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ON THE OPERATIONAL EXPONENTIAL FUNCTION

BY

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Abstract. This paper deals with the operational exponential function, which arise from solving a partial differential equation. The discussion of writing this exponential function in Bessel form and the difficulties arise are given.

1. Introduction. In the present paper M denotes the field of Mikusinski operators [1]. C is the ring of functions $f(t)$, which are defined and continuous in the interval $0 \leq t < \infty$. The function $f(t) \in C$ may have real or complex values. M contains ring C. The elements of M is isomorphic to $f(t) \in C$ is denoted by f or $\{f(t)\}$. $x(\lambda)$ stands for an operational function and $\{x(\lambda, t)\}$ for a parametric function ([1], page 179).

The operators of Mikusinski are very useful for solving the equations containing the operations of translation, integration ($l = \{1\}$), differentiation ($s = \frac{I}{l}$, I is identities operator), ... etc; since those belong to M. However, the theory of the operational differential equation has not yet been satisfactory developed.

J. M i k u s i n s k i [1] proved that it is impossible to give a general definition of the exponential function in the field M by means of power series, since the series of the function $exp(-\lambda s)$ with $\lambda > 0$, for instance is not convergent and not locally integrable function.

J.M i k u s i n s k i [1] analyzed the exponential function $exp(-\lambda\sqrt{s^2 + a^2})$, also he could find the analytic expression of $exp(-\lambda\sqrt{s})$ as follows:

$$exp(-\lambda\sqrt{s}) = \left\{ \frac{\lambda}{2\sqrt{\pi t^3}} exp\left(-\frac{\lambda^2}{4t}\right) \right\}, \quad (0 < \lambda < \infty).$$

D.N i k o l i c [2] analyzed the exponential function $exp\left(-\lambda\sqrt{s^m - \frac{\alpha}{s^n}}\right)$.

We shall investigate the exponential function

$$exp\left[-\lambda(s^n + f(t))^{\frac{1}{m}}\right], \quad f(t) \in C$$

which aries from solving the partial differential equation of type

(1) $$x_{\lambda^m}(\lambda, t) - x_{t^n}(\lambda, t) - f(t)x(\lambda, t) = \phi(\lambda, t) : \ 0 < n < m$$

with the conditions,

(2) $$x(\lambda, 0) = g_0(\lambda), x_t(\lambda, 0) = g_1(\lambda), \ldots, x_{t^{n-1}}(\lambda, 0) = g_{n-1}(\lambda); \ (0 < \lambda < \infty)$$

(3) $$x(0, t) = h_0(t), x_\lambda(0, t) = h_1(t), \ldots, x_{\lambda^{m-1}}(0, t) = h_{m-1}(t), \ , (0 < t < \infty)$$

where $x(\lambda, t)$ is a function which maps an interval $[o, A]$ into the field of Mikusinski operators [1] and $\phi(\lambda, t)$ is a given operational function.

Notes. (1) If a function $x = \{x(t)\}$ has i-th derevative $x(i) = \{x(i)(t)\}$, continuous in the interval $0 \leq t < \infty$, then

(4) $$s^i x = x^{(i)} + x^{(i-1)}(0) + sx^{(i-2)}(0) + \cdots + s^{i-1}x(0)$$

see (M i k u s i n s k i [1], page 30).

(2) Somtimes we find that the solution of (1) satisfies the Cauchy conditions

(5) $$\sum_{\nu=0}^{n-1} x_{t^\nu}(\lambda, 0) = g_\nu(\lambda)$$

and does not satisfies conditions (2). For this contradiction vanishes, the conditions on the λ - axis must be given in the form (5).

Under those conditions and condition (3), we obtaine the unique actual solution of (1). Conditions (2) and (5) are not equivalent. (2) implies (5) but not the vice versa.

2. On the solvability of equation 1. Using the equation (4), the corresponding operational equation of (1) has the form

(6) $$x^{(m)}(\lambda) - (s^n + f(t))x(\lambda) = F(\lambda)$$

where, $F(\lambda) = \phi(\lambda, t) - \psi(s, \lambda)$ and

$$\psi(s, \lambda) = \sum_{\nu=0}^{n-1} s^{n-\nu-1} g_\nu(\lambda).$$

To construct the solution of equation (6), we deal with the homogeneous equation

(7) $$x^{(m)}(\lambda) - (s^n + f(t))x(\lambda) = 0$$

with the conditions (3) and (5). We first seek the solutions in the form of exponential function $x = e^{\lambda u}$. We thus obtain the characteristic equation in the form

(8) $$u^m - (s^n + f(t)) = 0$$

we assume that equation (8) is a reducible equation i.e. it can be decomposed into m factors.

Remark: Since the field M of Mikusinski's operators [1] is not algebraically closed, equation (8) is not always solvable (see [3]) i.e. the operator u satisfying that equation does not always exist. This can be illustred as follows: If $u^m = q\{g(t)\}$, where $q \in M$ and $\{g(t)\} = \{1 + t^{n+1}f(t)\}$, such that

$$T_\alpha\{g(t)\} = -a(\alpha)\{g(t)\}; \ \alpha > 0$$

where $a(\alpha)$ is a numerical function, then equation (8) is irreducible (see [4]).

3. On the representation of the exponential function $exp[-\lambda(s^n + f(t))^{\frac{1}{m}}]$. In this section we shall investigate the exponential function

(9) $$exp\left[-\lambda(s^n + f(t))^{\frac{1}{m}}\right]$$

which aries from solving equation (7) under the conditions (3) and (5).

We start from the exponential function

(10) $$exp\left[\lambda(s^{\frac{n}{m}} - (s^n + f(t))^{\frac{1}{m}}\right] \ (\lambda > 0).$$

Muitiplying expression (10) by the exponential function $exp\left(-\lambda s^{\frac{n}{m}}\right)$, when this function exists, we obtain the representation of function (9).

Writing the operator

$$(11)\qquad s^{\frac{n}{m}} - [s^n + f(t)]^{\frac{1}{m}} = s^{\frac{n}{m}} - s^{\frac{n}{m}} \sum_{k=0}^{\infty} \binom{\frac{1}{m}}{k} \left(s^{-n} f(t)\right)^k$$

$$= -s^{\frac{n}{m}} \sum_{k=0}^{\infty} \binom{\frac{1}{m}}{k+1} \left(s^{-n} f(t)\right)^{k+1}, \quad (0 < n < m),$$

$$= -s^{\frac{n}{m}} \sum_{k=0}^{\infty} (-1)^k \frac{(mk)!(s^{-n} f(t))^{k+1}}{m^{2k+1} k!(k+i)! \Pi_{\nu=1}^{k} \{\Pi_{j=2}^{m-1} (m\nu - j)\}}$$

We can see that the above series is an operationaly convergent, (see M i k u s i n s k i [1], page 178). We consider the simple case when $m = 2$, we thus have

$$(12)\qquad s^{\frac{n}{2}} - \sqrt{s^n + f(t)} = -s^{\frac{n}{2}} \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (s^{-n} f(t))^k$$

which is an element of the class C, $0 < n \leq 2$; therfore the operator (12) is logarithmic and the exponential function

$$exp\left[\lambda\left(s^{\frac{n}{2}} - \sqrt{s^n + f(t)}\right)\right] \quad (\lambda > 0)$$

can be expanded in a convergent power series

$$exp\left[\lambda\left(s^{\frac{n}{2}} - \sqrt{s^n + f(t)}\right)\right] = 1 + g(\lambda)$$

where

$$(13)\qquad g(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \lambda^{k+1} s^{\frac{n}{2}(k+1)} \left(\sqrt{1 + s^{-n} f(t)} - 1\right)^{k+1}}{(k+1)!}$$

is a parametric function (for a fixed λ, $g(\lambda)$ is a function of class C). Using the hypergeometric expansion

$$(\sqrt{1+z} - 1)^{k+1} = (k+1) \sum_{i=0}^{\infty} \frac{(-1)^i (k+2i)! z^{k+i+1}}{2^{k+2i+1} i!(k+i+1)!} \quad k = 0, 1, 2, \dots$$

we can easily find that

$$(\sqrt{1 + s^{-n} f(t)} - 1)^{k+1} = -(-1)^k (k+1) \sum_{i=k}^{\infty} \frac{(-1)^i (2i-k)! s^{-n(i+1)} \{f(t)\}^{i+1}}{2^{2i-k+1} (i-k)!(i+1)!}$$

$$k = 0, 1, 2 \dots$$

Substituting this expression in (13), we obtain

$$g(\lambda) = -\sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \sum_{i=k}^{\infty} \frac{(-1)^i (2i-k)! \{f(t)\}^{i+1} s^{\frac{n}{2}k - ni - \frac{n}{2}}}{2^{2i-k+1} (i-k)!(k)!}$$

$$= \lambda \sum_{i=0}^{\infty} \frac{(-1)^{i+1} \{f(t)\}^{i+1}}{(i+1)! 2^{2i+1}} \sum_{k=0}^{i} \frac{(2\lambda)^k (2i-k)! s^{\frac{n}{2}k - ni - \frac{n}{2}}}{(i-k)!(k)!}.$$

In the particular case for $n = 2$, we have

$$g(\lambda) = \lambda \sum_{i=0}^{\infty} \frac{(-1)^{i+1} \{f(t)\}^{i+1}}{(i+1)! 2^{2i+1}} \sum_{k=0}^{i} \frac{(2\lambda)^k (2i-k)! s^{-(2i-k+1)}}{(i-k)!(k)!}.$$

$$= \lambda \sum_{i=0}^{\infty} \frac{(-1)^{i+1} \{f(t)\}^{i+1}}{(i+1)! i! 2^{2i+1}} \star \left\{ \sum_{k=0}^{i} \binom{i}{k} (2\lambda)^k t^{2i-k} \right\}$$

$$(14)\qquad g(\lambda) = \lambda \sum_{i=0}^{\infty} \frac{(-1)^{i+1} \{f(t)\}^{i+1}}{(i+1)! i! 2^{2i+1}} \star \{t^i (t + 2\lambda)^i\}.$$

If the function $f(t)$ is replaced by a numerical value a, then we have the results given by D.N i k o l i c [2]. If there exists a function $\{g(t)\}$ such that

$$(15)\qquad \{f(t)\}^{i+1} \star \{(t^2 + 2\lambda t)^i\} = \{g^i(t)\},$$

then, the exponential function can be reduced in the Bessel function

$$J_i = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} t^{i+2\nu}}{2^{i+2\nu} \nu!(i+\nu)!} \quad (i = 0, 1, 2, \dots).$$

To find relation (15), this is outside the scope of the analysis here, and we hope to discuss it later.

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CHERN FORMS AND C-PROJECTIVE CURVATURE OF NORMAL ALMOST CONTACT MANIFOLDS

BY

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1. Introduction. As it is known, many authors have studied the characteristic classes of real or complex manifolds endowed with remarkable geometric properties, obtaining in this way a large class of global invariants. Among the interesting results in this direction, we recall the following theorems:

Theorem 1.1 [1]*The Pontrjagin classes of projectively flat manifolds are trivial.*

Theorem 1.2 [8]*The Chern classes of complex H-projectively flat manifolds are powers of the first class.*

We notice that, as examples of projectively and H-projectively flat manifolds respectively, these theorems include the Riemannian manifolds of constant sectional curvature (see also [3]) and the Kaehlerian manifolds of constant holomorphic sectional curvature.

As regards the almost contact manifolds, in [9] the author proved that the Chern classes of the horizontal vector bundle of Sasakian manifolds of constant φ-sectional curvature are trivial.

On the other hand, the C-projective curvature tensor on normal almost contact manifolds has been defined in [7]; in particular, all Sasakian manifolds of constant φ-sectional curvature are C-projectively flat.

Therefore, it was a natural question to extend the previous theorems to normal almost contact manifolds with vanishing C-projective curvature tensor field; it is interesting to remark as the study of the Chern algebra of this kind of manifolds gave a natural generalization of the cited results with the following

Theorem 1.3 *The Chern classes of a C-projectively flat manifold are trivial.*