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CANONICAL AND JOIN HYPERGROUPS

BY

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1. Introduction. The canonical hypergroups are a special type of hypergroups (completely regular, according to M a r t y' s definition [15]). Initially they derived from the additive part of the hyperfield and the hyperring [12], [13] and later on from other different ways, as well (e.g. [36], [38]). The name canonical has been given to these hypergroups by J. M i t t a s, the first one who has studied them [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]. A **canonical hypergroup** is a hypergroupoid $(H,+)$, that for every $x, y, z \in H$ satisfies the axioms:

CH_1. $x+y=y+x$ (*commutativity*)

CH_2. $x+(y+z)=(x+y)+z$ (*associativity*)

CH_3. there exists an element $0 \in H$ for which $0+x=x$

CH_4. for every $x \in H$ there exists one and only one element $x' \in H$, denoted by $-x$, such that $0 \in x+(-x)$.

CH_5. $z \in x+y \Longrightarrow x \in z+(-y)$ (*reversibility*)

It has been proved that axiom CH_5 is equivalent to:

CH_5'. $(\forall(x,y) \in H^2)[-(x+y)=-x-y]$ as well as to:

CH_5''. $(\forall(x,y,z,w) \in H^4)[(x+y)\cap(z+w) \neq \emptyset \Longrightarrow (z-x)\cap(y-w) \neq \emptyset]$

If from the axioms of the canonical hypergroups, the commutativity is left out, there derives another class of very interesting hypergroups, the **quasicanonical hypergroups** (by Bonansinga and Corsini) or **polygroups** (by Comer). Examples of such hypergroups are:

i. The collection G/H of all double cosets of a subgroup H of a group G [10]

ii. The set of all the conjugacy classes of a group G as it is defined in [15], [3], [9].

Bonansinga and Corsini ([1], [2], [8]) studied the structure of these hypergroups and their subhypergroups, while Massouros ([17]) studied their congruence relations and their homomorphisms. A dual equivalence between the category of polygroups and the category of complete atomic integral relation algebras has been established by Comer in [4], where other examples of polygroups are given. Also in [6] the notion of conjugacy of polygroups is given and it is being proved that the conjugacy relations of a polygroup form a complete lattice with a very rich structure. The lattices are related to some very difficult problems involving groups. Moreover in [5] the notion of a partial multy-valued loop is given. This notion is like a "hypercategory" and it is being proved that a polygroup can be obtained as a special case. Furthermore Corsini and De Salvo have presented studies on the **feebly quasicanonical** hypergroups, which are a generalisation of quasicanonical ones [8].

Closely related to the canonical hypergroups are the join hypergroups. A *join hypergroup* is a commutative hypergroup $(H,+)$, which for every $x,y,z,w \in H$ satisfies the axiom:

$J.\quad a \div b \cap c \div d \neq \emptyset \Longrightarrow (a+d)\cap(b+c) \neq \emptyset$

where $a \div b = \{x | a \in x + b\}$ (induced *hypercomposition*).

It can be proved that every canonical hypergroup is also join and moreover that if a join hypergroup has a scalar neutral element, then it is a canonical one [7], [16].

The above axiom (J) has been introduced by W. Prenowitz, who has used a special type of join hypergroup, that he named join space, for a very important study of several types of geometries with the use of methods and techniques from the theory of the hypergroups [33], [34], [35], having produced many essential and interesting results [36], [37].

2. Canonical hypergroups. In this paragraph there appear certain fundamental properties of the canonical hypergroups deriving from the study of Mittas on this subject. We use the symbol "+" for the hypercomposition, as it appears in his study, due to the association of the canonical hypergroup with the hyperfield and with its valuation.

Example 2.1. A canonical hypergroup can derive from every set H, which is totally ordered and symmetrical around a center (denoted by $0 \in H$) in which the abvious symmetrical partition is considered:

$$H = H_1 \cup \{0\} \cup H_2 = H^- \cup \{0\} \cup H^+ \qquad (H_1 = H^-, H_2 = H^+)$$

such that: $x < 0 < y$, for every $x \in H^-$, $y \in H^+$ and $x \leq y \Longrightarrow -y \leq -x$ for every $x, y \in H$, where $-x$ is the symmetrical of $x \in H$ with regard to 0.

Such a set becomes a canonical hypergroup if the following, commutative by definition, hypercomposition is introduced:

$$y + x = x + y = y, \text{ if } |x| < |y|, \quad x + x = x, \quad x - x = [-|x|, |x|]$$

(where obviously $|x| = x$, if $x \in H^+$, $-x$ if $x \in H^-$ and 0, if $x = 0$).

A canonical hypergroup can possibly have more scalar elements, (i.e. elements for which the result of the hypercomposition with any other element of the hypergroup is a singleton) other than zero. Relatively we have:

Proposition 2.1. *$x \in H$ is a scalar element if and only if $x - x = 0$.*

Remark 2.1. The cancellation law is not generally valid in a canonical hypergroup. From axiom C_5'' it derives only that $x + y = x + z \Longrightarrow (x - -x) \cap (y - z) \neq \emptyset$. On the contrary, the cancellation law holds if x is a scalar element.

Proposition 2.2. *The set S of the scalar elements of H is an abelian group.*

The heart of a semiregular hypergroup H is the union of all the products of finite number of factors, each containing at least one left unit of H and at least one of its right units. The heart Ω of a canonical hypergroup $(H,+)$ is the union of all the sums $(x_1 - x_1) + \cdots + (x_i - x_i)$, where i is an arbitrary positive integer and the elements $x_1, \ldots, x_i$ are independently chosen from H. Ω is the least in the sense of inclusion subhypergroup, such that the quotient H/Ω is a group. If instead of H we consider an arbitrary subset $X \subseteq H$, we denote these unions by $\Omega(X)$. Also $\Omega(H) = \Omega$. Relatively we have:

Proposition 2.3. *For every non void subset X of a canonical hypergroup H, $\Omega(X)$ is a subhypergroup of H and $0 \in \Omega(X)$.*

With regard to the subhypergroups of H we have:

Definition 2.3. *A subhypergroup h of a canonical hypergroup H is called a* **canonical subhypergroup** *of h, if it is a canonical hypergroup itself and with the same zero (thus $0 \in h$).*

It has been proved that every subhypergroup of H containing the zero element is a canonical one. Therefore the above subhypergroups $\Omega(X)$ are canonical. Furthermore, greatly important in the study of the subhypergroups of a hypergroup are the closed and the invertible subhypergroups. Let us remind here that right (resp. left) **closed subhypergroup** of a hypergroup H is a subhypergroup h for which $ah \cap h = \emptyset$ (resp. $ha \cap h = \emptyset$), for every $a \in H \backslash h$. Moreover h is called **right invertible** if $ah \cap a'h = \emptyset$, for every $a, a' \in H$ with $ah \neq a'h$. (resp. is the definition of the left invertible). Also, since a closed (and much more an invertible) subhypergroup of a hypergroup contains all its units, in the canonical hypergroups, their unique unit (the zero element) belongs to such subhypergroups. Thus every closed subhypergroup of a canonical hypergroup is a canonical one. The converse is valid as well: Every canonical subhypergroup is invertible (and consequently closed).

The invertible subhypergroups now, define a partition in the hypergroup. So for the canonical hypergroups we have:

Proposition 2.4. *For every canonical subhypergroup h of H the set of the classes H/h is a canonical hypergroup with hypercomposition:*

$$(x+h)\oplus(y+h)=\bigcup_{z\in x+y}(z+h).$$

H/h is an abelian group if and only if $h\subseteq\Omega$.

In the theory of hypergroups, (contrarly to the theory of groups), the intersection of two subhypergroups, if not void, is generally a semi-subhypergroup. But the intersection of two closed subhypergroups, if again it is not void, is always a closed subhypergroup. Thus the set of the canonical subhypergroups of a canonical hypergroup H is a complete lattice, since the canonical subhypergroups are closed and their intersection always contains the zero element. From this derives that for a given subset X of H there always exists the least (in the sense of inclusion) canonical subhypergroup $\overline{X}$ of H which contains X. This subhypergroup $\overline{X}$, is the subhypergroup generated by X. If $X=\emptyset$, then, obviously, $\overline{X}=\{0\}$. For $X\neq\emptyset$ we have:

Proposition 2.5. *The canonical subhypergroup $\overline{X}$ of H, which is generated by a non empty subset X of H, is the union of the sums of finite number of elements of the union $-X\cup X$.*

Now, if X is a singleton, then the canonical hypergroup generated from it, is called **monogene**. So a canonical subhypergroup h of H is monogene if there exists $x\in H$ such that $h=\overline{\{x\}}$. If $H=\overline{\{x\}}$, then H itself is called monogene. Observing now that:

$$mx+nx=\begin{cases}(m+n)x & \text{if } mn>0\\ (m+n)x+\min\{|m|,|n|\}(x-x) & \text{if } mn<0\end{cases}$$

we have:

Proposition 2.6. $\overline{\{x\}}=mx+n(x-x)$, *for every $x\in H$ and $m,n\in Z$.*

Since $-(x-x)=x-x$, we can assume that $(m-n)\in Z\times N$ instead of $Z\times Z$. With the use of this Proposition can be derived the definition of the order of an element x. Two cases can appear, from which the one excludes the other:

I) For every $(m,n)\in Z\times N$ with $m\neq 0$, $0\notin mx+n(x-x)$, in which case x, as well as $\overline{\{x\}}$ is said to be of **infinite order** denoted by $\omega(x)=+\infty$.

Proposition 2.7. $\omega(x)=+\infty$ *if and only if $m'x\cap m''x=\emptyset$, for every $m'm''\in Z, m'\neq m''$.*

II) There exixts $(m,n)\in Z\times N$, with $m\neq 0$, such that $0\in mx+n(x-x)$. Let p be the minimum positive integer for which there exists $n\in N$, such that $0\in px+n(x-x)$.

Proposition 2.8. *For a given $m\in Z$ there exists $n\in N$ such that $0\in mx+n(x-x)$, if and only if m is divided by p.*

If $m=kp(k\in Z)$, let $q(k)$ be the minimum non negative integer such that $0\in kpx+q(k)(x-x)$. The q is a function from Z to N. Then the pair $\omega(x)=(p,q)$ has been called **order** of x and $\overline{\{x\}}$ and more precisely, p is the **principal** order of x and q is the **associative** order of x.

Moreover J. Mittas has studied the **valuation** of the canonical hypergroups, having developed a theory analogous to the one of the valuated groups, which is much more complicated though, due to the special character of the hypercomposition, which consists the basis for the study of the valuation. Later on J. Mittas has generalized his own theory with the introduction of the (**simply** and **strictly**) **hypervaluated** canonical hypergroups, which he has also studied. In the following we will see a few introductory elements of this theory, starting with the definition of the ultrametric space [11].

Definition 2.4. **Ultrametric distance** *over a set E, is a mapping $d\colon E\times E\longrightarrow R_+$ that satisfies the conditions:*

i. $d(x,y)=0\Longleftrightarrow x=y\ \ \forall x,y\in E$
ii. $d(x,y)=d(y,x)\ \ \forall x,y\in E$
iii. $d(x,y)\leq\max\{d(x,z),d(z,y)\}\ \ \forall x,y,z\in E$

The pair (E,d) is called **ultrametric space.**

Definiton 2.5. *A canonical* **hypergroup** *H is called* **ultrametric,** *if an ultrametric distance has been defined on H, satisfying the conditions:*

h_1. For every $x,y\in H$ the sum $x+y$ is a circle of the ultrametric space (H,d) with radius analogous to the $\max\{d(0,x),d(0,y)\}$, where 0 is the zero element of H. That is there exists a semi-real number [11] $0\leq p<1$ of type 0 or $-$, such that $x+y=C(z,p\max\{|x|,|y|\})$. z is an arbitrary element of $x+y$ and the valuation of the element $x\in H$, denoted by $|x|$, is the distance

$d(0, x)$. *The function* $|\cdot|: H \longrightarrow R_+$ *is called valuation of* H, *correlated to the ultrametric distance* d.

h_2. *For every* $x, y, a \in H$ *such that* $(x+a) \cap (y+a) = \emptyset$ *holds:*

$d(x, y) = d(x+a, y+a)$

Every such ultrametric distance on H is called *compatible* to the structure of the canonical hypergroup H.

Let $(H, +, d)$ be a canonical ultrametric hypergroup. Then the valuation $|\cdot|$, which is correlated to the ultrametric distance d, satisfies the properties:

i. $|x| = 0 \Longleftrightarrow x = 0$

ii. $|x| = |-x|$, for every $x \in H$

iii. For every $x, y \in H$, with $x \neq y$ the set $|x - y|$ is a singleton

iv. For every $x, y \in H$ holds: $|x + y| \leq \max\{|x|, |y|\}$

v. There exists a semi-real number $r \geq 0$ of type 0 or $-$ such that for every $x, y, z \in H$, with $z \in x + y$ and $|w - z| \leq p \max\{|x|, |y|\}$, w belongs to $x + y$.

The converse is also valid, that is every canonical hypergroup $(H, +)$ in which a function $|\cdot|: H \longrightarrow R_+$ has been defined and which satisfies the above properties (i) – (v) is ultrametric with ultrametric distance:

$$d(x, x) = 0 \text{ and } d(x, y) = |x - y|, \text{ for } x \neq y$$

This ultrametric function $|\cdot|$, is called **valuation** of H and d is called **ultrametric hyperdistance** correlated to the valuation $|\cdot|$. The canonical hypergroup is then called **valuated**. So the notions of the valuated and the ultrametric canonical hypergroup are the same.

The generalization of the notion of the valuation of the canonical hypergroups to the notion of the **hypervaluation** has been done in the same way, but with the use of a totally ordered set Ω with minimal element, denoted by 0, instead of the set R_+. This hypervaluation is called **simple** if for every $x, y \in H$ the $x + y$ is a circle of the ultrametric space (H, d), and **strict** if it satisfies the axiom (h_1), but with coefficient a semi-real number p of type 0 or $-$ of the K u r e p a's complement $\hat{\Omega}$ of Ω [14].

Proposition 2.9. *Every valuated (resp. strictly hypervaluated) canonical hypergroup* $(H, +)$ *satisfies the following purely algebraic properties:*

S_1. *if for* $x, y \in H$ *holds* $x \in x + y$, *then* $x + y = x$

S_2. *for every* $x, y, z, w \in H$, *such that* $(x + y) \cap (z + w) \neq \emptyset$ *either* $x + y \subseteq z + w$, *or* $z + w \subseteq x + y$ *is valid*

S_3. *for every* $x, y, z, w \in H$ *such that* $0 \notin x + y$ *and* $z, w \in x + y$ *holds* $z - z = w - w$

S_4. *if* $x \in z - z$ *and* $y \notin z - z$, *then* $x - x \subseteq y - y$

A canonical hypergroup that also satisfies the conditions (S_1) and (S_2) is called **strongly** canonical, while if it satisfies (S_1) - (S_4) is called **superiorly** canonical.

Theorem 2.1. *A necessary and sufficient condition for a canonical hypergroup to be strictly hypervaluatable (or valuatable) is to be superiorly canonical.*

3. Join hypergroups. Let us also start this part with two examples:

Example 3.1. Let $(V, +)$ be a vector space over a valuated field $(F, +, \cdot)$. If the hypercomposition:

$xy = \{kx + \lambda y | k, \lambda \in F, k, \lambda > 0, k + \lambda = 1\}$

is introduced in V, then V becomes a join hypergroup. This example gives a way similar to the one that W. P r e n o w i t z has used in order to introduce the hypercomposition into geometry.

Example 3.2. Let (S, A, s_0, F, δ) be an automaton. The language L that the automaton accepts is a special subset of the set of the words A^*, over the alphabet A, which is described by a regular expression. The definition of the regular expressions requires the use of the bisets $\{a, b\}$, where $a, b \in A$, introducing thus in A^* the hypercomposition $x + y = \{x, y\}$ with $x, y \in A^*$. With this hyprecoposition A^* becomes a join hypergroup.

The theory of the hypergroups has been introduced into the theory of laguages and automata, by G.G. M a s s o u r o s and J. M i t t a s [19]. The hypergroups that mainly appear in the theory of languages are the join ones, while in the theory of automata usually emerge canonical ones. We must mention though, that other hypercompositional structures, than the hypergroups, have also been introduced in languages and automata (e.g. see [20], [21]).

The properties of the join spaces and, next, of the join hypergroups have been studied from many researchers. So, except W. P r e n o w i t z's work, papers on this area have been written by J a n t o s c i a k, P. C o r s i n i, Ch. M a s s o u r o s, G. M a s s o u r o s and J. M i t t a s. Many and interesting results have derived, regarding both, its algebraic structure and its applications on geometry, as well as on the theory of languages and automata.

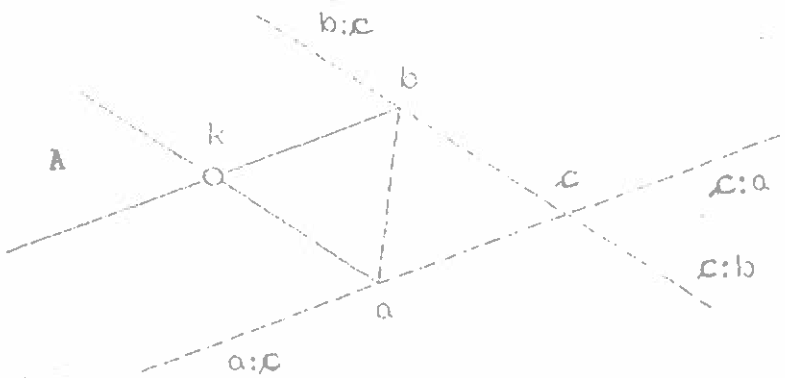
Let us start with a property regarding the relation between the two hypercompositions. As it is known [16], in every hypergroup we have: $(a:b):c = a:(c \cdot b)$. In the join hypergroups though, the following properties for the sets $(a \cdot b):c$ and $(a \cdot d):(b \cdot c)$ have also been proved [16]:

Proposition 3.1. *In a join hypergroup the followings statements are valid:*

i) $(a\colon c)\cdot b \cup (b\colon c)\cdot a \cup a\colon (c\colon b)\cup b\colon (c\colon a) \subseteq (a\cdot b)\colon c$

ii) $(a\colon b)\cdot(d\colon c)\cup(a\colon c)\cdot(d\colon b)\cup(a\colon b)\colon(c\colon d)\cup(a\colon c)\colon(b\colon d)\cup(d\colon c).(b\colon a)\cup$
$\cup(d\colon b)\colon(c\colon a)\subseteq(a\cdot d)\colon(b\cdot c)$

It is interesting to remark that in terms of the high school algebra, the first part of these relations describes all the possible ways that the second part can be written. Also seen on the plane and in simple words, (i) of this proposition implies that the second term of the relation gives all the points into the area A (defined by the half lines $a\colon c$, $b\colon c$ and the segment ab) of the plane, while the first term just leaves out point k of this area. Remember that [Example 3.1] ab is the open segment through a and b and $a\colon c$ is the open half line from a, as in the following figure:



Regarding the subsets of a join hypergroup, the ones with greatest interest are the semi–subhypergroups and the closed subhypergroups.

A **semi–subhypergroup** of a hypergroup H is a non empty subset h of H for which $xy \subseteq h$, for every $x, y \in h$ (while in a subhypergroup: $xh = hx = h$, for every $x \in h$). If E is a subset of H, $[E]$ denotes the least semi–subhypergroup which contains E. In the case of the first example the semi–subhypergroups correspond to the convex subsets of V. Thus from the study of the semi– subhypergroups of the join hypergroup there results for the convex sets of V. For instance, in the join hypergroups we have:

Proposition 3.2. *([16]) Let h_1, h_2 be two disjoint semi– subhypergroups of a join hypergroup $(H,\cdot)$ and let x be an idempotent element of H, which does not belong to the union $h_1 \cup h_2$. Then either $[h_1 \cup \{x\}] \cap h_2 = \emptyset$ or $[h_2 \cup \{x\}] \cap h_1 = \emptyset$.*

Moreover, since V can be endowed with the join hypergroup structure, as in Example 1, we can use the above Proposition, expressed in the terminology of the vector spaces, and get:

Corollary 3.1. *([16]) If A, B are disjoint convex sets in a linear space V, and x is a point not in their union, then either the convex envelope of $A \cup \{x\}$ and B are disjoint, or the convex envelope of $B \cup \{x\}$ and A are disjoint.*

This is nothing else but **K a k u t a n i's Lemma**. Analogously **Theorem of S t o n e** can be derived from the following Proposition ([16]):

Proposition 3.3. *Let H be a join hypergroup, every element of which is idempotent. Also let h_1, h_2 be two disjoint semi–subhypergroups. Then there exist disjoint semi–subhypergroups H_1, H_2, such that $h_1 \subseteq H_1$, $h_2 \subseteq H_2$ and $H = H_1 \cup H_2$.*

If now h_1 and h_2 are two semi–subhypergroups of a join hypergroup, then $h = h_1\colon h_2$ is also a semi–subhypergroup, while it is a closed subhypergroup if $h_1 = h_2$, which contains h_1 as well. As it is known [18] the closed subhypergroups of a hypergroup always contain the result of the induced hypercomposition. Thus in the case of the join hypergroups the join axiom is always valid inside the closed subhypergroups, and from this point of view the closed subhypergroups are join hypergroups themselves.

Proposition 3.4. *A subset h of a join hypergroup H is a closed subhypergroup if:*

i. $x\colon y \subseteq h$ *for every* $x, y \in h$ *ii.* $xx \subseteq h$ *for every* $x \in h$

Proposition 3.5. *Let h_1 and h_2 be two closed subhypergroups of a join hypergroup H such that $h_1 \cap h_2 \neq \emptyset$. Then:*

i. $h_1\colon h_2 = h_2\colon h_1$ *and* *ii.* $h_1\colon h_2$ *is a closed subhypergroup of* H.

Proposition 3.6. *If H is a join hypergroup and $A = \{a_1, \dots, a_n\}$ a subset of H, then the closed subhypergroup which is generated by A is:*

$$< a_1, \dots, a_n >= ([a_1]\dots[a_n])\colon([a_1]\dots[a_n])$$

where $[a_i]$, $i = 1, \dots, n$ is the semi–subhypergroup generated by a_i.

We remind that $[a] = a^1 \cup a^2 \cup \dots \cup a^k \cup \dots$, where $a^1 = \{a\}$, $a^2 = a\cdot a, \dots, a^s = a\cdot a^{s-1}$, for every integer $s \geq 2$.

The closed subhypergroups of a join hypergroup define a partition in the hypergroup. More precisely, if H i s a join hypergroup and h a closed subhypergroup of H, then $h\colon x \cap h\colon y \neq \emptyset \Longrightarrow h\colon x = h\colon y$. So we have:

Proposition 3.7. *Every closed subhypergroup h of a join hypergroup H defines an equivalence relation mod h in H (which is also normal) as follows: $x \equiv y(mod\ \ h) \Longleftrightarrow h{:}\, x = h{:}\, y$*

Proposition 3.8. *The set H/h equipped with the hypercomposition:*

$$h{:}\, x * h{:}\, y = \{h{:}\, w | \ \ w \in xy\}$$

is a canonical hypergroup, having h as a neutral element.

From the above Proposition becomes clear the close relation between the canonical and the join hypergroups. More precisely we have:

Proposition 3.9. *A join hypergroup is canonical if and only if it has a scalar neutral element.*

It is possible though for a join hypergroup to have a non scalar neutral element. Such hypergroups, which are very important for the study of the theory of the languages are the **fortified join hypergroups**, introduced by G. M a s s o u r o s and J. M i t t a s:

Definition 3.1. *A* **fortified join hypergroup** *is a join hypergroup $(H, +)$ with a unique element 0, called the zero element of H, such that $0 + 0 = 0, x \in x + 0$ for every $x \in H$ and for every $x \in H\backslash\{0\}$ there exists one and only one element $-x \in H\backslash\{0\}$, the opposite of x, such that: $0 \in$ $\in x + (-x)$.*

The canonical and the join hypergroups, as well as the area between them seem to be of great interest for both, pure algebraic study and developement of applications, that might, someday make this theory a necessary part of the technological research. In the meanwhile there remains a lot of work to be done.

REFERENCES

1. B o n a n s i n g a, P. – *Sugli ipergruppi quasicanonici*, Atti Soc, Pelor, Sc. Fis. Mat. Nat. Vol. 27,1981.

2. B o n a n s i n g a, P. – *Ipergruppi debolmente quasicanonici*, Atti Sem. Mat. Fis. Univ. Modena XXX pp. 286–298, 1981.

3. C a m p a i g n e, H. – *Partition hypergroups*, Amer. J. Math. 62, pp. 599–612, 1940.

4. C o m e r, S. – *Combinatorial aspects of relations*, Algebra Univ. 18, pp. 77–94, 1984.

5. C o m e r, S. – *The representation of 3–dimensional cylindric algebras*, Colloquia Math. Soc. Janos Bolyai, 54 Algebraic Logic, pp. 147–172, Budapest 1988.

6. C o m e r, S. – *Lattices of conjugacy relations*, Contemporary Mathematics, Vol. 131, Part 3, pp. 31–48, 1992.

7. C o r s i n i, P. – *Recenti risultate in teoria degli ipergruppi*, Bollettino U.M.I (6) 2–A, pp. 133-138, 1983.

8. C o r s i n í, P. – *Prolegomena of hypergroup theory*, Aviani Editori, 1993.

9. D i e t z m a n, A.P. – *On the multigroups of complete conjugate sets of elements of a group*, C. R. (Doklady) Acad. Sci. USSR (N. S.), 49, pp. 315–317, 1946.

10. D r e s h e r, M. – O r e, O. – *Theory of multigroups*, Amer. J. Math. 60, pp. 705–733, 1938.

11. K r a s n e r, M. – *Nombres semiréels et espaces ultramétriques*, C. R. Acad. Sci. Paris, Tome II, 219, pp. 433–437, 1944.

12. K r a s n e r, M. – *Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux de caractéristique 0*, Colloque d' Algebre Superieure (Bruxelles, Decembre 1956), CBRM, Bruxelles, 1957.

13. K r a s n e r, M. – *A class of hyperrings and hyperfields*, Internat. J. Math. and Math. Sci. 6:2, pp. 307–312, 1983.

14. K u r e p a, D. – *Ensembles ordonnes et ramifies*, These de doctorat, Paris 1935, et publ. Univ. Beograd 4, pp. 1–138, (1935).

15. M a r t y, F. – *Sur un généralisation de la notion de groupe*, Huitieme Congrès des mathématiciens Scad., pp. 45–49, Stockholm 1934.

16. M a s s o u r o s, C.G. – *Hypergroups and convexity*, Riv. di Mat. pura ed applicata 4, pp. 7–26, 1989.

17. M a s s o u r o s, C.G. – *Quasicanonical Hypergroups*, Proceedings of the 4th Internat. Cong. in Algebraic Hyperstructures and Applications. pp. 129–136, Xanthi 1990. World Scientific.

18. M a s s o u r o s, C.G. – *On the semi–subhypergroups of a hypergroup*, Internat. J.Math. & Math. Sci. Vol. 14, No 2, pp. 293–304, 1991.

19. M a s s o u r o s, G.G. and M i t t a s, J. – *Languages – Automata and hypercompositional structures*, Proceedings of the 4th Internat. Cong. in Algebraic Hyperstructures and Applications. pp. 137–147, Xanthi 1990. World Scientific.

20. M a s s o u r o s, G.G. – *Automata and Hypermoduloids*, Proceedings of the 5th Internat. Cong. in Algebraic Hyperstructures and Applications. pp. 251–266, Iaşi 1993. Hadronic Press 1994.

21. M a s s o u r o s, G.G. – *Hypercompositional Structures in the Theory of the Language and Automata*, An. Ştiinţifice Univ. Al.I.Cuza, Iaşi, Inform. t. iii, pp. 65–73, 1994.

22. M i t t a s, J. – *Sur une classe d' hypergroupes commutatifs*, C. R. Acad. Sci. (Paris) 269, Série A, pp. 485–488, 1969.

23. M i t t a s, J. – *Hypergroupes canoniques hypervalués*, C. R. Acad. Sci. (Paris) 271, Série A, pp. 4–7, 1970.

24. M i t t a s, J. – *Les hypervaluations strictes des hypergroupes canoniques*, C. R. Acad. Sci. (Paris) 271, Série A, pp. 69–72, 1970.

25. M i t t a s, J. – *Hypergroupes valués et hypergroupes fortement canoniques*, Πρακτικά της Ακαδημιας Αθηνών έτους, τομ. 44, σ. 304–312 Αθηνών, 1971.

26. M i t t a s, J. – *Hypergroupes canoniques valués et hypervalués*, Math. Balk, 1, pp. 181–185, 1971.

27. M i t t a s, J. – *Contributions a la théorie des hypergroupes, hyperanneaux et hypercorps hypervalués*, C. R. Acad. Sci. (Paris) 272, Serie A, pp. 3–4, 1971.

28. M i t t a s, J. – *Hypergroupes canoniques*, Mathematica Balkanica, 2, pp. 165–179, 1972.

29. M i t t a s, J. – *Hypergroupes fortement canoniques et supérieurement canoniques*, Proceedings II of the Inter. Symp. on applications of Math. in Syst. theory pp. 27–30, December 1978.

30. M i t t a s, J. – *Hypergroupes canoniques valués et hypervalués. Hypergroupes fortement et supérieurement canoniques*, Bull. of the Greek Math. Soc. 23, pp. 55–88, Athens, 1982.

31. M i t t a s, J. – *Hypergroupes polysymétriques canoniques*, Atti del convegno su ipergruppi, altre strutture multivoche e loro applicazioni, pp. 1–25, Udine 1985.

32. M i t t a s, J. – *Certaines remarques sur les hypergroupes canoniques hypervaluablés et fortement canoniques*, Riv. Mat. Pura e Appl. N. 9 pp. 61–67, 1991.

33. P r e n o w i t z, W. – *Projective Geometries as multigroups*, Amer. J. Math. 65, pp. 235–256, 1943.

34. P r e n o w i t z, W. – *Descriptive Geometries as multigroups*, Trans. Amer. Math. Soc. 59, pp. 333–380, 1946.

35. P r e n o w i t z, W. – *Spherical Geometries and multigroups*, Canad. J. Math. 2, pp. 100–119, 1950.

36. P r e n o w i t z, W. – *Contemporary Approach to Classical Geometry*, Amer. Math. 68, No 1, part II, pp. 1–67, 1961.

37. P r e n o w i t z, W. and J a n t o s c i a k, J. – *Geometries and Join Spaces*, Reine Angew. Math. 257, pp. 100–128, 1972.

38. R o t h, R.L. – *Character and conjugacy class hypergroups of a finite group*, Ann. Math. Pura Appl. 105, pp. 295–311, 1975.

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ON THE OPERATIONAL EXPONENTIAL FUNCTION

BY

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Abstract. This paper deals with the operational exponential function, which arise from solving a partial differential equation. The discussion of writing this exponential function in Bessel form and the difficulties arise are given.

1. Introduction. In the present paper M denotes the field of Mikusinski operators [1]. C is the ring of functions $f(t)$, which are defined and continuous in the interval $0 \leq t < \infty$. The function $f(t) \in C$ may have real or complex values. M contains ring C. The elements of M is isomorphic to $f(t) \in C$ is denoted by f or $\{f(t)\}$. $x(\lambda)$ stands for an operational function and $\{x(\lambda, t)\}$ for a parametric function ([1], page 179).

The operators of Mikusinski are very useful for solving the equations containing the operations of translation, integration ($l = \{1\}$), differentiation ($s = \frac{I}{l}$, I is identities operator), ... etc; since those belong to M. However, the theory of the operational differential equation has not yet been satisfactory developed.

J. M i k u s i n s k i [1] proved that it is impossible to give a general definition of the exponential function in the field M by means of power series, since the series of the function $exp(-\lambda s)$ with $\lambda > 0$, for instance is not convergent and not locally integrable function.

J.M i k u s i n s k i [1] analyzed the exponential function $exp(-\lambda\sqrt{s^2 + a^2})$, also he could find the analytic expression of $exp(-\lambda\sqrt{s})$ as follows:

$$exp(-\lambda\sqrt{s}) = \left\{ \frac{\lambda}{2\sqrt{\pi t^3}} exp\left(-\frac{\lambda^2}{4t}\right) \right\}, \quad (0 < \lambda < \infty).$$

D.N i k o l i c [2] analyzed the exponential function $exp\left(-\lambda\sqrt{s^m - \frac{\alpha}{s^n}}\right)$.