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LARGE TIME BEHAVIOUR FOR PARABOLIC STOCHASTIC VARIATIONAL INEQUALITIES

BY

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Abstract. Dependence on initial data and large time behaviour of the solution of parabolic stochastic variational inequality

$$\begin{cases} du + A_0 u dt + \partial\varphi(u) dt \ \ni \ F(t,u)dt + B(t,u)dW(t) \\ u(0) = u_0 \end{cases}$$

are studied. The existence and uniqueness of a invariante meaure for the solutions is proved.

Key words: stochastic variational inequalities, exponential stability, Markov property, invariant measure.

0. Introduction. In the paper [2] we studied the existence and uniqueness of te solution of the parabolic stochastic variational inequalities:

$$\begin{cases} du + A_0 u dt + \partial\varphi(u) dt \ \ni \ F(t,u)dt + B(t,u)dW(t) \\ u(0) = u_0, \ t \in [0,T] \end{cases} \tag{0.1}$$

Certainly to put in evidence some properties of the solution are neessary.

In this paper we start with several results on regular dependence of the solution on initial data. Then the existence and uniqueness of invariant measure is proved.

[1] The work of this author was done during the visit to INRIA-France, July 1995, whose generous support is gratefuly acknowledged.

Let $(H_0, |\cdot|_0)$, $(H, |\cdot|)$, $(V, \|\cdot\|)$ be real separable Hilbert spaces and $(B, \|\ \|_B)$ be a real separable Banach spaces. We assume:

(0.2) W is a H_0 - valued Wienner proces with respect to a complete right continuous stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ and $Q = \sum_i \lambda_i\,(\cdot, e_i)_0$, $(\lambda_i \geq 0, \sum_i \lambda_i < \infty, \{e_i\}$ an orthonormal basis in $H_0)$ is the covariance operator of W, and

(0.3) $V \subset B \subset H \cong H^* \subset B^* \subset V^*$ [1] where the injections are continuous with dese range and $\forall\, r > 0 \quad \exists\, n_r > 0$ such that $\|u\|_B \leq r\|u\|_V + n_r|u|_H, \quad \forall\, u \in V$ [2].

We shall denote

$$|D|_Q^2 = tr DQD^* = \sum_i \lambda_i |De_i|^2, \quad \forall\, D \in \mathcal{L}(H_0, H),$$

and $\Lambda^p(W, H_0, H)$, $p \geq 1$, the space of the point-wise $\mathcal{F}_t$-progressively measurable (t- a.e. adapted and measurable) stochastic processes $f : \Omega \times]0, T[\to \mathcal{L}(H_0, H)$ [3] such that

$$E \int_0^T |f(t)|_Q^2 dt)^{p/2} < \infty$$

If $\varphi : H \to]-\infty, +\infty]$ is a proper convex l.s.c. function then we denote

$$\begin{aligned}
&Dom\varphi = \{h \in H : \ \varphi(h) < \infty\},\\
&\partial\varphi(u) = \{h \in H : (h, v - u) + \varphi(u) \leq \varphi(v), \quad \forall\, v \in H\},\\
&J_\varepsilon(u) = (I + \varepsilon\partial\varphi)^{-1}(u), \quad \varepsilon > 0,\\
&\varphi_\varepsilon(u) = \inf\{\tfrac{1}{2}|u - v|^2 + \varepsilon\varphi(v) : v \in H\},\ \varepsilon > 0,\\
&D\varphi_\varepsilon(u) = \partial\varphi_\varepsilon(u) = u - J_\varepsilon u
\end{aligned}$$

(see V.Barbu & T.Precupanu [1] or H.Brezis [3] for the properties of $\partial\varphi$, J_ε, φ_ε, $D\varphi_\varepsilon$).

The equation (0.1) given, one assumes also:

1 H^*, B^*, V^* are the dual spaces.

2 This last condition is satisfied if, for example, the embedding $V \subset B$ is compact.

3 $\forall\, t \in [0, T], \quad \forall\, h_0 \in H_0, \quad \forall\, h \in H : (\omega, s) \mapsto (f(\omega, s)h_0, h) : \Omega \times [0, t] \to \boldsymbol{R}$ is $\mathcal{F}_t \times \mathcal{B}_{[0,t]}$ - measurable.

(0.4)
$$\begin{aligned}
&A_0 \in \mathcal{L}(V, V^*),\ A_0 = A_0^*,\\
&< A_0 v, v > \geq \alpha_0\|v\|^2 \qquad (\alpha_0 \text{ is a positive constant }),\\
&D(A_0) = \{v \in V : Av \in H\}.
\end{aligned}$$

(0.5)
$$\begin{aligned}
&F(t, \cdot) : V \to H, \qquad t \in [0, T]\\
&\exists\, L, K > 0 \text{ such that } \quad \forall\, u, v \in V :\\
&i) \quad |F(t, u) - F(t, v)| \leq L\|u - v\|, \quad t - a.e.,\\
&ii) \quad |F(t, u)| \leq K(1 + \|u\|), \quad t - a.e.,\\
&iii) \quad s \mapsto (F(s, u), v) : [0, T] \to \boldsymbol{R} \text{ is measurable.}
\end{aligned}$$

(t- a.e. means "dt – $a.e.\ t \in [0, T]$").

(0.6)
$$\begin{aligned}
&B(t, \cdot) : V \to \mathcal{L}(H_0, H), t \in [0, T],\\
&\exists\, L_1, K_1 > 0,\ \exists\, b_1, d_1 \in]0, \alpha_0[,\ s.t.\ \forall\, u, v \in V,\ \ \forall\, h \in H_0:\\
&i) \quad |B(t, u) - B(t, v)|_Q^2 \leq b_1\|u - v\|^2 + L_1|u - v|^2, \quad t - a.e.,\\
&ii) \quad |B(t, u)|_Q^2 \leq d_1\|u\|^2 + K_1(1 + |u|^2), \quad t - a.e.,\\
&iii) \quad s \mapsto (B(s, u)h, v) : [0, T] \to \boldsymbol{R} \text{ is measurable .}
\end{aligned}$$

(0.7)
$$\begin{aligned}
&\exists\, K_2 > 0, \quad \exists\, d_2 \in]0, \alpha_0[\quad s.t. \quad \forall\, v \in D(A_0)\cdot\\
&i) \quad B(t, v) \in \mathcal{L}(H_0, V), \quad t - a.e.,\\
&ii) \quad |A_0^{\frac{1}{2}} B(t, v)|_Q^2 \leq \tfrac{d_2}{\alpha_0}|A_0 v|^2 + K_2(1 + \|v\|^2), \quad t - a.e.
\end{aligned}$$

(0.8) i) $\varphi: H \to]-\infty, +\infty]$ is a proper lowe-semicontinuous convex function,
ii) $\varphi(u) \geq \varphi(0) = 0, \quad \forall\, u \in H.$

(The last assumption is made for convenience only, and can be removed).

(0.9)
$$(A_0 u, D\varphi_\varepsilon(u)) \geq 0, \quad \forall\, u \in D(A_0).$$

$$\exists\, v_0 \in H, \quad \exists\, r_0 > 0, \quad \exists\, M_0 > 0 \text{ such that:} \tag{0.10}$$
$$\varphi(v_0 + r_0 h) \le M_0 \quad \forall\, h \in \boldsymbol{B}, \|h\|_{\boldsymbol{B}} \le 1.$$

$$\text{(0.11)}\qquad \begin{aligned} &i) \quad D\varphi_\varepsilon(u) \in \boldsymbol{B}, \quad \forall\, u \in D(A_0),\\ &ii) \quad \forall\, r > 0,\ \exists\, \gamma_r > 0 \text{ s.t. } \forall\, u \in D(A_0), \quad \forall\, \varepsilon \in]0,1].\\ &\qquad \|D\varphi_\varepsilon(u)\|_{\boldsymbol{B}} \le r(1 + \|u\|) + \gamma_r |D\varphi_\varepsilon(u)|. \end{aligned}$$

In the sequel $L^2_{ad}(\cdots)$ will denote the closed linear subspaces of adapted (or a.e. adapted) stochastic processes f from the corresponding space $L^2(\cdots)$.

Definition 0.1. *A stochastic process u is a (strong) solution of Eq.(0.1) if*

$i)$ $u \in L^2_{ad}(\Omega\times]0,T[; V) \cap L^2_{ad}(\Omega; C([0,T]; \boldsymbol{B}))$,

$ii)$ $u(\omega,t) \in Dom(\varphi)$, (ω,t) – a.e. and $\varphi(u) \in L^1(\Omega\times]0,T[)$,

$iii)$ $\exists\, \eta \in L^2_{ad}(\Omega; C([0,T]; V^*)) \cap L^2(\Omega; BV(0,T; \boldsymbol{B}^*))$ such that $\eta(\omega,0) = 0$, ω – a.s., and $\forall\, t \in [0,T]$, ω – a.s :

$$u(t) + \int_0^t A_0 u(s)ds + \eta(t) = u_0 + \int_0^t F(s,u(s))ds + \int_0^t B(s,u(s))dW(s),$$

(0.12) $iv)$ (subdifferential inequality) $\forall v \in L^2(\Omega; C([0,T]; \boldsymbol{B}))$:

$$\boldsymbol{E}\int_0^T < u(s) - v(s), d\eta(s) > ds + \boldsymbol{E}\int_0^T \varphi(v(s))ds \ge \boldsymbol{E}\int_0^T \varphi(u(s))ds$$

$v)$ (Energy Equality)

$$\boldsymbol{E}\int_0^t < v^*(s) + A_0u(s) - F(s,u(s)), v(s) - u(s) > ds + \tfrac{1}{2}\boldsymbol{E}\int_0^t |\tilde v(s) - B(s,u(s))|^2_Q ds + \tfrac{1}{2}\boldsymbol{E}|v_0 - u_0|^2 + \boldsymbol{E}\int_0^t < v(s) - u(s), d\eta(s) > = \tfrac{1}{2}\boldsymbol{E}|v(t) - u(t)|^2$$

$\forall\, t \in [0,T]$, $\quad \forall\, v \in W_{ad}(\Omega\times]0,T[) \cap L^2(\Omega; C([0,T]; \boldsymbol{B}))$,

where $W_{ad}(\Omega\times]0,T[)$ is the space of stochastic processes

$$v \in L^2_{ad}(\Omega\times]0,T[; V) \cap L^2_{ad}(\Omega; C([0,T]; H))$$

of the form:

$$\text{(0.13)}\qquad \begin{aligned} &v(t) = v_0 + \int_0^t v^*(s)ds + \int_0^t \tilde v(s)dW(s)\\ &\text{with } \ v_0 \in L^2(\Omega, \mathcal{F}_0, P; H), v^* \in L^2_{ad}(\Omega\times]0,T[; V^*),\\ &\tilde v \in \Lambda^2(W; H_0, H). \end{aligned}$$

We call also (u,η) solution for (0.1) (from (0.12–iii) η is unique defined by u).

Theorem 0.2. *Let the hypotheses (0.2)–(0.11) be satisfied. One assumes also that there exists $p \ge 2$ such that*

$$\text{(0.14)}\qquad \begin{aligned} &i) \quad u_0 \in L^{2p}(\Omega, \mathcal{F}_0, P; V)\\ &ii) \quad \varphi(u_0) \in L^p(\Omega) \end{aligned}$$

and the constants b_1, d_1, d_2 from (0.6) and (0.7) are not too large[1]*. Then the equations (0.1) has one and only one (strong) solution (u,η). Moreover this solution satisfies:*

$$\text{(0.15)}\qquad \begin{aligned} &a) \quad u(\omega,t) \in D(A_0), \quad (w,t) - a.e.,\\ &b) \quad u \in L^{2p}(\Omega; L^\infty(0,T; V)),\\ &c) \quad \varphi(u) \in L^p(\Omega; L^1(0,T)),\\ &d) \quad A_0 u \in L^{2p}_{ad}(\Omega; L^2(0,T); H)),\\ &e) \quad \eta \in L^p(\Omega; BV(0,T; \boldsymbol{B}^*)) \cap L^2_{ad}(\Omega; C([0,T]; H)), \end{aligned}$$

and is obtained as limit for $\varepsilon \to 0$ of the approximate solution $(u_\varepsilon, \eta_\varepsilon)$ of:

$$\text{(0.16)}\qquad \begin{cases} du_\varepsilon + A_0u_\varepsilon dt + \frac{1}{\varepsilon}D\varphi_\varepsilon(u_\varepsilon)dt = F(t,u_\varepsilon)dt + B(t,u_\varepsilon)dW(t)\\ u_\varepsilon(0) = u_0,\\ \eta_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t D\varphi_\varepsilon(u_\varepsilon(s))ds, \quad \varepsilon \in]0,1], \quad t \in [0,T], \end{cases}$$

[1] for example $0 < d_1, d_2 < \frac{1}{2^{11}p^2}\alpha_0$, $0 < b_1 < \frac{1}{18}\alpha_0$

as follows:

$$(0.17)\quad \begin{aligned} a)\quad & \lim_{\varepsilon\to 0} u_\varepsilon = u \text{ strongly in } L^2_{ad}(\Omega; C([0,T], B))\cap \\ & \cap L^2_{ad}(\Omega\times]0,T[; V) \text{ and weak star in } L^{2p}(\Omega; L^\infty(0,T;V)), \\ b)\quad & \lim_{\varepsilon\to 0} \eta_\varepsilon = \eta \text{ strongly in } L^2_{ad}(\Omega; C([0,T]); V^*)) \text{ and} \\ & \text{weak star in } L^p(\Omega; BV(0,T;B^*)). \end{aligned}$$

Also there exists a constant $C = C_1 e^{C_2 T}$ with C_1, C_2 independent of ε, u_0, T, such that $\forall\, q\in[1,p]$ the following estimates hold[1]*:*

$$(0.18)\quad \begin{aligned} a)\quad & \mathbf{E}\sup_{t\in[0,T]} |u_\varepsilon(t)|^{2q} \le C(1+\mathbf{E}|u_0|^{2q}) \\ b)\quad & \mathbf{E}(\int_0^T \|u_\varepsilon(t)\|^2 dt)^q \le C(1+\mathbf{E}|u_0|^{2q}) \\ c)\quad & \mathbf{E}\sup_{t\in[0,T]} \|\eta_\varepsilon(t)\|^q_{B^*} \le C(1+\mathbf{E}|u_0|^{2q}) \\ d)\quad & \mathbf{E}\|\eta_\varepsilon\|^q_{BV(0,T;B^*)} \le C(1+\mathbf{E}|u_0|^{2q}) \\ e)\quad & \mathbf{E}\sup_{t\in[0,T]} \|u_\varepsilon(t)\|^{2q} \le C(1+\mathbf{E}\|u_0\|^{2q}) \\ f)\quad & \mathbf{E}(\int_0^T |A_0 u_\varepsilon(t)|^2 dt)^q \le C(1+\mathbf{E}\|u_0\|^{2q}) \end{aligned}$$

and

$$(0.19)\quad \mathbf{E}(\int_0^T \varphi(u(t))dt)^q \le C(1+\mathbf{E}|u_0|^{2q})$$

If u and $\tilde u$ are the solutions of Eq.(0.1) corresponding to initial data u_0 and $\tilde u_0$, respectively, then $\forall\, 0\le s\le t\le T$:

$$(0.20)\quad \int_s^t (u(\tau)-\tilde u(\tau), d\eta(\tau) - d\tilde\eta(\tau)) \ge 0, \quad P-\quad a.s.$$

[1] The same inequalities hold for u, η in the stead of $u_\varepsilon, \eta_\varepsilon$, respectively, obtained by passing to liminf as $\varepsilon\to 0$.

The proof of this theorem is given by A.Bensoussan & A.Răşcanu in [2] less the explicite estimates (0.18) and (0.19) which are easily obtained from the estimates given in [2], Section 2. Remark also that (0.20) follows from (0.12) by a standard calculus.

1. Dependence on initial data and Markov property. Let us remark that Theorem 0.2 obtained for the time interval $[0,T]$ can be generalized in a natural way to intervals $[s,T], s\in[0,T]$ with σ-algebras $\mathcal{F}_s$ playing the role of the σ-algebra $\mathcal{F}_0$, and $W(t)-W(s), s\le t$, the role of the Wiener process.

We shall denote $u(t;s,\xi)$ the solution $u(t)$ of Eq.(0.1) on $[s,T]$ with the initial condition $u(s)=\xi$. Denote also $u(t;\xi) = u(t;0,\xi)$.

Proposition 1.1. *Let the assumptions of Th.0.2 be satisfied for the initial data $u_0, \tilde u_0$ and let $(u,\eta), (\tilde u, \tilde\eta)$ be the corresponding solutions. Then $\exists\, C = C(\alpha_0, L, L_1, b_1, T) > 0$ a constant independent of u_0 and $\tilde u_0$ such that:*

$$(1.1)\quad \begin{aligned} a)\quad & \mathbf{E}\sup_{t\in[0,T]} |u(t)-\tilde u(t)|^2 \le C\mathbf{E}|u_0-\tilde u_0|^2, \\ b)\quad & \mathbf{E}\int_0^T \|u(t)-\tilde u(t)\|^2 dt \le C\mathbf{E}|u_0-\tilde u_0|^2, \\ c)\quad & \mathbf{E}\sup_{t\in[0,T]} \|\eta(t)-\tilde\eta(t)\|^2_{V^*} \le C\mathbf{E}|u_0-\tilde u_0|^2. \end{aligned}$$

Proof. From (0.17) is sifficiently to prove (1.1) for $u_\varepsilon, \tilde u_\varepsilon$ in stead of u and $\tilde u$, respectively; then (1.1) follows passing to limit for $\varepsilon\to 0$. By Energy Equality for $u_\varepsilon(t) - \tilde u_\varepsilon(t)$ we have

$$|u_\varepsilon(t)-\tilde u_\varepsilon(t)|^2 + \int_s^t G_\varepsilon(\tau, u_\varepsilon(\tau), \tilde u_\varepsilon(\tau))d\tau =$$

$$= |u_\varepsilon(s)-\tilde u_\varepsilon(s)|^2 + 2\int_s^t (u_\varepsilon - \tilde u_\varepsilon, (B(\tau,u_\varepsilon) - B(\tau,\tilde u_\varepsilon))dW(\tau))$$

for all $0\le s\le t\le T$, $P-a.s.$, where

$$(1.3)\quad \begin{aligned} G_\varepsilon(\tau,u,v) = \; & 2 < A_0(u-v), u-v > + \tfrac{2}{\varepsilon}(D\varphi_\varepsilon(u) - D\varphi_\varepsilon(v), u-v) - \\ & -2(F(\tau,u)-F(\tau,v), u-v) - |B(\tau,u)-B(\tau,v)|^2 \end{aligned}$$

Since

$$
\begin{aligned}
G_\varepsilon(\tau,u,v) &\geq 2\alpha_0\|u-v\|^2 - 2L\|u-v\||u-v| - \\
&\quad - b_1\|u-v\|^2 - L_1|u-v|^2 \\
&\geq \alpha_0\|u-v\|^2 - \left(\frac{L^2}{\alpha_0-b_1}+L_1\right)|u-v|^2
\end{aligned}
$$

then

$$
(1.4)\qquad
\begin{aligned}
|u_\varepsilon(t)-\tilde u_\varepsilon(t)|^2 \quad &+ \alpha_0 \int_s^t \|u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)\|^2 d\tau \leq |u_\varepsilon(s)-\tilde u_\varepsilon(s)|^2 \\
&+\left(\frac{L^2}{\alpha_0-b_1}\right)\int_s^t |u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)|^2 d\tau + \\
&+2\int_s^t (u_\varepsilon-\tilde u_\varepsilon,(B(\tau,u_\varepsilon)-B(\tau,\tilde u_\varepsilon))dW(\tau))
\end{aligned}
$$

which yields, using Gronwall's lemma:

$$
(1.5)\qquad
\begin{aligned}
&(a)\quad E(|u_\varepsilon(t)-\tilde u_\varepsilon(t)|^2|\mathcal{F}_s) \leq e^{M(t-s)}|u_\varepsilon(s)-\tilde u_\varepsilon(s)|^2 \\
&b)\quad E\int_s^t \|u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)\|^2 d\tau \leq \tfrac{1}{\alpha_0}e^{2M(t-s)}E|u_\varepsilon(s)-\tilde u_\varepsilon(s)|^2
\end{aligned}
$$

with $M=\frac{L^2}{\alpha_0-b_1}+L_1$.

By Burkholder-Davis-Gundy inequality and (1.5) we have:

$$
\begin{aligned}
2E\sup_{\tau\in[s,t]}\left|\int_s^\tau (u_\varepsilon-\tilde u_\varepsilon,(B(u_\varepsilon)-B(\tilde u_\varepsilon))dW)\right| &\leq \\
\leq 6E\sqrt{\int_s^t |u_\varepsilon-\tilde u_\varepsilon|^2|B(u_\varepsilon)-B(\tilde u_\varepsilon)|_Q^2 d\tau} & \\
\leq \frac{1}{2}E\sup_{\tau\in[s,t]}|u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)|^2 + 18E\int_s^t |B(u_\varepsilon)-B(\tilde u_\varepsilon)|_Q^2 d\tau &
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}E\sup_{\tau\in[s,t]}|u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)|^2 + 18b_1 E\int_s^t \|u_\varepsilon-\tilde u_\varepsilon\|^2 d\tau \\
&\qquad +18L_1\int_s^t E|u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)|^2 d\tau \\
&\leq \frac{1}{2}\leq \sup_{\tau\in[s,t]}|u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)|^2 + C_1(s,t)E|u_\varepsilon(s)-\tilde u_\varepsilon(s)|^2
\end{aligned}
$$

where

$$
C_1(s,t) = 18\left(\frac{b_1}{\alpha_0}+L_1\right)e^{(2M+1)(t-s)}
$$

Now from (1.4) we obtain:

$$
(1.6)\qquad E\sup_{\tau\in(s,t)}|u_\varepsilon(\tau)-\tilde u_\varepsilon(\tau)|^2 \leq C\cdot e^{(2M+1)(t-s)}E|u_\varepsilon(s)-\tilde u_\varepsilon(s)|^2
$$

where $C=2+36\left(\frac{b_1}{\alpha_0}+L_1\right)$. Passing to limit for a subsequence $\varepsilon=\varepsilon_n\to 0$ in (1.5) and (1.6) we have for all $0\leq s\leq t\leq T$:

$$
(1.7)\qquad
\begin{aligned}
&a)\quad E(|u(t)-\tilde u(t)|^2|\mathcal{F}_s) \leq e^{M(t-s)}|u(s)-\tilde u(s)|^2,\ a.s., \\
&b)\quad E\int_s^t \|u(\tau)-\tilde u(\tau)\|^2 d\tau \leq e^{2M(t-s)}E|u(s)-\tilde u(s)|^2, \\
&c)\quad E\sup_{\tau\in[s,t]}|u(\tau)-\tilde u(\tau)|^2 \leq Ce^{(2M+1)(t-s)}E|u(s)-\tilde u(s)|^2,
\end{aligned}
$$

where $M=\frac{L^2}{\alpha_0-b_1}+L_1$ and $C=2+36(\frac{b_1}{\alpha_0}+L_1)$.

The inequality (1.1-c) follows now from (0.12-iii). The proof is finished. ■

Let

$$
(1.8)\qquad
\begin{aligned}
\mathcal{D}_s = \{\xi\in L^2(\Omega,\mathcal{F}_s,P;H):\ &\exists\,\xi_n\in L^4(\Omega,\mathcal{F}_s,P;V), \\
E|\varphi(\xi_n)|^2<\infty,\quad &E|\xi_n-\xi|^2\longrightarrow 0\}.
\end{aligned}
$$

Let us remark that

$$
(1.9)\qquad
\begin{aligned}
&a)\quad \{0\}\subset\mathcal{D}_0\subset\mathcal{D}_s\subset\mathcal{D}_T,\quad \forall\, 0\leq s\leq T, \\
&b)\quad \mathcal{D}_s \text{ is a closed set in } L^2(\Omega,\mathcal{F}_s,P;H), \\
&c)\quad \xi\in\mathcal{D}_s \Longrightarrow E(\varphi(\xi))^2<\infty, \\
&d)\quad Dom(\varphi)\cap V\subset\mathcal{D}_s,\quad \forall\, s\in[0,T].
\end{aligned}
$$

Moreover if the function φ satisfies

$$(1.10)\qquad \begin{aligned}&\forall\, x \in \overline{Dom\varphi} \quad \exists\, x_n \in V \cap Dom(\varphi) \text{ such}\\ &\text{that } x_n \longrightarrow x \text{ in } H\end{aligned}$$

then

$$(1.11)\qquad \overline{Dom\varphi} \subset \mathcal{D}_s, \quad \forall\, s \in [0,T].$$

The estimations (1.1) show that we can generalize the notion of the solution for Eq.(0.1) extending by continuity to $u_0 \in \mathcal{D}_0$.

If $\xi \in \mathcal{D}_s$, $s \in [0,T]$ fixed, then

$$u(\cdot;s,\xi_n) \in L^4(\Omega; L^\infty(s,T;V)) \cap L^2_{ad}(\Omega; C([s,T];B)),$$

$$\varphi(u(\cdot;s,\xi_n)) \in L^2(\Omega; L^1(s,T))$$

$$\eta(\cdot;s,\xi_n) \in L^2(\Omega; BV([s,T];B^*)) \cap L^2_{ad}(\Omega; C([s,T];H))$$

and from (1.7), (0.18), (0.19) we have:

$$\boldsymbol{E} \sup_{t\in[s,T]} |u(t;s,\xi_n) - u(t;s,\xi_n)|^2 \le C_T \boldsymbol{E}|\xi_n - \xi_m|^2$$

$$\boldsymbol{E}\int_s^T \|u(t;s,\xi_n) - u(t;s,\xi_m)\|^2 dt \le C_T \boldsymbol{E}|\xi_n - \xi_m|^2$$

$$\boldsymbol{E}\int_s^T \varphi(u(t;s,\xi_n))dt \le C_T(1 + \boldsymbol{E}|\xi_n|^2)$$

$$\boldsymbol{E} \sup_{t\in[s,T]} \|\eta(t;s,\xi_n) - \eta(t;s,\xi_m)\|^2_{V^*} \le C_T|\xi_n - \xi_m|^2$$

$$\boldsymbol{E}\|\eta(\cdot;s,\xi_n)\|_{BV(s,T;B^*)} \le C_T(1 + \boldsymbol{E}|\xi_n|^2)$$

where C_T is a constant independent of $s \in [0,T]$ and the sequence $\{\xi_n : n \in \boldsymbol{N}^*\}$.

Hence there exist

$$(1.12)\qquad \begin{aligned}&u(\cdot;s,\xi) \in L^2_{ad}(\Omega; C([s,T];H)) \cap L^2_{ad}(\Omega\times]s,T[;V) \equiv X\\ &\eta(\cdot;s,\xi) \in L^2_{ad}(\Omega; C([s,T];V^*)) \cap L^1(\Omega; BV(s,T;B^*))\end{aligned}$$

such that

$$(1.13)\qquad \begin{aligned}
a)&\quad \lim_{n\to\infty} u(\cdot;s,\xi_n) = u(\cdot;s,\xi) \text{ in } X,\\
b)&\quad \lim_{n\to\infty} \eta(\cdot;s,\xi_n) = \eta(\cdot;s,\xi) \text{ in } L^2_{ad}(\Omega; C([s,T];V^*))\\
&\quad \text{and } P-a.s. \text{ in } BV([s,T];B^*),\\
c)&\quad \boldsymbol{E}\int_s^T \varphi(u(t;s,\xi))dt \le C_T(1 + \boldsymbol{E}|\xi|^2),\\
d)&\quad \boldsymbol{E}\|\eta(\cdot;s,\xi)\|_{BV([s,T],B^*)} \le C_T(1 + \boldsymbol{E}|\xi|^2).
\end{aligned}$$

Definition 1.2. *The couple (u,η) from (1.12), (1.13) with $s = 0$ corresponding to initial datum $\xi = u_0 \in \mathcal{D}_0$ is called the generalized solution of (0.1).*

Remark 1.3. *The generalized solution $u(t;u_0)$ is unique since*

$$(1.14)\qquad \begin{aligned}
a)&\quad \boldsymbol{E} \sup_{t\in[0,T]} |u(t,u_0) - u(t,\tilde{u}_0)|^2 \le C_T \boldsymbol{E}|u_0 - \tilde{u}_0|^2\\
b)&\quad \boldsymbol{E} \sup_{t\in[0,T]} \|\eta(t;u_0) - \eta(t;\tilde{u}_0)\|^2_{V^*} \le C_T \boldsymbol{E}|u_0 - \tilde{u}_0|^2
\end{aligned}$$

Moreover the generalized solution satisfies:

(1.15)

a) $\xi \in \mathcal{D}_s \Rightarrow u(t;s,\xi) \in \mathcal{D}_t$

b) $0 \le s \le \tau \le t \le T \Rightarrow u(t;s,\xi) = u(t;\tau,u(\tau;s,\xi))$

c) $$u(t) + \int_s^t A_0 u(\tau)d\tau + \eta(t) = u(s) + \int_s^t F(\tau,u(\tau))d\tau + \int_s^t B(\tau,u(\tau))dW(\tau), \quad \forall\, 0 \le s \le t \le T,\ P-a.s.,$$

d) $$\begin{aligned}&\int_s^t < v^*(\tau) + A_0 u(\tau) - F(\tau,u(\tau)), v(\tau) - u(\tau) > d\tau +\\
&+\frac{1}{2}\int_s^t |\tilde{v}(\tau) - B(\tau,u(\tau))|^2_Q d\tau - \int_s^t (\tilde{v}(\tau) - u(\tau), B(\tau,u(\tau))dW(\tau))\\
&+\frac{1}{2}\boldsymbol{E}|v(s) - u(s)|^2 + \int_s^t \varphi(v(\tau))d\tau \ge \frac{1}{2}\boldsymbol{E}|v(t) - u(t)|^2 + \int_s^t \varphi(u(\tau))d\tau,\end{aligned}$$

$\forall\, 0 \le s \le t \le T,\ P-a.s.,\ \forall\, v \in W_{ad}(0,T),$

where $W_{ad}(0,T)$ is the space of stochastic processes $v \in L^2_{ad}(\Omega; C([0,T]; H)) \cap L^2_{ad}(\Omega\times]0,T[; V)$ of the form

$$v(t) = v_0 + \int_0^t v^*(\tau)d\tau + \int_0^t \widetilde{v}(\tau)dW(\tau)$$

with $v_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$, $v^* \in L^2_{ad}(\Omega\times]0,T[; V^*)$, $\widetilde{v} \in \Lambda^2(W; H_0, H)$.

Also by continuity with respect to initial data – (1.14 a) – the inequality (1.7–a) is preserved. Hence for $M = \frac{L^2}{\alpha_0 - b_1} + L_1$ we have:

Corollary 1.4. *If $u_0, \widetilde{u} \in \mathcal{D}_0$ then $e^{-Mt}|u(t; u_0) - u(t; \widetilde{u}_0)|^2$ is a super–martingale.*

To the end of this section we assume that (1.10) is satisfied. Denote $K = \overline{Dom\varphi}$. Let $B_b(K)$ and $C_b(K)$ be the Banach spaces of all real bounded Borel and respectively continuous functions $g: K \to R$ endowed with the sup–norm.

Define for $0 \le s \le t \le T$ and $g \in B_b(K)$

$$(P_{st}g)(x) = Eg(u(t; s, x)) \tag{1.16}$$

Let us remark that if $g \in C_b(K)$ then $P_{st}g(\cdot) \in C_b(K)$ since the convergence in $L^2(\Omega; H)$ implies the convergence in probability and in law. Hence the (generalized) solution u has the Feller property.

The transition probability of the solution $u(t; s, x)$ is defined by

$$P(s, x; t, \Gamma) = P_{st}1_\Gamma(x) = P(\{\omega : u(\omega, t; s, x) \in \Gamma\}) \tag{1.17}$$

for $\Gamma \in \mathcal{B}_H$ and is concentrated on K:

$$P(s, x; t, K) = 1, \quad P(s, x, ; t, H \setminus K) = 0$$

Proposition 1.5. *Let the assumptions (0.2)–(0.11) and (1.10) be satisfied. Then the stochastic process $u(t; s, x)$, $0 \le s \le t \le T$, $x \in K = \overline{Dom\varphi}$. is a Markov process with transition probability P_{st}, $0 \le s \le t \le T$ that is:*

$$E(g(u(t; s, x))|\mathcal{F}_\tau) = P_{\tau t}g(u(\tau; s, x)), \quad \text{a.s.} \tag{1.18}$$

for all $0 \le s \le \tau \le t \le T$, $\forall\, g \in B_b(K)$.

The proof is the same as in [5] – Da Prato & Zabczyk, p.249. Important is to observe that $u(t; s, x)$ satisfies (1.15–b) and $u(t; s, x)$ is measurable with respect to $\sigma(\{w(r) - w(s), r \in [s, t]\})$ and therefore in independent of $\mathcal{F}_s$.

Proposition 1.6. *Assume (0.2)–(0.11) and (1.10) are satisfied. Then $\forall\, g \in B_b(K)$ and $0 \le s \le \tau \le t \le T$ hold:*

$$\begin{aligned} &a) \quad P_{s\tau}(P_{\tau t}g)(x) = P_{st}g(x), \quad \forall\, x \in K, \\ &b) \quad P(s, x; t, \Gamma) = \int_K P(s, x; \tau, dy)P(\tau, y; t, \Gamma) \end{aligned} \tag{1.19}$$

Moreover if $F(t, u) \equiv F(u), B(t, u) \equiv B(u)$ then

$$P_{st} = P_{0,t-s}, \quad 0 \le s \le t \le T \tag{1.20}$$

Proof. (see [5] – Da Prato & Zabczyk p.251):

$$(P_{st}g)(x) = Eg(u(t; s, x)) = E[E(g(u(t; s, x))|\mathcal{F}_\tau)] =$$

$$= EP_{\tau,t}g(u(\tau; s, x)) = P_{s\tau}(P_{\tau t}g)(x)$$

that is (1.19–a), and for $g = 1_\Gamma$, follows (1.19–b). The equality (1.20) is obtained from

$$Eg(u_\varepsilon(t; s, x_n)) = Eg(u_\varepsilon(t - s; 0, x_n))$$

passing to limit for $\varepsilon \to 0$ and then $x_n \to x$. ∎

The next results will give supplimentary informations on continuity in time for the solution started from $u_0 = 0$. Also the estimates finded will be used for asymptotically behaviour problems.

Let

$$\gamma_0 \ge \sup\{|u| / \|u\| : u \in V, u \ne 0\} \tag{1.21}$$

Since the inclusion $V \subset H$ is continuous then the $\sup\{\ldots\}$ is finit. Denote

$$\begin{aligned} \beta &= \tfrac{1}{\gamma_0^2}(2\alpha_0 - b_1) - 2\tfrac{L}{\gamma_0} - L_1, \\ b_0 &= \tfrac{b_1}{\gamma_0^2} + L_1 + 1, \\ C(\lambda, \delta) &= \beta - \left(\tfrac{\sqrt{\lambda}}{\gamma_0} - \tfrac{L}{\sqrt{\lambda}}\right)^2 - \delta b_0 \end{aligned} \tag{1.22}$$

where $\lambda \in]0, 2\alpha_0 - b_1[$, $\delta \in]0, \frac{2\alpha_0 - b_1 - \lambda}{b_1 + 1}]$ (recall that by the assumption (0.6): $0 \le b_1 < \alpha_0$).

Proposition 1.7. *Assume (0.2) – (0.11) be satisfied with F and B do not depend on time t. Then* $\forall\, 0 \le s \le t$ *we have*

$$\text{(1.23)}\quad \begin{aligned} &a)\quad \boldsymbol{E}|u(t;0)|^2 \le \left(1+\frac{1}{\delta}\right)\left[|F(0)|^2+|B(0)|_Q^2\right]\int_0^t e^{-C(\lambda,\delta)\tau}d\tau,\\ &b)\quad \boldsymbol{E}|u(t;0)-u(s;0)|^2 \le \left(1+\frac{1}{\delta}\right)\left[|F(0)|^2+|B(0)|_Q^2\right]e^{-\delta b_0 s}\times\\ &\qquad\qquad \times\int_s^t e^{-C(\lambda,\delta)\tau}d\tau, \end{aligned}$$

for all $\lambda \in]0, 2\alpha_0 - b_1[$, $\forall\, \delta \in]0, \frac{2\alpha_0-b_1-\lambda}{b_1+1}]$.

Proof. We shall use an idea from [5]–Da Prato & Zabczyk, Theorem 11.21, p.327. Let $\widetilde{W}$ a H_0–valued Wiener process with Q as covariance operator and $\widetilde{W}$ is independent of W. We extend $W(t)$ for $t<0$ by

$$W(t) = \widetilde{W}(-t), \quad t \le 0$$

and consider $u_{\varepsilon,\tau}(t) = u_{\varepsilon,\tau}(t;-\tau,0), \tau \ge 0$ the solution of the equation:

$$\text{(1.24)}\quad \begin{aligned} &du_{\varepsilon,\tau}(t) + A_0 u_{\varepsilon,\tau}(t) + \tfrac{1}{\varepsilon}D\varphi_\varepsilon(u_{\varepsilon,\tau}(t))dt = F(u_{\varepsilon,\tau}(t))dt +\\ &\qquad\qquad\qquad\qquad\qquad + B(u_{\varepsilon,\tau}(t))dW(t)),\\ &u_{\varepsilon,\tau}(-\tau) = 0, \quad t \ge -\tau \end{aligned}$$

Then it is easily seen that:

$$\mathcal{L}(u_{\varepsilon,\tau}(0)) = \mathcal{L}(u_\varepsilon(\tau;0,0))$$

(the random variables $u_{\varepsilon,\tau}(0), u_\varepsilon(\tau;0,0)$ have the same law).

By Energy Equality we have $\forall\, t \ge -\tau$:

$$\boldsymbol{E}|u_{\varepsilon,\tau}(t)|^2 + \int_{-\tau}^t \boldsymbol{E}L_\varepsilon(u_{\varepsilon,\tau}(s))ds = 0,$$

where

$$L_\varepsilon(v) = 2 < A_0 v, v > + \frac{2}{\varepsilon}(D\varphi_\varepsilon(v), v) - 2(F(v),v) - |B(v)|_Q^2$$

Hence

$$\text{(1.25)}\quad \frac{d}{dt}E|u_{\varepsilon,\tau}(t)|^2 + EL_\varepsilon(u_{\varepsilon,\tau}(t)) = 0, \quad t \ge -\tau.$$

But

$$\begin{aligned} L_\varepsilon(v) \ge\ & 2\alpha_0 \| v \|^2 - 2|F(v)-F(0)||v| - 2|F(0)||v| -\\ & -(1+\delta)|B(v)-B(0)|_Q^2 - (1+\tfrac{1}{\delta})|B(0)|_Q^2 \ge\\ & \ge (2\alpha_0 - \lambda - (1+\delta)b_1)\| v \|^2 - \left(\tfrac{L^2}{\lambda} + \delta + (1+\delta)L_1\right)|v|^2 -\\ & -(1+\tfrac{1}{\delta})\,|B(0)|_Q^2 - \tfrac{1}{\delta}|F(0)|^2 \ge\\ & \ge \left(\tfrac{2\alpha_0 - b_1 - \lambda - \delta b_1}{\gamma_0^2} - \tfrac{L^2}{\lambda} - \delta - (1+\delta)L_1\right)|v|^2 -\\ & -(1+\tfrac{1}{\delta})\left(|F(0)|^2 + |B(0)|_Q^2\right) =\\ & = C(\lambda,\delta)|v|^2 - (1+\tfrac{1}{\delta})\left(|F(0)|^2 + |B(0)|_Q^2\right) \end{aligned}$$

for all $\lambda \in]0, 2\alpha_0 - b_1[, \delta \in]0, \frac{2\alpha_0-b_1-\lambda}{b_1+1}]$.

Then from (1.25) it follows

$$\frac{d}{dt}\boldsymbol{E}|u_{\varepsilon,\tau}(t)|^2 + C(\lambda,\delta)\boldsymbol{E}|u_{\varepsilon,\tau}(t)|^2 \le (1+\frac{1}{\delta})(|F(0)|^2 + |B(0)|_Q^2)$$

and by integrationfrom $-\tau$ to t we have:

$$\text{(1.26)}\quad e^{C(\lambda,\delta)t}\boldsymbol{E}|u_{\varepsilon,\tau}(t)|^2 \le (1+\frac{1}{\delta})(|F(0)|^2 + |B(0)|_Q^2)\int_{-\tau}^t e^{C(\lambda,\delta)r}dr$$

If we put $t = 0$ one obtains:

$$E|u_\varepsilon(\tau;0)|^2 \le \left(1+\frac{1}{\delta}\right)(|F(0)|^2 + |B(0)|_Q^2)\int_0^\tau e^{-C(\lambda,\delta)r}dr$$

which yields (1.23-a), for $\varepsilon \longrightarrow 0$.

Let now $0 < \theta < \tau$. Applying Itô's formula to $|u_{\varepsilon,\theta}(t) - u_{\varepsilon,\tau}(t)|^2$, for $t \ge -\theta$ one has

$$E|u_{\varepsilon,\theta}(t)-u_{\varepsilon,\tau}(t)|^2+\int_{-\theta}^{t} EG_\varepsilon(u_{\varepsilon,\theta}(r),u_{\varepsilon,\tau}(r)dr=E|u_{\varepsilon,\theta}(-\theta)|^2$$

or more

$$\frac{d}{dt}E|u_{\varepsilon,\theta}(t)-u_{\varepsilon,\tau}(t)|^2+EG_\varepsilon(u_{\varepsilon,\theta}(t),u_{\varepsilon,\tau}(t))=0,$$

where

$$\begin{aligned}G_\varepsilon(u,v)=\ & 2<A_0(u-v)),u-v>+\tfrac{2}{\varepsilon}(D\varphi_\varepsilon(u)-D\varphi_\varepsilon(v),u-v)-\\ & -2(F(u)-F(v),u-v)-|B(u)-B(v)|_Q^2\ge\\ & \ge 2\alpha_0\|u-v\|^2-2L\|u-v\|\,|u-v|-\\ & -b_1\|u-v\|^2-L_1|u-v|^2\ge\\ & \ge(2\alpha_0-\lambda-b_1)\|u-v\|^2-(\tfrac{L^2}{\lambda}+L_1)|u-v|^2\ge\\ & \ge(\tfrac{2\alpha_0-\lambda-b_1}{\gamma_0^2}-\tfrac{L^2}{\lambda}-L_1)|u-v|^2=\\ & =(C(\lambda,\delta)+\delta b_0)|u-v|^2\end{aligned}$$

for all $\lambda\in]0,2\alpha_0-b_1[$.

As above we obtain by integration from $-\theta$ to 0:

$$E|u_{\varepsilon,\theta}(0)-u_{\varepsilon,\tau}(0)|^2\le e^{-(C(\lambda,\delta)+\delta b_0)\theta}E|u_{\varepsilon,\tau}(-\theta)|^2 \tag{1.27}$$

and using (1.26) for $t=-\theta>-\tau$ one follows

$$E|u_\varepsilon(\theta;0)-u_\varepsilon(\tau;0)|^2\le e^{-\delta b_0\theta}(1+\frac{1}{\delta})\,[|F(0)|^2+$$

$$+|B(0)|_Q^2]\int_\theta^\tau e^{-C(\lambda,\delta)r}dr$$

which yields (1.23-b) passing to limit for $\varepsilon\longrightarrow o$.

Clearly from (1.23-b) we have:

Corollary 1.8. *If the assumptions of Proposition 1.7 are satisfied then* $\forall\, R>0\ \ \exists\, C_R>0$ *such that* $\forall\, s,t\in[0,R]$:

$$E|u(t;0)-u(s,0)|^2\le C_R(t-s)[|F(0)|^2+|B(0)|_Q^2] \tag{1.28}$$

(the constant C_R depeds on $R,\alpha_0,\gamma_0,b_1,L,L_1$)

Also from Proposition 1.7 follows:

Corollary 1.9. *Let the assumptions of Proposition 1.7 be satisfied. If*

$$\beta=\frac{1}{\gamma_0^2}(2\alpha_0-b_1)-2\frac{L}{\gamma_0}-L_1>0$$

then there exists $\delta_0=\delta_0(\alpha_0,\gamma_0,L,b_1,L_1)>0$ such that $\forall\ 0\le s\le t$ we have:

$$\begin{aligned}&a)\quad E|u(t;0)|^2\le\tfrac{2M_0}{\beta}\\ &b)\quad E|u(t;0)-u(s,0)|^2\le M_0(t-s)\\ &c)\quad E|u(t;0)-u(s,0)|^2\le\tfrac{2M_0}{\beta}e^{-\delta_0 b_0 s}\left(e^{-\frac12\beta s}-e^{-\frac12\beta t}\right),\end{aligned}\tag{1.29}$$

where

$$M_0=(1+\frac{1}{\delta_0})(|F(0)|^2+|B(0)|_Q^2)$$

Proof. Since $\beta>0$ then $\lambda_0=\gamma_0L<2\alpha_0-b_1$. Let

$$\delta_0=\min\left\{\frac{2\alpha_0-b_1-\gamma_0L}{b_1+1},\frac{\beta}{2b_0}\right\} \tag{1.30}$$

We have $C(\lambda_0,\delta_0)=\beta-\delta_0b_0\ge\frac{\beta}{2}>0$ and obviously (1.23) yields (1.29) for $\lambda=\lambda_0$ and $\delta=\delta_0$.

2. Stability invariant measure. We shall consider $F:[0,\infty[\times V\longrightarrow H$ and $B:[0,\infty[\times V\longrightarrow\mathcal{L}(H_0,h)$ satisfying (0.5), (0.6), (0.7) on each interval $[0,T]$ with the constants L,b_1,L_1 independent on T.

If $u_{[0,T]}(t;u_0)$ is the solution of Eq.(0.1) on $[0,T]$, then by uniqueness

$$u_{[0,T_1]}(t;u_0)=u_{[0,T_2]}(t;u_0),\qquad\forall\, t\in[0,T_1\wedge T_2],\qquad P-a.s..$$

Thus we can define $u(\cdot;u_0):\Omega\times[0,\infty]\longrightarrow H$ by $u(t;u_0)=u_{[0,n]}(t;u_0)$ for $t\in[0,n]$, $n\in N^*$.

Proposition 2.1. *Let the assumptions (0.2)–(0.11) be satisfied and the initial data $u_0,\widetilde{u_0}\in\mathcal{D}_0$([1]). If u and $\tilde u$ are the corresponding (generalized) solutions and*

$$\beta=\frac{1}{\gamma_0^2}(2\alpha_0-b_1)-2\frac{L}{\gamma_0}-L_1>0$$

[1] $\mathcal{D}_0$ is defined in (1.8)

then for all $0 \le s_0 \le s \le t$ hold:

$$(2.1)\quad \begin{aligned} &a)\quad \boldsymbol{E}(|u(t) - \widetilde{u}(t)|^2 \,|\mathcal{F}_{s_0} \le e^{-\beta(t-s)} \boldsymbol{E}(|u(s) - \widetilde{u}(s)|^2 \,|\mathcal{F}_{s_0}), \quad a.s; \\ &b)\quad \boldsymbol{E}\int_s^t \|u(\tau) - \widetilde{u}(\tau)\|^2 d\tau \le \frac{1}{\beta\gamma_0^2} \boldsymbol{E}|u(s) - \widetilde{u}(s)|^2. \end{aligned}$$

Proof. It is sufficiently to prove (2.1) for $u_0, \widetilde{u}_0$ satisfying (0.14) since by (1.12) and (1.13) we can pass to limit and (2.1) is preserved for generalized solution.

Itô's formula for $|u_\varepsilon(t) - \widetilde{u}_\varepsilon(t)|^2$ gives:

$$|u_\varepsilon(t) - \widetilde{u}_\varepsilon(t)|^2 + \int_s^t G_\varepsilon(\tau, u_\varepsilon(\tau), \widetilde{u}(\tau))d\tau = |u_\varepsilon(s) - \widetilde{u}_\varepsilon(s)|^2 + M_\varepsilon(s,t),$$

where

$$M_\varepsilon(s,t) = 2\int_s^t (u_\varepsilon - \widetilde{u}_\varepsilon, (B(\tau, u_\varepsilon) - B(\tau, \widetilde{u}_\varepsilon))dW(\tau)$$

and

$$\begin{aligned} G_\varepsilon(\tau, u, v) = 2 < A_0(u - v), u - v > &+ \frac{2}{\varepsilon}(D\varphi_\varepsilon(u) - D\varphi_\varepsilon(v), u - v) - \\ &- 2(F(\tau, u) - F(\tau, v), u - v) - |B(\tau, u) - B(\tau, v)|^2 \ge \\ &\ge 2\alpha_0\|u - v\|^2 - 2L\|u - v\|\,|u - v| - b_1\|u - v\|^2 - L_1|u - v|^2 \ge \\ &\ge (2\alpha_0 - 2L\gamma_0 - b_1 - L_1\gamma_0^2)\|u - v\|^2 = \gamma_0^2\beta\|u - v\|^2. \end{aligned}$$

Hence

$$(2.2)\quad |u_\varepsilon(t) - \widetilde{u}_\varepsilon(t)|^2 + \gamma_0^2\beta\int_s^t \|u_\varepsilon(\tau) - \widetilde{u}_\varepsilon(\tau)\|^2 d\tau \le |u_\varepsilon(s) - \widetilde{u}_\varepsilon(s)|^2 + M_\varepsilon(s,t)$$

and more

$$(2.3)\quad |u_\varepsilon(t) - \widetilde{u}_\varepsilon(t)|^2 + \beta\int_s^t |u_\varepsilon(\tau) - \widetilde{u}_\varepsilon(\tau)|^2 d\tau \le |u_\varepsilon(s) - \widetilde{u}_\varepsilon(s)|^2 + M_\varepsilon(s,t)$$

From (2.2) we obtain:

$$\boldsymbol{E}\int_s^t \|u_\varepsilon(\tau) - \widetilde{u}_\varepsilon(\tau)\|^2 d\tau \le \frac{1}{\gamma_0^2\beta}\boldsymbol{E}|u_\varepsilon(s) - \widetilde{u}_\varepsilon(s)|^2$$

and for $\varepsilon \longrightarrow 0$ by (0.17-a) the inequality (2.1-b) is yielded.

Define

$$\psi_\varepsilon(t) = \boldsymbol{E}(|u_\varepsilon(t) - \widetilde{u}_\varepsilon(t)|^2|\mathcal{F}_{s_0})$$

Then from (2.3) it follows

$$\psi_\varepsilon(t) + \beta\int_s^t \psi_\varepsilon(\tau)d\tau \le \psi_\varepsilon(s)$$

for all $0 \le s \le t$, wich implies

$$\psi_\varepsilon(t) \le e^{-\beta(t-s)}\psi_\varepsilon(s)$$

Once again for $\varepsilon \longrightarrow 0$ by (0.17-a) the last inequality gives us (2.1-a).

Clearly fro Proposition 2.1 we have:

Corollary 2.2. *Assume that the conditions of Proposition 2.1 hold. Then:*

a) $M(t) = |u(t;u_0) - u(t;\widetilde{u}_0)|^2$ and $N(t) = e^{\beta t}|u(t;u_0) - u(t;\widetilde{u}_0)|^2$ are nonnegative supermartingales;

b) the solution u of Eq.(0.1) is exponentially 2–mean stable:

$$(2.4)\quad \boldsymbol{E}|u(t;u_0) - u(t;\widetilde{u}_0)|^2 \le e^{-\beta t}\boldsymbol{E}|u_0 - \widetilde{u}_0|^2;$$

c) $\forall\, a \in [0, \beta[$ one gets $\lim\limits_{t\to\infty} e^{at}|u(t;u_0) - u(t;\widetilde{u}_0)|^2 = 0$ in $L^1(\Omega, \mathcal{F}, P, \mathbb{R})$ and $P - a.s.$ $\omega \in \Omega$;

d) the solution u of Eq.(0.1) is 2-stable

$$(2.5)\quad \boldsymbol{E}\int_0^\infty \|u(t;u_0) - u(t;\widetilde{u}_0)\|^2 dt \le \frac{1}{\beta\gamma_0^2}\boldsymbol{E}|u_0 - \widetilde{u}_0|^2$$

e) the solution u of Eq.(0.1) is almost surely exponentially asymptotically stable: $\forall\, x, \widetilde{x} \in H \cap D_0$, $\forall\, a \in [0, \beta[$ there exists a random variable $\tau(\omega) = \tau(\omega, a, x - \widetilde{x}) < \infty$, a.s. such that

$$(2.6)\quad \begin{aligned} &|u(\omega,t;x) - u(\omega,t;\widetilde{x})|^2 \le e^{-at}|x - \widetilde{x}|^2, \\ &\qquad \forall\, t \ge \tau(\omega),\ P - a.s.\ \omega \in \Omega. \end{aligned}$$

The next result extend the results from (1.23-a) and (1.29-a) to the case when the initial datum is a random variable $u_0 \in \mathcal{D}$ and F and B can depend on time; But we do not know prove estimates of the form (1.23-b), (1.29-b,c) which give some supplimentary informations on the continuity of the mapping $t \longmapsto u(t;u_0), u_0 \in \mathcal{D}_0$.

Proposition 2.3. *Let the assumptions (0.2)-(0.11) be satisfied and $u_0 \in \mathcal{D}_0$. If $u(t) = u(t;u_0)$ is the corresponding (generalized) solution and*

$$\beta = \frac{1}{\gamma_0^2}(2\alpha_0 - b_1) - 2\frac{L}{\gamma_0} - L_1 > 0$$

then $\exists\, \delta_0 = \delta_0(\alpha_0, \gamma_0, L, L_1, b_1) > 0$ *s.t.* $\forall\, 0 \le s_0 < s < t$:

$$\text{(2.7)}\quad \begin{aligned} &a)\quad E(|u(t)|^2|\mathcal{F}_{s_0}) \le e^{-\frac{1}{2}\beta(t-s)}E(|u(s)|^2|\mathcal{F}_{s_0}) + \\ &\qquad +(1+\frac{1}{\delta_0})\int_s^t e^{-\frac{1}{2}\beta(t-\tau)}(|F(\tau,0)|^2 + |B(\tau,0)|_Q^2)d\tau, \\ &b)\quad E(\int_s^t \|u(\tau)\|^2 d\tau|\mathcal{F}_{s_0}) \le \frac{2}{\beta\gamma_0^2}\,[E(|u(s)|^2 \mid \mathcal{F}_{s_0}) + \\ &\qquad +(1+\frac{1}{\delta_0})\int_s^t (|F(\tau,0)|^2 + |B(\tau,0)|_Q^2)d\tau] \end{aligned}$$

Proof. To prove (2.7) is sufficiently to consider u_0 satisfying (0.14). We replay partly the proof from Proposition 1.7. By Energy Equality we have

$$\text{(2.8)}\qquad |u_\varepsilon(t)|^2 + \int_s^t L_\varepsilon(\tau, u_\varepsilon(\tau))d\tau = |u_\varepsilon(s)|^2 + M_\varepsilon(s,t)$$

for all $0 \le s \le t$, $P-$ a.s. , where

$$M_\varepsilon(s,t) = 2\int_s^t (u_\varepsilon(\tau), B(\tau, u_\varepsilon(\tau))dW(\tau))$$

and

$$L_\varepsilon(\tau, v) = 2 < A_0 v, v > + \frac{2}{\varepsilon}(D\varphi_\varepsilon(v), v) - 2(F(\tau,v),v) - |B(\tau,v)|_Q^2.$$

Let $0 \le s_0 \le s \le t$ and

$$\psi_\varepsilon(t) = E(|u_\varepsilon(t)|^2|\mathcal{F}_{s_0}).$$

Applying conditional expectation in (2.8) one follows:

$$\psi_\varepsilon(t) + \int_s^t E(L_\varepsilon(\tau, u_\varepsilon(\tau))|\mathcal{F}_{s_0}) = \psi_\varepsilon(s)$$

or more

$$\frac{d}{dt}\psi_\varepsilon(t) + E(L_\varepsilon(t, u_\varepsilon(t))|\mathcal{F}_{s_0}) = 0, \quad t \ge s.$$

As in the proof of Proposition 1.7 and Corollary 1.9 we find:

$$L_\varepsilon(t,v) \ge \gamma_0^2\, C(\lambda_0, \delta_0)\|v\|^2 - (1+\frac{1}{\delta_0})(|F(t,0)|^2 + |B(t,0)|_Q^2) \ge$$

$$\ge \frac{1}{2}\gamma_0^2\beta\|v\|^2 - g_0(t) \ge \frac{1}{2}\beta|v|^2 - g_0(t)$$

with

$$\text{(2.9)}\qquad \begin{aligned} &\lambda_0 = \gamma_0 L, \\ &\delta_0 = \min\left\{\tfrac{2\alpha_0 - b_1 - \gamma_0 L}{b_1 + 1}, \tfrac{\beta\gamma_0^2}{2(b_1 + \gamma_0^2 L_1 + \gamma_0^2)}\right\}, \\ &g_0(t) = (1+\tfrac{1}{\delta_0})(|F(t,0)|^2 + |B(t,0)|_Q^2). \end{aligned}$$

Hence

$$\text{(2.10)}\qquad \frac{d}{dt}\psi_\varepsilon(t) + \frac{1}{2}\beta\gamma_0^2 E(\|u_\varepsilon(t)\|^2|\mathcal{F}_{s_0}) \le g_0(t)$$

and

$$\frac{d}{dt}\psi_\varepsilon(t) + \frac{1}{2}\beta\psi_\varepsilon(t) \le g_0(t)$$

which gets

$$e^{\frac{1}{2}\beta t}\psi_\varepsilon(t) - e^{\frac{1}{2}\beta s}\psi_\varepsilon(s) \le \int_s^t e^{\frac{1}{2}\beta\tau}g_0(\tau)d\tau$$

that yields (2.7-a) for $\varepsilon \longrightarrow 0$.

Now from (2.10) we have

$$\int_s^t E(\|u_\varepsilon(t)\|^2|\mathcal{F}_{s_0})d\tau \le \frac{2}{\beta\gamma_0^2}\left[\psi_\varepsilon(s) + \int_s^t g_0(\tau)d\tau\right]$$

that yields (2.7-b) for $\varepsilon \longrightarrow 0$.

Corollary 2.4. *Assume that the conditions of Proposition 2.3 hold. Then:*

a) $M(t) = e^{\frac{1}{2}\beta t}|u(t)|^2 - (1+\frac{1}{\delta_0})\int_0^t e^{\frac{1}{2}\beta\tau}(|F(\tau,0)|^2 + |B(\tau,o)|_Q^2)d\tau$ *is a supermartingale.*

b) The solution $u(t;u_0)$ is exponentially ultimated bounded:

$$E|u(t;u_0)|^2 \le C + e^{-\frac{1}{2}\beta t}E|u_0|^2$$

with $C = (1+\frac{1}{\delta_0})\frac{2}{\beta}(K^2+K_1)$.

c)

$$\frac{1}{T}E\int_0^T \|u(t)\|^2 dt \le \frac{1}{\gamma_0^2}C_0 + \frac{2}{T\beta\gamma_0^2}E|u_0|^2. \tag{2.11}$$

In the sequel of this section we shall study the behaviour of laws $\mathcal{L}(u(t;u_0))$, $u_0 \in \mathcal{D}_0$, for $t \longrightarrow \infty$ under the assumptions:

$$\begin{array}{ll} a) & ((0.2)\text{-}(0.11) \text{ and } (1.10) \text{ are satisfied,} \\ b) & F \text{ and } B \text{ do not do not depend on } t. \end{array} \tag{2.12}$$

Let

$$P(x;t,\Gamma) = \mathcal{L}(u(t;x))(\Gamma) = P(\{\omega: u(\omega,t;0,x) \in \Gamma\}) = P(0,x;t,\Gamma),$$

$$(P_t g)(x) = Eg(u(t;x)) = P_{0t}g(x),$$

where $t \ge 0$, $g \in B_b(K)$, $x \in K = \overline{Dom\varphi}$, $\Gamma \in \mathcal{B}_K$. Denote also $\mathcal{M}(K)$ the space of bounded measure on $(K, \mathcal{B}_K)$ and $\mathcal{M}_1^+(K) \subset \mathcal{M}(K)$ the probability measures.

For all $\mu \in \mathcal{M}(K)$ and $g \in B_b(K)$ define

$$<\mu, g> = \int_K g(x)d\mu(x)$$

and the dual semigroup $P_t^*: \mathcal{M}(K) \longrightarrow \mathcal{M}(K)$ by

$$<P_t^*\mu, g> = <\mu, P_t g>$$

Definition 2.5. *A probability measure $\mu \in \mathcal{M}_1^+(K)$ is an invariant measure for the solutions of Eq.(0.1) if it is satisfied one of the following equivalent conditions:*

$$P_t^*\mu = \mu, \qquad \forall\, t > 0; \tag{2.13}$$

$$\int_K (P_t g)(x)d\mu(x) = \int_K g(x)d\mu(x), \quad \forall\, t>0, \quad \forall\, g \in B_b(K) \tag{2.14}$$

(*or for all* $g \in C_b(K)$);

$$\int_K Eg(u(t;x))d\mu(x) = \int_K g(x)d\mu(x), \quad \forall\, t>0, \quad \forall\, g \in B_b(K) \tag{2.15}$$

(*or* $\forall\, g \in C_b(K)$);

$$\int_K P(x;t,\Gamma)d\mu(x) = \mu(\Gamma), \quad \forall\, t>0, \quad \forall\, \Gamma \in \mathcal{B}_K. \tag{2.16}$$

From [5]-Ch.11, we collect the following nice properties based on Propositions 1.5. & 1.6.

Proposition 2.6.
A. $\mathcal{L}(u_0) = \nu \Rightarrow P_t^*\nu = \mathcal{L}(u(t;u_0))$, *in particular* $P_t^*\delta_x = \mathcal{L}(u(t;x))$;
B. If $\exists\, u_0 \in \mathcal{D}_0$ *such that*

$$\lim_{t\to\infty} Eg(u(t;u_0)) = \int g(x)d\mu(x)$$

for all $g \in C_b(K)$, *then* μ *is an invariant measure for Eq (0.1);*
C. If $\exists\, u_0 \in \mathcal{D}_0$, $\exists\, t_n \nearrow \infty$ *such that*

$$\lim_{n\to\infty} \frac{1}{t_n}E\int_0^{t_n} g(u(s;u_0))ds = \int_K g(x)d\mu(x)$$

for all $g \in C_b(K)$, *then* μ *is an invariant measure for Eq. (0.1);*
D. If $\exists\, \mu \in \mathcal{M}_1(K)$ *such that*

$$\lim_{t\to\infty} <P_t^*\delta_x, g> = \lim_{t\to\infty} Eg(u(t;x)) = \int_K g(y)d\mu(y)$$

for all $x \in \boldsymbol{K}$, $\forall\, g \in C_b(\boldsymbol{K})$ *then*

$$\lim_{t\to\infty} < P_t^* \nu, g > = < \mu, g >, \ \forall\, g \in C_b(\boldsymbol{K}),\ \forall\, \nu \in \mathcal{M}_1^+(\boldsymbol{K})$$

E. If $\mathcal{L}(u_0) = \mu$ *and* μ *is an invariant measure then the stochastic process* $u(t) = u(t; u_0)$ *is stationary i.e.*

$$\boldsymbol{E}[g_1(u(t_1+\delta))\dots g_n(u(t_n+\delta))] = \boldsymbol{E}[g_1(u(t_1))\dots g_n(u(t_n))]$$

for all $t_1, t_2, \cdots, t_n \geq 0$, $\delta > 0$ *and* $g_1, \dots, g_n \in \mathcal{B}_b(\boldsymbol{K})$.

Proposition 2.7. *If the assumptions (2.12) are satisfied, then there exist* $\eta \in L^2(\Omega, \mathcal{F}, P; H)$ *and* $C_0 = C_0(\gamma_0, \alpha_0, L, b_1, L_1) > 0$ *s.t.:*

$$\text{(2.17)} \qquad \begin{aligned} &a) \quad \eta \in \boldsymbol{K} \text{ a.s.,}\\ &b) \quad \boldsymbol{E}|u(t;u_0) - \eta|^2 \leq C_0\left(1 + \boldsymbol{E}|u_0|^2\right) e^{-C_0 t}, \end{aligned}$$

for all $u_0 \in \mathcal{D}_0$ *and* $t \in [0, \infty[$.

Proof. From Corollary 1.9 one follows that $\exists\, \eta \in L^2(\Omega, \mathcal{F}, P; H)$ such that

$$\lim_{t\to\infty} u(t;0) = \eta \quad \text{in } L^2(\Omega, \mathcal{F}, P; H)$$

and

$$\boldsymbol{E}|\eta - u(s,0)|^2 \leq \frac{2M_0}{\beta} e^{-(\delta_0 b_0 + \frac{1}{2}\beta)s}$$

for all $s \geq 0$.

From Corollary 2.2, the inequality (2.4), we have

$$\boldsymbol{E}|u(t,u_0) - u(t,0)|^2 \leq e^{-\beta t}\boldsymbol{E}|u_0|^2.$$

Hence

$$\boldsymbol{E}|u(t;u_0) - \eta|^2 \leq 2\boldsymbol{E}|u(t;u_0) - u(t;0)|^2 + 2\boldsymbol{E}|u(t;0) - \eta|^2 \leq$$

$$\leq 2e^{-\beta t}\boldsymbol{E}|u_0|^2 + 2\frac{M_0}{\beta} e^{-(\delta_0 M_0 + \frac{1}{2}\beta)t}$$

that is (2.17) since $u(t;u_0) \in \boldsymbol{K}$ a.s.. ∎

Corollary 2.8. *Under the assumptions (2.12), the Equation (0.1) has a unique invariant measure. This measure is the law of* η.

Proof. Let $\mu = \mathcal{L}(\eta)$. Since the convergence in L^2 implies the convergence in law we have:

$$\lim_{t\to\infty} \boldsymbol{E}g(u(t;u_0)) = \boldsymbol{E}g(\eta) = \int_{\boldsymbol{K}} g(x)d\mu(x)$$

for all $g \in C_b(\boldsymbol{K})$, $u_0 \in \mathcal{D}_0$, and by Proposition 2.6-B one follwos that μ is an invariant measure. The uniqueness is easily obtained as follows: Let $g \in C_b(\boldsymbol{K})$ and μ_1, μ_2 two invariant measures. From (2.15) for $\mu = \mu_i$, $i = 1, 2$ and $t \to \infty$ we have $\boldsymbol{E}g(\eta) = < \mu_i, g >$, for $i = 1, 2$. Hence $\mu_1 = \mu_2$. ∎

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