c}\\
&(c)\ g_{i\ |k}^{\ b} = \delta_kg_i^{\ b} - g_h^{\ b}F_{i\ k}^{\ h} - g^{db}F_{idk} - g_{ih}F^{bh}{}_k - g_i^{\ d}F^{b}_{dk}\\
&(d)\ g_i^{\ b}|^c = \partial^cg_i^{\ b} - g_h^{\ b}C_i^{\ hc} - g^{db}C_{id}^{\ \ c} - g_{ih}C^{bhc} - g_i^{\ d}C^{b\ c}_{\ d}\\
&(e)\ g^{ab}{}_{|k} = \delta_kg^{ab} - g_h^{\ b}F^{ah}{}_k - g^{db}F^{a}_{dk} - g^{a}_{\ h}F^{bh}_{\ \ k} - g^{ad}F^{b}_{dk}\\
&(f)\ g^{ab}|^c = \partial^cg^{ab} - g_h^{\ b}C^{ahc} - g^{db}C^{a\ c}_{\ d} - g^{a}_{\ h}C^{bhc} - g^{ad}C^{b\ c}_{\ d}.
\end{aligned}
$$

The statement follows from (2.9) and (3.1). The calculation in this direct proof is very long. The shorter way is if we use the relation

$$
\text{(3.5)}\qquad G_{|k}dx^b + G|^c\delta p_c = \overline{G}_{\overline{|}k}dx^k + \overline{G}\overline{|}^c\overline{\delta}p_c,
$$

where (see (1.14)) we have

$$
\text{(3.6)}\qquad
\begin{aligned}
G_{|k} =\ & g_{ij|k}dx^i\otimes dx^j + g_{i\ |k}^{\ b}dx^i\otimes\delta p_b+\\
& g^{a}_{\ j|k}\delta p_a\otimes dx^j + g^{ab}{}_{|k}\delta p_a\otimes\delta p_a.
\end{aligned}
$$

$G|^c$ is obtained from (3.6) if $|_k$ is substituted by $|^c$; $\overline{G}_{\overline{|}k}$ and $\overline{G}\overline{|}^c$ are obtained from (3.2) in the similar way. If the expressions $G_{|k}$, $G|^c$, $\overline{G}_{\overline{|}k}$, $\overline{G}\overline{|}^c$ and $\overline{\delta}p_c = \delta p_c + A_{ck}dx^k$ (see (2.4)(b)) are substituted into (3.5) we easely obtain (3.3).

Definition 3.1. *The connection ∇ is recurrent with respect to the metric tensor expressed in the basis B^* if exists 1-form field*

$$\lambda = \lambda_k(x,p)dx^k + \lambda^c(x,p)\delta p_c \tag{3.7}$$

such that

$$\begin{aligned} &g_{ij|k} = \lambda_k g_{ij}, \quad g_i{}^b{}_{|k} = \lambda_k g_i{}^b, \quad g^a{}_{j|k} = \lambda_k g^a_j, \quad g^{ab}{}_{|k} = \lambda_k g^{ab} \\ &g_{ij}|^c = \lambda^c g_{ij}, \quad g_i{}^b|^c = \lambda^c g_i{}^b, \quad g^a_j|^c = \lambda^c g^a_j, \quad g^{ab}|^c = \lambda^c g^{ab}. \end{aligned} \tag{3.8}$$

The connection ∇ is metric connection if in (3.8) $\lambda_k = 0$ and $\lambda^c = 0$. In both cases the covariant differentials of the metric tensor are determined by (3.4).

From (3.3) follows the wellknown

Theorem 3.3. *If ∇ is a metric connection with respect to the basis B^*, then it will be a metric connection also in the basis $\overline{B}^*$.*

Theorem 3.4. *If ∇ is a recurrent connection in the basis B^* with respect to the metric tensor G ((3.8) are valid), then in the basis $\overline{B}^*$ the following relations are valid:*

$$\begin{aligned} &\overline{g}_{ij\overline{|}k} = (\lambda_k - \lambda^c A_{ck})\overline{g}_{ij} \quad \overline{g}^a{}_{j\overline{|}k} = (\lambda_k - \lambda^c A_{ck})\overline{g}^a{}_j \\ &\overline{g}_i{}^b{}_{\overline{|}k} = (\lambda_k - \lambda^c A_{ck})\overline{g}_i{}^b \quad \overline{g}^{ab}{}_{\overline{|}k} = (\lambda_k - \lambda^c A_{ck})\overline{g}^{ab} \\ &\overline{g}_{ij}\overline{|}^c = \lambda^c \overline{g}_{ij} \quad \overline{g}^a{}_j\overline{|}^c = \lambda^c \overline{g}^a{}_j \\ &\overline{g}_i{}^b\overline{|}^c = \lambda^c \overline{g}_i{}^b \quad \overline{g}^{ab}\overline{|}^c = \lambda^c \overline{g}^{ab}. \end{aligned} \tag{3.9}$$

The proof follows from (3.1), (3.3) and (3.8).

The coefficients of the recurrent and metric connection coefficients in the space (E^*, N, G, ∇, T) are given in [8].

REFERENCES

1. A t a n a s i u, Gh.: – *Connexions de Finsler générales presque cosimplectiques.* C. R. Acad. Sci. Paris Sér. A. 290 (1980), 969–972.
2. A t a n a s i u, Gh..: – *Ŝtructures et connexions Finsler presque complexes et presque Hermitiennes.* Proc. Nat. Sem. Finsler Sp. Brasov. (1980), 41–53.
3. A t a n a s i u, Gh. and K l e p p, F.C.: –*Almost contact Finsler connections.* Bul. St. Teh. Inst. Pol. T.V. Tim. Tom 26 (40), 1(1981), 45–51.
4. A t a n a s i u, Gh. and K l e p p, F.C.: –*p*$(3,-1)$ *Finsler structures and their lifts.* Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fac. Ser. Mat. 19, 2, (1989), 1–21.
5. A t a n a s i u, Gh., and H a s h i g u c h i, M.: –*Semi-symmetric Miron connections in dual Finsler spaces.* Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) Nr. 20, (1987).
6. Č o m i ć, I.: – *Generalized Mirons d-connection in the recurrent K. Hamilton spaces.* Publ. Inst. Math. (Beograd) 52(66) (1992) 136–152.
7. Č o m i ć, I., K a w a g u c h i, T. and K a w a g u c h i, H.: – *A theory of dual vector bundles and their subbundels for general dynamical systems or the information geometry.* Tensor N.S. Vol. 52 (1993) 286–300.
8. Č o m i ć, I.: – *Recurrent Hamilton spaces with generalized Miron,s d-connection.* Annals of the Univ. Craiova XVIII (1990) 90–104.
9. G h i n e a, I.: –*Conformal Finsler connections.* Proc. Nat. Sem. Finsler Spaces.Brasov (1980), 55–68.
10. G h i n e a, I.: –*Conformal almost symplectical Hamilton connections.* Proc. of the 22nd Conf. on Diff. Geom. and Topology. Polytechnic Inst. Bucharest, (1991) 133–137.
11. K l e p p, F.C.: –*Structuri Finsler metrice pe fibrate vectoriale Finsler.* Lucrările Sesiunii St. Inst. Inv. Sup. Constanta (1981), Matem.Fiz., 37–43.
12. K l e p p, F.C.: –*Structuri Finsler remarcabile si geometria Finsler a fibratelor vectoriale.* Doctoral Thesis. Univ. "Al.I.Cuza" Iasi (1982).
13. K l e p p, F.C.: –*Remarkable Finsler structures and connections of Finsler vector bundles.* The Proc. of the Nat.Sem. on Finsler spaces Brasov 1983, 125–140.
14. M a t s u m o t o, M.: –*The theory of Finsler connections.Publ. of the Study Group of Geometry 5.* Dep. of Math. Okayama Univ. (1970).
15. M i r o n, R.: –*On transformation groups of Finsler connections.* Tensor N.S. vol. 35 (1981), 235–241.
16. M i r o n, R. and H a s h i g u c h i, M.: –*Metrical Finsler connections.* Rep. Mat. Sci. Kagoshima Univ. (Math. Phys. Chem.) 12 (1979), 21–35.
17. M i r o n, R. and H a s h i g u c h i, M.: –*Almost symplectic Finsler structures.* Rep. Fac. Sci. Kagoshima Univ. (Math., Phys. Chem.) 14 (1981), 9–19.

18. M i r o n, R. and A n a s t a s i e i, M.: -*Fibrate vectoriale Spatii Lagrange.* Academy Press Romana 1987

19. M i r o n, R.: - *The geometry of Hamilton spaces.* Proc. of the Fifth Nat. Sem. of Finsler and Lagrange spaces (Brasov) (1988) 249–278.

20. O p r i s, D.: -*Fibrés vectoriels de Finsler et connexions associés.* Proc. Nat. Sem. Finsler Spaces. Braşov (1980), 185–193.

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Tomul XLII, s.I.a, Matematică, 1996, f1.

ON INDUCED NON–LINEAR CONNECTIONS

BY

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1. Introduction. The starting point of this paper is [2], where Professor V.Crucianu establishes a very interesting joining of the theory of non–holonomic spaces and the classical theory of riemannian submanifolds. The background here, as there, are two supplementary vector subbundles ξ' and ξ'' of a vector bundle ξ. The aim of this paper is to induce non–linear connections on ξ' and ξ'' using a non–linear connection on ξ, the linear case fitting on the case in [2]. The content of the paper is as follows:

In *Section2* the basic constructions used in the paper are made. Two supplementary vector subbundles $\mathcal{V}'\xi$ and $\mathcal{V}''\xi$ of $V\xi$ are defined and some results concerning these vector bundles and the vertical bundles $V\xi, V\xi'$ and $V\xi''$ are given.

In *Section 3* a canonical method to induce non–linear connections on ξ' and ξ'', using a non–linear connection on ξ, is given.

In *Section 4*, some coordinates, which we call *adapted*, are induced by the vector bundles structures on the vector bundles ξ', ξ'' and ξ. They are considered, in order to give a more simple local description of the constructions made in the previous sections. Using such coordinates it is shown that, in the case of a linear connection on ξ, the induced connections on ξ' and ξ'' are linear connections and they are the same as those induced in [2].

We intend to use the results obtained in this paper in order to study induced non–linear connections on supplementary vector subbundles and submanifolds, using some ideas from [1,2,3]. This will be done in a subsequent paper.