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Department of Mathematics
University of Kalyani
Kalyani – 741 235
West Bengal, INDIA.

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
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THEORY OF FINSLER SPACES WITH m–th ROOT METRIC III: STRONGLY NON-RIEMANNIAN SPACES

BY

MAKOTO MATSUMOTO

Theory of Finsler spaces with $m-th$ root metric has been recently made rapid progress ([4], [7]) on the basis of two papers ([6], [8]).

The m–th root metric $L(x,y)$ which is given by

$$L^m = a_{i_1 \dots i_m}(x) y^{i_1} \dots y^{i_m}$$

is seemingly a direct generalization of a Riemannian metric, because the case $m = 2$ is nothing but a Riemannian metric. It may be, however, said that all the attempts to establish the Riemannian geometry–like of spaces with m–th root metric had come to nothing.

Nevertheless, to show that the m–th metrics are not so much removed from Riemannian metrics in the sense that was introduced by R.M i r o n and the present author about eighteen years ago [5], is the purpose of the present paper. The main results are stated as Theorems 3 and 4.

1. Derived F–tensors. We consider an n–dimensional Finsler space $F^n = (M^n, L(x,y))$ with the fundamental function $L(x,y)$ which is defined on the tangent bundle TM^n over a differentiable n–manifold M^n. We put $F(x,y) = L^2/2$ and define

$$F_{ij\dots k}(x,y) = \dot\partial_i \dot\partial_j \cdots \dot\partial_k F,$$

called the *derived* F-*tensors*, where $\dot\partial_i = \partial/\partial y^i$. We have first

$$F_i = L l_i, \quad F_{ij} = g_{ij}, \quad F_{ijk} = 2C_{ijk}, \tag{1.1}$$

where l_i, g_{ij} and C_{ijk} are the normalized supporting element, the fundamental tensor and the C-tensor respectively.

To write the fourth derived F-tensor F_{hijk}, we shall recall the T-tensor:

$$T_{hijk} = LC_{hij}|_k + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}, \tag{1.2}$$

where $(|)$ denotes the v-covariant differentiation in the C a r t a n connection $C\Gamma = (\Gamma^{*i}_{j\ k}, G^i_j, C_j{}^i{}_k)$, that is, for a contravariant vector field $X^i(x,y)$ we have

$$X^i|_j = \dot\partial_j X^i + X^r C_r{}^i{}_j, \qquad C_{ijk} = g_{jr} C_i{}^r{}_k.$$

The T-tensor is completely symmetric and has been introduced by H.K a w a g u c h i and the present author in 1972 ([3], Remark 28.6; [1], [10])

Throughout the present paper we shall use often the abbreviations as follows:

$$a_{hir} G_j{}^r{}_k + (i,j,k) = a_{hir} G_j{}^r{}_k + a_{kjr} G_k{}^r{}_i + a_{hkr} G_i{}^r{}_j,$$

$$l_h C_{ijk} + (4) = l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij},$$

$$F_h F_{ijkl} + (5) = F_h F_{ijkl} + F_i F_{hjkl} + F_j F_{hikl} + F_k F_{hijl} + F_l Fhijk,$$

$$l_h l_i C_{jkl} + (10) = l_h l_i C_{jkl} + l_h l_j C_{ijk} + \cdots + l_k l_l C_{hij},$$

where the right-hand side of the last one consists of ten terms.

Now, if we put

$$C^2_{hijk} = C_{hir} C_j{}^r{}_k + (i,j,k), \tag{1.3}$$

then it is easy to show

$$\frac{L}{2} F_{hijk} = T_{hijk} - \{l_h C_{ijk} + (4)\} + LC^2_{hijk}.$$

Next we are concerned with the fifth derived F-tensor F_{hijkl}, which is written in terms of the covariant derivative as

$$F_{hijkl} = F_{hijk}|_l + F_{ijkr} C_h{}^r{}_l + F_{hjkr} C_i{}^r{}_l + F_{hikr} C_j{}^r{}_l + F_{hijr} C_k{}^r{}_l.$$

From (1.4) a long calculation leads to

$$\begin{aligned} \frac{L^2}{4} F_{hijkl} = {} & T_{hijkl} - \tfrac{1}{2}\{l_h T_{ijkl} + (5)\} + \tfrac{L}{2}\{C_h{}^r{}_i T_{rjkl} \\ & + (10\} - L\{l_h C^2_{ijkl} + (5)\} + \{l_h l_i C_{jkl} + (10)\}, \end{aligned} \tag{1.5}$$

where we put

$$\begin{aligned} 2T_{hijkl} = {} & LT_{hijk}|_l - (h_{lh} C_{ijk} + h_{li} C_{hjk} + h_{lj} C_{hik} + \\ & + h_{lk} C_{hij}) + L^2 (C_l{}^r{}_h C^2_{rijk} + C_l{}^r{}_i C^2_{rhjk} + \\ & C_l{}^r{}_j C^2_{rhik} + C_l{}^r{}_k C^2_{rhij}), \end{aligned} \tag{1.5$_a$}$$

and $h_{lh} = g_{lh} - l_l l_h$ is the angular metric tensor.

The completely symmetrical property of T_{hijkl} is easily verified from

$$\begin{aligned} T_{hijk}|_l - T_{hijl}|_k = {} & -L\{C_{hir} S_j{}^r{}_{kl} + (h,i,j)\} + \frac{1}{L}[\{h_{lh} C_{ijk} + \\ & + (h,i,j)\} - \{h_{kh} C_{ijl} + (h,i,j)\}], \end{aligned}$$

where $S_j{}^r{}_{kl}$ is the v-curvature tensor of $C\Gamma$.

We are interested in the two-dimensional case in particular. The we have the Berwald frame $(1,m)$ and the main scalar $I(x,y)$ ([4], [7]), defined by

$$h_{ij} = \varepsilon m_i m_j, \qquad LC_{hij} = I m_h m_i m_j,$$

where $\varepsilon = \pm 1$ is the signature of the space.

In the following we shall use the symbols $m_{hi\cdots k} = m_h m_i \cdots m_k$, so that we have $LC_{hij} = I m_{hij}$. The T- tensor is written as

$$LT_{hijk} = I_{;2} m_{hijk}, \tag{1.2$_2$}$$

where we put $LS|_i = S_{;1} l_i + S_{;2} m_i$. It is obvious that $S_{;1} = 0$ for a scalar field S positively homogeneous of degree zero in y^i such as I ([3], Σ28). Since we have $C^2_{hijk} = 3\varepsilon (I/L)^2 m_{hijk}$, we get

$$\frac{L^2}{2} F_{hijk} = (I_{;2} + 3\varepsilon I^2) m_{hijk} - I\{l_h m_{ijk} + (4)\}. \tag{1.4$_2$}$$

Next, from (1.2$_2$) and the differentiation formula $Lm_i|_j = -l_i m_j$ we have

$$2LT_{hijkl} = (I_{;2;2} + 12 I^3 - 4\varepsilon I) m_{hijkl} - I_{;2}\{l_h m_{ijkl} + (5)\}. \tag{1.5$_{a2}$}$$

Consequently (1.5) gives

$$\frac{L^3}{4} F_{hijkl} = \frac{1}{2}(I_{;2;2} + 10\varepsilon I I_{;2} + 12 I^3 - 4\varepsilon I) m_{hijkl} \tag{1.5$_2$}$$

$$-(I_{;2} + 3\varepsilon I^2)\{l_h m_{ijkl} + (5)\} + I\{l_h l_i m_{jkl} + (10)\}.$$

2. Cubic metrics and quartic metrics. We consider an $F^n = (M^n, L(x,y))$ with m-th root metric L:

$$L^m = a_{i_1 \cdots i_m}(x) y^{i_1} \cdots y^{i_m},$$

where $a_{i_1\cdots i_m}(x)$ are components of a completely symmetric tensor field on M^n. Throughout the following we suppose $m > 2$.

We define $m-l$ symetric tensor fields with the components

$$a_{i_1\cdots i_r}(x,y) = \frac{1}{L^{m-r}} a_{i_1\cdots i_r j_1 \cdots j_{m-r}}(x) y^{j_1} \cdots y^{j_{m-r}},$$

where $1 \leq r \leq m-1$. a_i, a_{ij} and a_{ijk} are specially important, because we have ([6], [8])

$$l_i = a_i, \quad h_{ij} = (m-1)(a_{ij} - a_i a_j), \quad g_{ij} = (m-1)a_{ij} - (m-2)a_i a_j,$$

$$2LC_{ijk} = (m-1)(m-2)[a_{ijk} - \{a_{ij}a_k + (i,j,k)\} + 2a_i a_j a_k].$$

Since $\det(g_{ij}) = (m-1)^{n-1}\det(a_{ij})$ is easily shown, we suppose $\det(a_{ij}) \neq 0$ (*regularity*). Then we have $(a^{ij}) = (a_{ij})^{-1}$ and $l^i = a^i (= a^{ir}a_r)$, $\quad g^{ij} = \frac{l}{m-1}\{a^{ij} + (m-2)a^i a^j\}$. Hence we obtain [8]

$$(2.1) \quad \begin{cases} 2LC_j{}^i{}_k = (m-2)\{a_j{}^i{}_k - (\delta^i_j a_k + \delta^i_k a_j) + a^i(2a_j a_k - a_{jk})\}, \\ 2LC_i = (m-2)(a_i{}^r{}_r - na_i), \end{cases}$$

where $a_j{}^i{}_k = a^{ir}a_{rjk}$ and $C_i = C_i{}^r{}_r$.

From the definition of $a_{i_1\cdots i_r}$ it follows that

$$L\dot\partial_j a_{i_1\cdots i_r} = (m-r)(a_{i_1\cdots i_r j} - a_{i_1\cdots i_r} a_j).$$

Hence we obtain

$$La_i|_j = (m-1)(a_{ij} - a_i a_j),$$

$$La_{ij}|_k = (m-2)(a_i a_{jk} + a_j a_{ik} - 2a_i a_j a_k),$$

$$La_{hij}|_k = (m-3)a_{hijk} + \frac{m}{2}a_{hij}a_k - \frac{m-2}{2}\{a_{hir}a_j{}^r{}_k + a_{hi}(2a_j a_k - a_{jk}) - a_{hik}a_j + (h,i,j)\}.$$

Consequently a long calculation leads to

$$(2.2) \quad 4LT_{hijk} = 2(m-1)(m-2)(m-3)a_{hijk} - (m-1(m-2)^2\{a_{hir}a_j{}^r{}_k + (3)\} + m(m-1)(m-2)[\{a_h a_{ijk} + (4)\} - \{a_{hi}a_{jk} + (3)\}].$$

which is also written in the form

$$(2.2') \quad \begin{aligned} 2LT_{hijk} = {} & (m-3)[(m-1)(m-2)(a_{hijk} - l_h l_i l_j l_k) - 2L\{l_h C_{ijk} \\ & + (4)\} - (m-2)\{l_h l_i h_{jk} + (6)\}] - 2L^2\{C_{hir}C_j{}^r{}_k \\ & + (3)\} - (m-2)\{h_{hi}h_{jk} + (3)\}. \end{aligned}$$

The third root metric ($m=3$) is called a *cubic metric*. For a cubic metric the three terms with the coefficient $m-3$ in (2.2′) vanish. Hence we get

$$(2.3) \quad 2LT_{hijk} = -\{2L^2 C_{hir}C_j{}^r{}_k + h_{hi}h_{jk} + (3)\},$$

as already shown in the papers ([6], [8]).

Further (2.3) shows more important fact as follows: It is easy to show that for a general F^n we have [4]

$$\frac{16}{3}L^5\dot\partial_h\dot\partial_i\dot\partial_j\dot\partial_k L^3 = 8F^3 F_{hijk} + 4F^2\{F_{hij}F_k + (4)\} + 4F^2\{F_{hi}F_{jk} + (3)\} - 2F\{F_{hi}F_j F_k + (6)\} + 3F_h F_i F_j F_k.$$

Then (1.1) and (1.4) yields

$$(2.4) \quad \frac{16}{3}L\dot\partial_h\dot\partial_i\dot\partial_j\dot\partial_k L^3 = 2LT_{hijk} + 2L^2 C^2_{hijk} + \{h_{hi}h_{jk} + (3)\}.$$

It is obvious that L is a cubic metric if and only $\dot\partial_h\dot\partial_i\dot\partial_j\dot\partial_k L^3 = 0$. Then (2.4) shows that this condition is given by (2.3).

Theorem 1 ([6]). *A Finsler space is with cubic metric, if and only if the T-tensor is written as (2.3).*

Next we consider a *quartic metric* ($m=4$) similarly to the above. For a general F^n we have [4]

$$\frac{1}{8}\dot\partial_h\dot\partial_i\dot\partial_j\dot\partial_k\dot\partial_l L^4 = FF_{hijkl} + \{F_{hijk}F_l + (5)\} + \{F_{hij}F_{kl} + (10)\}.$$

Then (1.1), (1.4) and (1.5) lead to

$$(2.5) \quad \frac{1}{8}\dot\partial_h\dot\partial_i\dot\partial_j\dot\partial_k\dot\partial_l L^4 = 2T_{hijkl} + \{l_h T_{ijkl} + (5)\} + \{LC_h{}^r{}_i T_{rjkl} + 2C_{hij}h_{kl} + (10)\}.$$

Therefore we have

Theorem 2. *A Finsler space is with quartic metric, if and only if the T-tensor satisfies*

$$2T_{hijkl} + \{1_h T_{ijkl} + (5)\} + \{LC_h{}^r{}_i T_{rjkl} + 2C_{hij}h_{kl} + (10)\} = 0, \tag{2.6}$$

where T_{hijkl} is defined by (1.5_a).

Remark: For the two-dimensional case (2.3) and (2.6) lead to the beautiful characterization of cubic and quartic metrics respectively which are given by Theorem 4 of [4].

3. Strongly non-Riemannian spaces. The notion of the intrinsic orthonormal frame field in an F^n was defined by R. M i r o n and the present author [5] as a generalization of the B e r w a l d frame of the two-dimensional case and the M o ó r frame of the three-dimensional case ([3], ΣΣ 27–29).

For a general $F^n = (M^n, L(x,y))$, if we put

$$L_{i_1\cdots i_{2r-1}} = \frac{1}{2^r}\dot\partial_{i_1}\cdots\dot\partial_{i_{2r-1}}L^2, \quad r \geq 1,$$

then a sequence of covariant vector fields is given by

$$L_{1)i} = L_i, \qquad L_{s+1)i} = L_{ij_1\cdots j_{2s}}g^{j_1j_2}\cdots g^{j_{2s-1}j_{2s}}, \qquad s \geq 1.$$

$L_{1)i}$ and $L_{2)i}$ are equal to $L1_i$ and $C_i(= C_i{}^r{}_r)$ respectively. Thus $L_{2)i}$ vanishes for a Riemannian metric.

Definition. *A Finsler space of dimension $n > 2$ is called strongly non-Riemannian space, if $n-1$ vectors $L_{r)i}, r = 1, \ldots, n-1$, are linearly independent.*

Thus, we can construct locally an intrinsic orthonormal frame field from $L_{r)i}, r = 1, \ldots, n-1$ for a strongly non- Riemannian space. It is obvious that a three-dimensional space is strongly non-Riemannian, if and only if C_i does not vanish ([3], Σ29).

As it has been shown in [2], the third vector $L_{3)i}$ is written in the form

$$\begin{aligned} 2L_{3)i} &= \frac{1}{L}(T^r{}_i|_r + 2C^rT_{ri} + 4C^{rst}T_{rsti}) + 3C^rD_{ri} \\ &\quad + (C_rC_s + 8D_{rs} + \frac{4}{L}T_{rs})C^{rs}{}_i - \frac{n-2}{L^2}C_i \\ &\quad - \frac{1}{L}(2C^2 + 4D + \frac{T}{L})1_i, \end{aligned} \tag{3.1}$$

where we put

$$\begin{cases} C^2 = C^rC_r, \quad D_{ij} = C_i^{rs}C_{rsj}, \quad D = C^{rst}C_{rst}, \\ T_{ij} = g^{rs}T_{rsij} = LC_i|_j + 1_iC_j + 1_jC_i, \quad T = g^{rs}T_{rs} = LC^r|_r. \end{cases} \tag{3.1$_a$}$$

In the notation of the present paper we have

$$L_{s+1)i} = \frac{1}{2^s}F_{ij_1\cdots j_{2s}}g^{j_1j_2}\cdots g^{j_{2s-1}j_{2s}}.$$

Then $L_{3)i} = F_{ihjkl}g^{hj}g^{kl}/4$ and it is easy to show that (3.1) can be also derived from (1.5) together with (1.5_a). The definition above leads to

Proposition 1 ([2]). *We put*

$$M_i = \frac{1}{L}(T^r{}_i|_r + 2C^rT_{ri} + 4C^{rst}T_{rsti}) + 3C^rD_{ri}$$

$$+ (C_rC_s + 8D_{rs} + \frac{4}{L}T_{rs})C^{rs}{}_i.$$

If l_i, C_i and M_i are linearly dependent, then $F^n, n > 3$, is not strongly non-Riemannian.

Remark: We have [2]

$$T^r{}_i|_r = LC_i|_j|_kg^{jk} + \frac{1}{L}\{T1_i + (n-2)C_i\}. \tag{3.2}$$

Since $1^iC_i|_j|_kg^{jk} = -2C^r|_r/L$, we have $1^iT^r{}_i|_r = -C^r|_r = -T/L$. Consequently, if $1_i, C_i$ and M_i are linearly dependent, then we have

$$M_i = -\frac{1}{L^2}(T1_i + SC_i), \qquad T = LC^r|_r, \tag{3.3}$$

where S is a scalar positively homogeneous of degree zero in y^i.

Now we shall deal with the problem: Whether is an F^n with cubic metric strongly non-Riemannian or not? To do so, we have to find M_i for a cubic metric.

The expression (2.3) of the T-tensor gives

$$T_{ij} = -L(C^rC_{rij} + 2D_{ij}) - \frac{n+1}{2L}h_{ij}.$$

From this and (2.3) we obtain

$$\begin{aligned} T_{ij}|_k &= \{(\frac{n+1}{2L^2}h_{ij} + C^rC_{rij} + 2d_{ij}) + (i,j,k)\} \\ &\quad -2(C^{rs}{}_iT_{rsjk} + C^{rs}{}_jT_{rsik}) - C_i{}^r{}_jT_{rk} - C^rT_{rijk}. \end{aligned}$$

Consequently we get

$$T^r{}_i|_r = (\frac{n^2-1}{2L^2} + C^2 + 2D)l_i - 3C^{rs}{}_iT_{rs} - 2C^{rst}T_{rsti} - C^rT_{ri},$$

and M_i is written in the form

$$M_i = \frac{1}{L}(\frac{n^2-1}{2L^2} + C^2 + 2D)l_i + 3C^rD_{ri} + (C_rC_s + 8D_{rs})C^{rs}{}_i$$

$$+\frac{1}{L}(C^rT_{ri} + C_i^{rs}T_{rs} + 2C^{rst}T_{rsti}).$$

It is observed from (2.3) that

$$T = -L(\frac{n^2-1}{2L^2} + C^2 + 2D),$$

$$C^rT_{ri} = -L(C_rC_sC^{rs}{}_i + 2C^rD_{ri}) - \frac{n+1}{2L}C_i,$$

$$C^{rs}{}_iT_{rs} = -L(2C^{rs}{}_iD_{rs} + C^rD_{ri}) - \frac{n+1}{2L}C_i,$$

$$C^{rst}T_{rsti} = -3LC^{rs}{}_iD_{rs} - \frac{3}{2L}C_i.$$

Finally we obtain M_i of a cubic metric in the form

$$M_i = -\frac{1}{L^2}\{Tl_i + (n+1)C_i\}, \tag{3.4}$$

which is of the form (3.3). Therefore we have one of the main results as follows:

Theorem 3. *A Finsler space with cubic metric of dimension more than three is not a strongly non-Riemannian space.*

Next we consider a Finsler space F^n with quartic metric and deal with the problem similar to the above. For this case we have (2.6) where the v-covariant derivative of the T-tensor is contained. From (2.6) we get

$$2T_{hijkl}g^{hj}g^{kl} = -Tl_i - 2L(C^rT_{ri} + 2C^{rs}{}_iT_{rs} + 2C^{rst}T_{rsti})$$

$$-4(n+3)C_i.$$

On the other hand, (1.5_a)gives

$$2T_{hijkl}g^{hj}g^{kl} = LT^r{}_i|_r + L^2(C_rC_sC^{rs}{}_i + 3C^rD_{ri} + 8C^{rs}{}_iD_{rs})$$

$$-(n+2)C_i.$$

From these equations it follows that

$$LT^r{}_i|_r = -Tl_i - (3n+10)C_i - L(2C^rT_{ri} + 3LC^rD_{ri} + 4C^{rst}T_{rsti})$$

$$-L^2(C_rC_s + 8D_{rs} + \frac{4}{L}T_{rs})C^{rs}{}_i.$$

Consequently we have M_i of a quartic metric in the form

$$M_i = -\frac{1}{L^2}\{Tl_i + (3n+10)C_i\},$$

which is of form (3.3). Therefore we have

Theorem 4. *A Finsler space with quartic metric of dimension more than three is not a strongly non-Riemannian space.*

Remark: Theorems 3 and 4 show that a cubic metric and a quartic metric are similar to a Riemannian metric in a sense, as naturally conjectured. We have already such metrics, (α,β)–metrics, as shown in 1977 [2].

As a consequence of Theorem 3 we have the possibility of "strongly non-Riemannian" for three-dimensional spaces with cubic metric. Such a space has the non-zero C_i. From (2.1) it follows that a Finsler space with m–th root metric has $C_i = 0$, if and only if $a^{rs}a_{rsi} = na_i$, where $(a^{rs}) = (a_{rs})^{-1}$.

We treat of three-dimensional case in particular. First we have

$$a(=\ \det\,(a_{ij})) = \frac{1}{6L^3}e^{abc}e^{pqr}a_{api}a_{bqj}a_{crk}y^iy^jy^k,$$

$$a^{rs} = \frac{1}{2aL^2}e^{rhi}e^{sjk}a_{hja}a_{ikb}y^ay^b,$$

where $e^{abc} = \delta^{abc}_{123}$. Thus, if we put

$$A_{ijk} = e^{abc}e^{pqr}a_{api}a_{bqj}a_{crk}, \tag{3.6}$$

then we get

$$a = \frac{1}{6L^3}A_{ijk}y^iy^jy^k, \qquad a^{rs}a_{rsi} = \frac{1}{2aL^2}A_{ijk}y^jy^k. \tag{3.7}$$

It is noted that A_{ijk} are functions of position x^i alone. Consequently, if we use the index, we have $a^{rs}a_{rsi} - 3a_i = 0$ in the form

$$A_{000}a_{i00} = a_{000}A_{i00}, \tag{3.8}$$

which is a homogeneous polynominal of order five in y^i. It is easy toshow that A_{ijk} may be written as

$$A_{ijk} = \begin{vmatrix} a_{11i} & a_{12i} & a_{13i} \\ a_{21j} & a_{22j} & a_{23j} \\ a_{31k} & a_{32k} & a_{33k} \end{vmatrix} + (6), \tag{3.6$'$}$$

in our abbreviation.

Proposition 2. *A Finsler space with cubic metric of dimension three has $C_i = 0$, if and only if (1) $a^{rs}a_{rsi} = 3a_i$, or (2) (3.8) is satisfied.*

Example 1: We consider again the three-dimensional cubic metrics

$$L^3 = c_1(y^1)^3 + c_2(y^2)^3 + c_3(y^3)^3 + 6by^1y^2y^3,$$

of the normal form [4]. We have $a_{iii} = c_i$, $i = 1,2,3$, and $a_{123} = b$. Hence it is obvious that $A_{iii} = -6b^2c_i$, $A_{iik} = 0$, $i, k = 1,2,3, \neq$, and $A_{123} = c_1c_2c_3 + 2b^3$. Thus we get $a_{100} = c_1p^2 + 2bqr$, $(p,q,r) = (y^1, y^2, y^3)$, and

$$A_{100} = -6b^2c_1p^2 + 2(c_1c_2c_3 + 2b^3)qr,$$

$$A_{000} = -6b^2(c_1p^3 + c_2q^3 + c_3r^3) + 6(c_1c_2c_3 + 2b^3)pqr.$$

Consequently we get

$$A_{000}a_{100} - a_{000}A_{100} = 2(c_1c_2c_3 + 8b^3)(2c_1p^3 - c_2q^3 - c_3r^3)qr.$$

Therefore we have $C_i = 0$, if and only if

$$(1)\quad c_2c_2c_3 + 8b^3 = 0, \qquad \text{or (2)}\quad c_1 = c_2 = c_3 = 0.$$

The former is the distinctive condition as has been seen in [4]. The latter leads to $L^3 = 6by^1y^2y^3$ which was indicated by A.M o ó r in 1954 ([3], Proposition 24.2).

Example 2: We consider also the exceptional cubic metric [4]:

$$L^3 = 3ay^1(y^3)^2 + b(y^2)^3, \qquad ab \neq 0.$$

We have $(a_{133}, a_{222}) = (a, b)$ and

$$L^2(a_1, a_2, a_3) = (ar^2, bq^2, 2apr),$$

$$L(a_{13}, a_{22}, a_{33}) = (ar, bq, ap),$$

$$\frac{1}{L}(a^{11}, a^{13}, a^{22}) = (\frac{-p}{ar^2}, \frac{1}{ar}, \frac{1}{bq}),$$

$$\frac{1}{L}(a_1{}^i{}_i, a_2{}^i{}_i, a_3{}^i{}_i) = (0, \frac{1}{q}, \frac{1}{r}).$$

Consequently $a_i{}^r{}_r \neq 3a_i$ and $C_i \neq 0$. Therefore the Finsler space with this L is strongly non-Riemannian.

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15, Zenbu-cho, Shimogamo,
Sakyo-ku, Kyoto,
606 - JAPAN

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LARGE TIME BEHAVIOUR FOR PARABOLIC STOCHASTIC VARIATIONAL INEQUALITIES

BY

A. BENSOUSSAN and A. RĂŞCANU[1]

Abstract. Dependence on initial data and large time behaviour of the solution of parabolic stochastic variational inequality

$$\begin{cases} du + A_0 u dt + \partial\varphi(u) dt \ \ni \ F(t,u)dt + B(t,u)dW(t) \\ u(0) = u_0 \end{cases}$$

are studied. The existence and uniqueness of a invariante meaure for the solutions is proved.

Key words: stochastic variational inequalities, exponential stability, Markov property, invariant measure.

0. Introduction. In the paper [2] we studied the existence and uniqueness of te solution of the parabolic stochastic variational inequalities:

$$\begin{cases} du + A_0 u dt + \partial\varphi(u) dt \ \ni \ F(t,u)dt + B(t,u)dW(t) \\ u(0) = u_0, \ t \in [0,T] \end{cases} \tag{0.1}$$

Certainly to put in evidence some properties of the solution are neessary.

In this paper we start with several results on regular dependence of the solution on initial data. Then the existence and uniqueness of invariant measure is proved.

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