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ON INDUCED NON–LINEAR CONNECTIONS

BY

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1. Introduction. The starting point of this paper is [2], where Professor V.Crucianu establishes a very interesting joining of the theory of non–holonomic spaces and the classical theory of riemannian submanifolds. The background here, as there, are two supplementary vector subbundles ξ' and ξ'' of a vector bundle ξ. The aim of this paper is to induce non–linear connections on ξ' and ξ'' using a non–linear connection on ξ, the linear case fitting on the case in [2]. The content of the paper is as follows:

In *Section2* the basic constructions used in the paper are made. Two supplementary vector subbundles $\mathcal{V}'\xi$ and $\mathcal{V}''\xi$ of $V\xi$ are defined and some results concerning these vector bundles and the vertical bundles $V\xi, V\xi'$ and $V\xi''$ are given.

In *Section 3* a canonical method to induce non–linear connections on ξ' and ξ'', using a non–linear connection on ξ, is given.

In *Section 4*, some coordinates, which we call *adapted*, are induced by the vector bundles structures on the vector bundles ξ', ξ'' and ξ. They are considered, in order to give a more simple local description of the constructions made in the previous sections. Using such coordinates it is shown that, in the case of a linear connection on ξ, the induced connections on ξ' and ξ'' are linear connections and they are the same as those induced in [2].

We intend to use the results obtained in this paper in order to study induced non–linear connections on supplementary vector subbundles and submanifolds, using some ideas from [1,2,3]. This will be done in a subsequent paper.

2. Basic notations and constructions. All the manifolds and maps are assumed to be C^∞. On a manifold $M, \mathcal{F}(\mathcal{M})$ is the real algebra of real C^∞-functions and $\mathcal{X}(\mathcal{M})$ is the $\mathcal{F}(\mathcal{M})$-module of vector fields on M. For a vector bundle $\xi = (E, \pi, M)$, E is the total space, M is the base space, π is the canonical projection, $S(\xi)$ is the $\mathcal{F}(\mathcal{M})$-module of sections on ξ and $E_x \overset{not}{=} \pi^{-}1(x)$ is the fibre in $x \in M$, which is a finite dimensional vector space. For a manifold $M, \tau M = (TM, p, M)$ is the tangent (vector) bundle and for a map $f: M \to N, \tau f: \tau M \to \tau N$ is the f-morphism of the differential of f. For a vector bundle $\xi = (E, \pi, M), V\xi \overset{not}{=} Ker\tau\pi = (VE, p_1, E)$ is the vertical (vector) bundle of ξ. It is wellknown (see [3] that there is a canonical isomorphism $V\xi \cong \pi * \xi$. For a vector bundle f_0- morphism $f: \xi \to \xi'$ there is denoted by the same letter f the map induced between the total spaces.

Consider, as in [2], a vector bundle $\xi = (E, \pi, M)$ and two supplementary vector subbundles of $\xi: \xi' = (E', \pi', M)$ and $\xi'' = (E'', \pi'', M)$. It means that $E_x = E'_x \oplus E''_x$ (the direct sum is as vector spaces), for every $x \in M$; equivalently, the vector bundle ξ can be reduced as the Whitney sum $\xi' \oplus \xi''$. Denote as P' and P'' the projectors of ξ on ξ' and ξ''. It is easy to see that these are the canonical projections ot two vector bundles $\eta' = (E, P', E')$ and $\eta'' = (E, P'', E'')$ which have the fibres isomorphic with those of ξ'' and ξ' respectively.

We make the following notations concerning the vertical bundles of η' and η'' :

$$V\eta' \overset{not}{=} \mathcal{V}''\xi = (\mathcal{V}''E, \mathcal{V}''P', E'), V\eta'' \overset{not}{=} \mathcal{V}'\xi = (\mathcal{V}'E, \mathcal{V}'P'', E'')$$

Proposition 1. *There are canonical isomorphisms*

$$\mathcal{V}'\xi \cong \pi^*\xi', \mathcal{V}''\xi \cong \pi^*\xi''$$

Proof. From symmetry, it suffices to prove the first isomorphism. For this, observe firstly that the vector bundle η'' is isomorphic to $\pi'')^*\xi'$. Since $\mathcal{V}'\xi = V\eta'' \cong (P'')^*\eta''$, we have:

$$\mathcal{V}'\xi \cong (P'')^*(\pi'')^*\xi' \cong (\pi'' \circ P'')^*\xi' = \pi^*\xi'$$

and the assertion follows. ∎

Proposition 2. *$\mathcal{V}'\xi$ and $\mathcal{V}''\xi$ are supplementary vector subbundles of $V\xi$.*

Proof. We have $\pi = \pi' o P' = \pi'' o P''$, thus $\tau\pi = \tau\pi' o \tau P' = \tau\pi'' o \tau P''$. It follows that $Ker\tau P'' = \mathcal{V}'\xi$ and $Ker\tau P' = \mathcal{V}''\xi$ are vector subbundles of $Ker\tau\pi = V\xi$. It remains to show that they are supplementary vector subbundles.

For this, consider an arbitrary $e = e' + e'' \in E, e' \in E', e'' \in E'', \pi(e) = x$. The projections P' and P'', as maps, are surjective submersions. Their fibres, $(P')^{-1}(e') = e' + E''_x$ and $(P'')^{-1}(e'') = e'' + E'_x$, have in common only e and their tangent subspaces of T_eE are $Ker(\tau P')_e$ and $Ker(\tau P'')_e$ respectively. It follows that $Ker(\tau P')_e \cap Ker(\tau P'')_e = 0_e = e$. Taking into account the dimensions, we have the direct sum

$$T_eE = Ker(\tau P'')_e \oplus Ker(\tau P')_e = (\mathcal{V}'E)_e \oplus (\mathcal{V}''E)_e$$

which prove the assertion. ∎

According to this *Proposition*, we denote as Q' and Q'' the projectors of $V\xi$ on $\mathcal{V}'\xi$ and $\mathcal{V}''\xi$, respectively.

The manifolds E' and E'' are submanifolds of E and denote as $I': E' \to E$ and $I'': E'' \to E$ the inclusion maps; they are also inclusion morphisms $I': \xi' \to \xi$ and $I'': \xi'' \to \xi$.

Proposition 3. *For every $e' \in E'$ and $e'' \in E''$ we have*

$$\tau I')_{e'} = (\mathcal{V}'E)'_e, (\tau I'')_{e''}(VE'')_{e''} = (\mathcal{V}''E)_{e''}.$$

Proof. We have $P'' o I' = f_o$, where $f_o: E' \to E''$ is the null morphism which can be identified with the projection $\pi': E' \to M$, thus $Ker\tau f_o = Ker\tau\pi' = VE'$. For every $e' \in E'$ we have

$$(\tau P'')_{e'} \circ (\tau I')_{e'} = (\tau f_o)_{e'}$$

But $(\tau I')_{e'}$ is an injection and the fibres of $V\xi'$ and $\mathcal{V}'\xi$ have the same dimension. Using the definitions of $V\xi'$ and $\mathcal{V}'\xi$, it follows that

$$(\tau I')_{e'}(VE')_{e'} = Ker(\tau P'')_{e'} = (\mathcal{V}'E)_{e'}$$

which proves the assertion. ∎

Consider now the I'-morphism of the tangent bundles $\tau I': \tau E' \to \tau E$ which is an injection. From *Proposition 3* it follows that the restriction I'-morphism

$$\tau I'_{|VE'}: V\xi' \to \mathcal{V}'\xi \tag{1}$$

is an isomorphism on fibres. We have $P' \circ I' = id_{E'}$(i.e. I' has a left inverse). Consider the P'-morphism $P_1' : \mathcal{V}'\xi \to v\xi'$ given by $P_1'(X') = \tau P'(X'), (\forall) X' \in \mathcal{V}'E$. We must prove that the map P_1' is well defined, i.e. for every $X' \in \mathcal{V}'E$ then $P_1'(X') \in VE'$. But $X' \in \mathcal{V}'E \leftrightarrow \tau P''(X') = 0$ and we have :

$$\tau\pi'(P_1'(X')) = \tau\pi'(P'(X')) = \tau(\pi' \circ P')(X') =$$
$$= \tau\pi(X') = \tau(\pi'' \circ P'')(X') = \tau\pi''(P''(X')) = 0$$

which means that $P_1'(X') \in VE'$, since $ker \tau\pi' = V\xi'$.

From $\tau P' \circ \tau I' = id_{TE'}$ it follows

$$P_1' \circ \tau I'_{|VE'} = id_{VE'},$$

that is, the P'-morphism P_1' is a left inverse for the I'-morphism $\tau I'_{|VE'}$.

Similary, it can be defined a P''-morphism $P_1'' : \mathcal{V}''\xi \to V\xi''$ which is a left inverse for the I''-morphism $\tau I''_{|VE''} : V\xi'' \to \mathcal{V}''\xi$.

3. Induced non–linear connections. In this section we show that a non–linear connection on the vector bundle ξ induces canonicaly some non–linear connections on the supplementary vector subbundles ξ and ξ''.

Let $C : \tau E \to V\xi$ be a non-linear connection on the vector bundle ξ, i.e. (as, for example in [3], a vector bundle morphism such that $C \circ i = id_{VE}$, where $i : V\xi \to \tau E$ is the inclusion morphism.

Consider now the following sequence of vector bundles morphisms

$$\tau E' \xrightarrow{\tau I'} \tau E \xrightarrow{C} V\xi \xrightarrow{Q'} \mathcal{V}'\xi \xrightarrow{P_1'} V\xi'$$

which corresponds to the following sequence of base maps

$$E' \xrightarrow{I'} E \xrightarrow{id_E} E \xrightarrow{id_E} E \xrightarrow{P'} E$$

that give $P' \circ id_E \circ id_E \circ I' = P' \circ I' = id_{E'}$.

Proposition 4. *$C' = P_1' \circ Q' \circ C \circ \tau I' : \tau E' \to VE'$ is a non-linear connection on the vector bundle ξ'.*

Proof. Take $e' \in E'$ and $v' \in (VE')_{e'}$. From *Proposition 2.* and *3.* it follows that

$$(\tau I')_{e'}(v') \in (\mathcal{V}'E)_{e'} \subset (VE)_{e'}$$

thus the vector $(\tau I')_{e'}(v')$ is invariated by $C_{e'}$ and $Q'_{e'}$. Using (2) and the above remark, denoting as $i' : V\xi' \to \tau E'$ the inclusion morphism, we have:

$$(C' \circ i')_{e'}(v') = (P_1' \circ Q' \circ C \circ \tau I')_{e'}(v)' = P_1'(Q'(C(\tau I'_{|VE'}(v')))) =$$
$$= (P_1' \circ \tau I'_{|VE'})_{e'}(v') = v'$$

which prove the assertion. ∎

Similary, using the formula

$$C'' = P_1'' \circ Q'' \circ C \circ \tau I''$$

we can define a non-linear connection on the vector bundle ξ''.

Definition 1. *We call as induced the non-linear connections C' and C''.*

4. Adapted coordinates. In this section we use a local calculus, in order to give some explicit local formulas to the constructions made in the previous sections. In the case of a linear connection on ξ, using *Proposition 4*, there are induced also *linear connections* on ξ' and on ξ'', which are the same as those induced in [2].

Consider now vectorial coordinates on E, adapted to the vector bundle structure of ξ, which induce also on E' and E'' vectorial coordinates, adapted in a same way to the vector bundle structures of ξ' and ξ''; we say that it is an *adapted system of coordinates* on E

More precisely, for every $e \in E, \pi(e) = x, P'(e) = e', P''(e) = e''$ we have as adapted coordinates around $x \in M, e' \in E', e'' \in E'', e \in E$

$$x \to (x^i), e' \to (x^i, y^\alpha), e'' \to (x^i, y^u), e \to (x^i, (y^\alpha, y^u))$$

where $i = \overline{1,m}, \alpha = \overline{1,k_1}, u = \overline{1,k_2}, k_1 + k_2 = k$ (k_1, k_2 and k are the dimensions of the fibres of E', E'' and of E respectively). For some adapted coordinates on the total space on a vector bundle, we use, in the sequel, the convention to list them separated in two sets: the coordinates on the base space and the coordinates on the fibre. The change rules of such adapted coordinates are

$$x^{i'} = x^{i'}(x^i), y^{\alpha'} = h^{\alpha'}_\alpha(x^i) y^\alpha, y^{u'} = h^{u'}_u(x^i) y^u \tag{3}$$

These coordinates induce local bases of sections on the vector bundles ξ', ξ'' and ξ :

$$\{\widehat{s}_\alpha\}_{\alpha=\overline{1,k_1}}, \{\widetilde{s}_u\}_{u=\overline{1,k_2}}, \{s_A\}_{A=\overline{1,k}} = \{\widehat{s}_\alpha\}_{\alpha=\overline{1,k_1}} \cup \{\widetilde{s}_u\}_{u=\overline{1,k_2}}$$

and the change rules are:

$$\widehat{s}_\alpha = h^{\alpha'}_\alpha(x^i)\widehat{s}_{\alpha'}, \widetilde{s}_u = h^{u'}_u(x^i)\widetilde{s}_{u'}$$

Remark 1. *We do not use local sections in the vector bundles involved in our paper have not always the same base, therefore, in this case, it is not induced a map between sections.*

The adapted coordinates on E induce adapted coordinates on the total spaces of the other vector bundles.

For $\tau E'$, on TE' there are the coordinates $((x^i, y^\alpha), (X^j, Y^\beta))$, which have the change rule for x^i and y^α as in (3) and for X^i and Y^β :

$$X^{j'} = \frac{\partial x^{j'}}{\partial x^j}(x^i)X^j, Y^{\beta'} = h^{\beta'}_{\beta,i}(x^j)y^\beta X^i + h^{\beta'}_\beta(x^j)Y^\beta. \tag{4}$$

For $\tau E''$, on TE'' there exist the coordinates $((x^i, y^u), (X^j, Y^v))$, which have the change rule for x^i and $y^{u'}$ as in (3), for X^j as in (4) and for Y^v :

$$Y^{v'} = h^{v'}_{v,i}(x^j)y^v X^i + h^{v'}_v(x^j)Y^v. \tag{5}$$

For τE, on TE there are the coordinates $((X^i, y^\alpha, y^u), (X^j, Y^\beta, Y^v))$, which have the change rule for x^i, y^α and y^u as in (3) and for X^j, Y^α and Y^u as in (4) and (5).

We have also the following sets of coordinates: on VE': $((x^i, y^\alpha), Y^\beta)$; on VE'': $((x^i, y^u), Y^v)$; on VE: $((x^i, y^\alpha, y^u), (Y^\beta, Y^v))$; on $\mathcal{V}'E$: $((x^i, y^\alpha, y^u), Y^\beta)$; $\mathcal{V}''E$: $((x^i, y^\alpha, y^u), Y^v)$.

Using the adapted coordinates, we have the following local expresions of the maps indicated on arrows:

$$(x^i, (y^\alpha, y^u)) \xrightarrow{P'} (x^i, y^\alpha); (x^i, y^\alpha, y^u)) \xrightarrow{P''} (x^i, y^u)$$

$$(x^i, y^\alpha) \xrightarrow{I'} (x^i, (y^\alpha, 0)); (x^i, y^u \xrightarrow{I''} (x^i, (0, y^u))$$

$$((x^i, y^\alpha), (X^j, Y^\beta)) \xrightarrow{\tau I'} ((x^i, y^\alpha, 0), (X^j, Y^\beta, 0));$$

$$((x^i, y^u), (X^j, Y^v)) \xrightarrow{\tau I''} ((x^i, 0, y^u), (X^j, 0, Y^v))$$

$$((x^i, y^\alpha, y^u), (Y^\beta, Y^v)) \xrightarrow{Q'} ((x^i, y^\alpha, y^u), Y^\beta);$$

$$((x^i, y^\alpha, y^u), (Y^\beta, Y^v)) \xrightarrow{Q''} ((x^i, y^\alpha, y^u), Y^v)$$

$$((x^i, y^\alpha, y^u), (X^j, Y^\beta, Y^v)) \xrightarrow{\tau P'} ((x^i, y^\alpha), (X^j, Y^\beta)),$$

$$((x^i, y^\alpha, y^u), (X^j, Y^\beta, Y^v)) \xrightarrow{\tau P''} ((x^i, y^u), (X^j, Y^v))$$

$$((x^i, y^\alpha, y^u), Y^\beta) \xrightarrow{P'_1} ((x^i, y^\alpha), Y^\beta);$$

$$((x^i, y^\alpha, y^u) Y^v) \xrightarrow{P''_1} ((x^i, y^u), Y^v)$$

According to [3] and using adapted coordinates, for a non-linear connection C we have: $((x^i, y^\alpha, y^u), (X^j, Y^\beta, Y^v)) \xrightarrow{C} (Y^\beta + X^j N^\beta_j(x^i, y^\alpha, y^u), Y^v + X^j N^v_j(x^i, y^\alpha, y^u))$ where the set of local functions

$$\{N^\beta_j(x^i, y^\alpha, y^u), N^v_j(x^i, y^\alpha, y^u)\}$$

are called the local components of C.

Using Proposition 4, which defines the induced non-linear connection C', we have

$$(x^i, y^\alpha), (X^j, Y^\beta)) \xrightarrow{\tau I'} ((x^i, y^\alpha, 0), (X^j, Y^\beta, 0)) \xrightarrow{C}$$

$$((x^i, y^\alpha, 0), (Y^\beta + X^j N^\beta_j(x^i, y^\alpha, 0), X^j N^v_j(x^i, y^\alpha, 0))) \xrightarrow{Q'}$$

$$((x^i, y^\alpha, 0), Y^\beta + X^j N^\beta_j(x^i, y^\alpha, 0)) \xrightarrow{P'_1} ((x^i, y^\alpha), Y^\beta + +X^j N^\beta_j(x^i, y^\alpha, 0))$$

which gives finally:

$$((x^i, y^\alpha), (X^j, Y^\beta)) \xrightarrow{C'} ((x^i, y^\alpha). Y^\beta + X^j N^\beta_j(x^i, y^\alpha, 0)).$$

Thus, the local components of C' are:

$$\widehat{N}^\beta_j(x^i, y^\alpha) = N^\beta_j(x^i, y^\alpha, 0).$$

In a similar way, the local expression of C'' is:

$$((x^i, y^u), (X^j, Y^v)) \xrightarrow{C''} ((x^i, y^u), Y^v + X^j N^v_j(x^i, y^u, 0)).$$

and the local components of C'' are $\widetilde{N}^v_j(x^i, y^u) = N^v_j(x^i, 0, y^u)$.

If C is a linear connection, the components of C, C' and C'' have the local forms:

$$N_j^{\beta}(x^i, y^{\alpha}, y^u) = \Gamma_{j\gamma}^{\beta}(x^i)y^{\gamma} + \Gamma_{jw}^{\beta}(x^i)y^w,$$
$$N_j^{v}(x^i, y^{\alpha}, y^u) = \Gamma_{j\gamma}^{v}(x^i)y^{\gamma} + \Gamma_{jw}^{v}(x^i)y^w$$
$$\widetilde{N}_j^{\beta}(x^i, y^{\alpha}) = \Gamma_{j\gamma}^{\beta}(x^i)y^{\gamma}, \widetilde{N}_j^{v}(x^i, y^u) = \Gamma_{jw}^{v}(x^i)y^w$$

and it follows that C' and C'' are linear connections, too.

In [2] there is proved that if D is a linear connection on ξ, then the formulas

$$D'_X u' = P'(D_X u'), D''_X u'' = P''(D_X u'')$$

for $X \in \mathcal{X}(M), u' \in S(\xi'), u'' \in S(\xi'')$, define linear connections D' and D'' on ξ' and ξ'' respectively.

In the adapted coordinates considered in our paper; if D has the local components

$$\{\Gamma_{i\alpha}^{\beta}, \Gamma_{i\alpha}^{v}, \Gamma_{iu}^{\beta}, \Gamma_{iu}^{v}\}$$

then D' and D'' have as local components $\{\Gamma_{i\alpha}^{\beta}\}$ and $\{\Gamma_{iu}^{v}\}$ respectively. Therefore, for a linear connection, the connections induced by *Proposition 4* are the same as the linear connections induced in [2].

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ON JORDAN PLANE CURVES WHICH ARE ISOPTICS OF AN OVAL

BY

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Let Γ be an oval [3] and let Λ be a regular Jordan plane curve. In the paper we give the sufficient condition for Λ to be an isoptic of the oval Γ.

Let σ be a region bounded by Λ. We assume that:

$$(1°)\qquad \Gamma \subset \sigma,$$

$$(2°)\qquad \Gamma \cap \Lambda = \emptyset.$$

Let X be an arbitrary point of Λ. We denote the straight line tangent to Λ at the point X by l_X. Let l_A and l_B tangents to Γ at points A and B and passing by X, see Fig.1.