REFERENCES

1. M i k u s i n s k i, J. – *Operational Calcculus*, Pergamon Press, 1959
2. N i k o l i c, D. and D e s p o t o v i c, D. – *The representation of an exponential function*, Publ. Inst. Math. Beograd, 1976, 434–441
3. S t a n k o v i c, B. – *On the solvability of algebraic equation in the field of Mikusinski's operators*, Bechnk Math.8(23), 1971, 391–394
4. S h a r k a w i, E.I., El-A f i f i, M.M. and A h m e d, A.R. – *On the solvability of a diferential equation*, under publication, 1994

Received: 3.II.1995

Departament of Mathematics
Faculty of Science
Al-Azhar University
Nasr-City
Departament of Mathematics
Faculty of Women
Ain Shams University
Cairo
EGYPT

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLII, s.I.a, Matematică, 1996, f1.

CHERN FORMS AND C-PROJECTIVE CURVATURE OF NORMAL ALMOST CONTACT MANIFOLDS

BY

P. MATZEU

1. Introduction. As it is known, many authors have studied the characteristic classes of real or complex manifolds endowed with remarkable geometric properties, obtaining in this way a large class of global invariants. Among the interesting results in this direction, we recall the following theorems:

Theorem 1.1 [1] *The Pontrjagin classes of projectively flat manifolds are trivial.*

Theorem 1.2 [8] *The Chern classes of complex H-projectively flat manifolds are powers of the first class.*

We notice that, as examples of projectively and H-projectively flat manifolds respectively, these theorems include the Riemannian manifolds of constant sectional curvature (see also [3]) and the Kaehlerian manifolds of constant holomorphic sectional curvature.

As regards the almost contact manifolds, in [9] the author proved that the Chern classes of the horizontal vector bundle of Sasakian manifolds of constant φ-sectional curvature are trivial.

On the other hand, the C-projective curvature tensor on normal almost contact manifolds has been defined in [7]; in particular, all Sasakian manifolds of constant φ-sectional curvature are C-projectively flat.

Therefore, it was a natural question to extend the previous theorems to normal almost contact manifolds with vanishing C-projective curvature tensor field; it is interesting to remark as the study of the Chern algebra of this kind of manifolds gave a natural generalization of the cited results with the following

Theorem 1.3 *The Chern classes of a C-projectively flat manifold are trivial.*

The manifolds and the tensor fields considered in this paper are C^∞. The usual notations from the differential geometry are used throughout this paper.

The author wishes to thank V.Oproiu for many useful discussions.

2. C-projective transformations of normal almost contact manifolds. Let M be a $(2n+1)$-dimensional almost contact manifold with field of endomorphisms φ, fundamental 1-form η and vector field ξ; then on M the equations

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \eta \circ \varphi = 0, \tag{2.1}$$

hold, where I is the identity transformation on TM [2].

The vector subbundle H of TM defined as $H = Ker\eta$ has the fibres of dimension $2n$, and the restriction of φ to H defines a complex structure on H; if V denotes the trivial subbundle of TM generated by ξ, we have $TM = H \oplus V$, and we shall denote by $h = -\varphi^2 = I - \eta\otimes\xi$, $v = \eta\otimes\xi$ the projection operators on H and V respectively.

According to [4], the normality of the structure given by the vanishing of the (2,2)-tensor [2]

$$N = N_\varphi + d\eta \otimes \xi, \tag{2.2}$$

where N_φ denotes the Nijenhuis tensor of φ, is equivalent to the existence of a special torsion-free connection ∇, said "adapted", which satisfies the system

$$\begin{cases} (\nabla_X\varphi)Y = \eta(Y)hX + \frac{1}{2}d\eta(\varphi X, Y)\xi, \\ (\nabla_X\eta)(Y) = \frac{1}{2}d\eta(X,Y) \\ \nabla_X\xi = \varphi X \qquad\qquad X, Y \in \mathcal{X}(M). \end{cases} \tag{2.3}$$

In [5], [6] several examples of adapted connections have been constructed; anyway, a clear example of adapted connection is given by the Levi-Civita connection of a Sasakian manifold.

Considering the curvature tensor field R of ∇

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X,Y,Z \in \mathcal{X}(M)$$

from (2.3), we obtain

$$hR(X,Y)\varphi Z - \varphi R(X,Y)Z = \frac{1}{2}d\eta(X,Z)hY - \frac{1}{2}d\eta(Y,Z)hX + \tag{2.4}$$

$$+\{\frac{1}{2}d\eta(X,\varphi Z) + \eta(X)\eta(Z)\}\varphi Y - \{\frac{1}{2}d\eta(Y,\varphi Z) + \eta(Y)\eta(Z)\}\varphi X.$$

Now, if we denote by D the linear connection induced by the adapted connection ∇ on H:

$$D_X Z = h\nabla_X Z \quad Z \in H, \qquad X \in \mathcal{X}(M), \tag{2.5}$$

it easily follows that D is an almost complex connection on H; in other words, if φ still denotes the restriction of φ to H, we have $D_X\varphi = 0$.

Then, taking into account (2.3), the relation between the curvature K of D

$$K(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \qquad Z \in H, \quad X, Y \in \mathcal{X}(M),$$

and R is given by:

$$hR(X,Y)Z = K(X,Y)hZ + \{-\eta(X)hY + \eta(Y)hX\}\eta(Z) + \tag{2.6}$$

$$+\frac{1}{2}d\eta(Y,Z)\varphi X + \frac{1}{2}d\eta(X,Z)\varphi Y.$$

Next for the Ricci (0,2)-tensors of R, $\rho(R) : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$, and K, $\rho(K) : \mathcal{X}(M) \times \Gamma(H) \to C^\infty(M)$, defined respectively by

$$\rho(R)(Y,Z) = \mathrm{tr}\,(X \to R(X,Y)Z),$$

$$\rho(K)(Y,hZ) = \mathrm{tr}\,(X \to K(X,Y)hZ)$$

we have

$$\rho(R)(Y,Z) = \rho(K)(Y,hZ) + 2n\eta(Y)\eta(Z). \tag{2.7}$$

Notice that $K(X,Y)$ can be considered as the restriction to H of the operator $K(X,Y)h$ which defines a tensor field on M.

In [7] the authors introduced the definition of C-flat curve; a smooth path γ on M, $\gamma : J \to M$, with $J \subseteq \mathbf{R}$, is C-flat with respect to an adapted symmetric connection ∇, if its tangent vector field T satisfies the following equation

$$\nabla_T T = \lambda(t)T + \mu(t)\varphi T \tag{2.8}$$

where $\lambda, \mu \in C^\infty(M)$.

Then two symmetric adapted connections $\nabla, \tilde{\nabla}$ are said to be C-projectively equivalent iff they have the same C-flat paths or, equivalently, iff they are related by a C-projective transformation given by [7]

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X)hY + \alpha(Y)hX - \beta(X)\varphi Y - \beta(X)\varphi X, \tag{2.9}$$

where α is a 1-form such that $\alpha(\xi) = 0$ and $\beta = \alpha \circ \varphi$.

Moreover, in [7] the authors constructed a good Weyl type C-projective curvature tensor (invariant under C-projective transformations) which, taking into account (2.6) and (2.7), can be written as

$$W(X,Y)Z = K(X,Y)Z - S(X,Y)Z \qquad Z \in H, \quad X, Y \in \mathcal{X}(M), \tag{2.10}$$

where

$$\begin{aligned} S(X,Y)Z = \{L(X,\varphi Y) - L(Y,\varphi X) + d\eta(X,Y)\}\varphi Z + \\ + L(Y,Z)hX - L(X,Z)hY + \{L(X,Y) - L(Y,X)\}hZ + \\ + L(X,\varphi Z)\varphi Y - L(Y,\varphi Z)\varphi X, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} L(X,Y) = \frac{1}{2(n+1)}\{\rho(K)(X,hY) + \frac{1}{2(n-1)}(\rho(K)(hX,hY) + \\ + \rho(K)(hY,hX) - \rho(K)(\varphi X,\varphi Y) - \rho(K)(\varphi Y,\varphi X)) + d\eta(X,\varphi Y)\}. \end{aligned} \tag{2.12}$$

Furthermore, taking into account (2.6), the condition for an adapted symmetric connection to be C-projectively flat [7] can be expressed by the following

Theorem 2.1 *An adapted symmetric connection ∇ is C-projectively flat iff it is C-projectively equivalent to an adapted symmetric connection $\tilde{\nabla}$ for which*

$$\tilde{K}(X,Y)Z = d\eta(X,Y)\varphi Z. \tag{2.13}$$

Examples of C-projectively flat manifolds are the Sasakian and the cosymplectic manifolds with constant φ-sectional curvature and the totally η-umbilical hypersurfaces complex space forms.

It's important to remark that the curvature tensors W, K, S have the property

$$\begin{cases} W(X,Y)\varphi Z = \varphi W(X,Y)Z, \\ K(X,Y)\varphi Z = \varphi K(X,Y)Z, \\ S(X,Y)\varphi Z = \varphi S(X,Y)Z, \end{cases} \tag{2.14}$$

while the first Bianchi identity only holds for W, i.e. we have

$$\begin{cases} \underset{X,Y,Z}{\sigma} W(X,Y)Z = 0, \text{ and} \\ \underset{X,Y,Z}{\sigma} K(X,Y)Z = \underset{X,Y,Z}{\sigma} S(X,Y)Z = \underset{X,Y,Z}{\sigma} d\eta(X,Y)\varphi Z; \end{cases} \tag{2.15}$$

moreover, as W is a curvature tensor field of Weyl type, all its traces vanish, and in particular

$$\operatorname{trace}(\varphi W(R)(X,Y)) = 0. \tag{2.16}$$

3. The Chern forms and the C-projective curvature tensor field. Let E be a complex vector bundle with complex structure operator F and suppose that D is a complex connection on E with curvature operator R. It has been shown in [8] that the Chern ring of E is generated by the de Rham cohomology classes represented by the closed $2k$-forms trace $(FR)^k$, $k = 0,1,2,\ldots\ldots$ In our case the complex vector bundle is H with complex structure φ and the complex connection D is the restriction to H of an adapted connection ∇ with curvature operator $K(X,Y)h$. Our aim is to construct the Chern ring of H represented in the de Rham cohomology of M by the $2k$-forms trace$(\varphi K(.,.)h)^k$, $\quad k = 0,1,2,\ldots.,n$, for C-projectively flat manifolds.

Then, suppose that (U,χ) is a local chart on M, with local coordinates x^i, and denote by $\varphi^j_i, \xi^i, \eta_i, h^j_i, g_{ij}, d\eta_{ij}, K^h_{ijk}, S^h_{ijk}, W^h_{ijk}, \rho(K)_{ij}$ the local components in this chart of $\varphi, \xi, \eta, h, g, d\eta, K, S, W, \rho(K)$ respectively; furthermore, let Ω, Π and Θ be the matrices associated to the tensor fields FK, FW and FS respectively, i.e.

$$\begin{cases} \Omega^l_k = \frac{1}{2}\varphi^l_h K^h_{ijk} dx^i \wedge dx^j, \\ \Pi^l_k = \frac{1}{2}\varphi^l_h W^h_{ijk} dx^i \wedge dx^j, \\ \Theta^l_k = \frac{1}{2}\varphi^l_h S^h_{ijk} dx^i \wedge dx^j. \end{cases} \tag{3.1}$$

If we denote by c_p the Chern forms of D and by $\tilde{c}_p$ the forms constructed in a similar way using the matrix Π, i.e.:

$$c_p = \operatorname{trace} \Omega^p, \qquad \tilde{c}_p = \operatorname{trace} \Pi^p; \tag{3.2}$$

then it follows by a straightforward computation, by using (2.14), (2.15) and (2.16) the expression of c_1 and $\tilde{c}_1$

$$c_1 = -\frac{1}{2}\{\rho(K)_{is}\varphi^s_j - \rho(K)_{js}\varphi^s_i + 2(n-1)d\eta_{ij}\}dx^i \wedge dx^j, \quad \tilde{c}_1 = 0. \tag{3.3}$$

In order to find the relations between $c_p = \mathrm{trace}\Omega^p$ and $\widetilde{c}_p = \mathrm{trace}\Pi^p$ we need some auxiliary results.

Proposition 3.1 *Suppose that ∇ is an adapted symmetric connection with symmetric Ricci tensor field; for (2.7) this condition is equivalent with the assumption that the complex connection D, restriction to H of ∇, has the Ricci tensor $\rho(K)$ symmetric too (and in particular $\rho(K)(\xi, hY) = 0$). Moreover the following relations hold:*

$$\begin{cases} \Omega \wedge \Theta = \Omega \wedge \sigma + (\Lambda + \Gamma) \wedge d\eta, \\ \Omega \wedge \Theta = \Omega \wedge \sigma, \end{cases} \tag{3.4}$$

where

$$\begin{cases} \sigma = \frac{1}{2(n+1)}(c_1 - d\eta) \text{ with } d\eta = d\eta_{ij} dx^i \wedge dx^j, \\ \Lambda^l_k = \frac{1}{2(n+1)}\{-\frac{1}{2}(\varphi^l_i(\rho(K)_{jk} + \frac{1}{(n-1)}(\rho(K)_{jk} - \\ \quad -\rho(K)_{sr}\varphi^r_j\varphi^s_k + d\eta_{jr}\varphi^r_k)\}dx^i \wedge dx^j, \\ \Gamma^l_k = \frac{1}{2(n+1)}\{-\frac{1}{2}h^l_i((\rho(K)_{jr}\varphi^r_k + \frac{1}{(n-1)}(\rho(K)_{jr}\varphi^r_k + \\ \quad +\rho(K)_{rk}\varphi^r_j - d\eta_{jk})\}dx^i \wedge dx^j. \end{cases} \tag{3.5}$$

Note that in the left hand side of (3.4) we have the matrix multiplication and in the right side we have the multiplication of matrix forms with scalar forms.

The formula for the product $\Theta \wedge \Theta$ is similar to the first one in (3.4).

Proof. Substituting equation (2.11) in the equalities

$$\begin{cases} \Omega^l_r \wedge \Theta^r_k = -\frac{1}{4}K^l_{ijr}S^r_{abk}dx^i \wedge dx^j \wedge dx^a \wedge dx^b, \\ \Pi^l_r \wedge \Theta^r_k = -\frac{1}{4}W^l_{ijr}S^r_{abk}dx^i \wedge dx^j \wedge dx^a \wedge dx^b, \\ \Theta^l_r \wedge \Theta^r_k = -\frac{1}{4}S^l_{ijr}S^r_{abk}dx^i \wedge dx^j \wedge dx^a \wedge dx^b, \end{cases} \tag{3.6}$$

taking into account (2.15), we obtain the proposition by a straightforward computation.

From now on, we will consider only adapted symmetric connection with symmetric Ricci tensor. Anyway, apart from the several known examples of adapted connection of this kind [5], [6], it will be clear later on that, in the special case of C-projectively flat connections, this condition doesn't mean a restriction.

Proposition 3.1 directly implies

Theorem 3.2 *The forms c_p and $\widetilde{c}_p$ are related by the following formula*

$$c_p = \widetilde{c}_p + \binom{p}{1}\sigma \wedge \widetilde{c}_{p-1} + \binom{p}{2}\sigma^2 \wedge \widetilde{c}_{p-2} + \cdots + \sigma^p, \tag{3.7}$$

where σ^p denotes the p-exterior power of σ.

Proof. Considering the relation $\Omega = \Pi + \Theta$, taking also into account the previous results, we have

$$\begin{cases} \mathrm{trace}\ \Pi^p = \widetilde{c}_p, \\ \mathrm{trace}\ (\Omega \wedge \Pi^{p-1}) = \widetilde{c}_p + \sigma \wedge \widetilde{c}_{p-1}, \\ \mathrm{trace}\ (\Omega^2 \wedge \Pi^{p-2}) = \widetilde{c}_p + \sigma \wedge \widetilde{c}_{p-1} + \sigma^2 \wedge \widetilde{c}_{p-2}. \end{cases} \tag{3.8}$$

Then the following formula

$$\mathrm{trace}\ (\Omega^r \wedge \Pi^{p-r}) = \widetilde{c}_p + \binom{r}{1}\sigma \wedge \widetilde{c}_{p-1} + \cdots + \sigma^r \wedge \widetilde{c}_{p-r} \tag{3.9}$$

is obtained by usual induction. For $r = p$ in (3.9) we get the theorem.

Recalling the expression of σ (see (3.5)), Theorem 3.2 allows us to obtain some important consequences.

Corollary 3.3 *If the connection D with symmetric Ricci tensor field has the first Chern form proportional to $d\eta$, then all its Chern forms are C-projectively invariant.*

On the other hand, Theorem 2.1 assures that every C-projectively flat adapted connection is C-projectively equivalent to an adapted connection $\widetilde{\nabla}$ such that $\rho(\widetilde{K})(Y, Z) = d\eta(\varphi Y, Z)$; then, recalling that on every normal almost contact manifold the equality $d\eta(X, Y) = d\eta(\varphi X, \varphi Y)$ always holds, we have in this case $c_1 = -nd\eta$, so that we can state

Corollary 3.4 *All the Chern forms of a C-projectively flat adapted connection are C-projectively invariant.*

Moreover, we easily deduce from the equation (3.7) that all Chern forms of a C-projectively flat manifold are powers of $d\eta$ and, since (see [9]) $d\eta^k = d(\eta \wedge d\eta^{k-1})$ for every $k \in N$, we finally obtain Theorem 1.3.

REFERENCES

1. A v e z, A., – *Remarques sur les formes de Pontrjagin*, C.R. Acad. Sci. Paris, 270, (1970), 1248.

2. B l a i r, D.E., – *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., 509, Springer-Verlag, Berlin and New York, 1976.

3. C h e r n, S.S., - *On curvature and characteristic classes of a Riemannian manifold*, Abhandlungen Math. Sem. Univ. Hamburg, 20, (1956), 117-126.

4. M a t z e u, P., - *Su una caratterizzazione delle varietà normali*, Boll. U.M.I., VI, 4 (1985), 153-160.

5. M a t z e u, P., - *Decomposition of curvature tensors on normal almost contact manifolds*, Rend. Sem. Mat. Univers. Politecn. Torino, 47, 2 (1989).

6. M a t z e u, P., - *Trasformazioni C-proiettive e gruppi di olonomia di varietà di quasi contatto normali*, to appear on Rend. Circ. Mat. Palermo, 44, (1995).

7. M a t z e u, P. and O p r o i u, V., - *C-projective curvature of normal almost contact manifolds*, Rend. Sem. Mat. Univers. Politec. Torino, 45, 2 (1987), 41-58.

8. N e a g u, A. and O p r o i u, V. - *Chern forms and H-projective curvature of complex manifolds*, An. St. Univ. "Al. I. Cuza" Iasi, 24, 1 (1978), 39-46.

9. O p r o i u, V., - *Chern forms of Sasakian manifolds of constant φ-sectional curvature*, Rend. Sem. Fac. Sci. Univ. Cagliari, 55, 1 (1985), 39-46.

Received: 15.XII.1995

Università di Cagliari
Dipartimento di Matematica
Via Ospedale 72
09124 Cagliari
ITALY

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLII, s.I.a, Matematică, 1996, f1.

BOOK REVIEWS *RECENZII*

B.D. BOJANOV, H.A. HAKOPIAN and A.A. SAHAKIAN - *Spline Functions and Multivariate Interpolations*, Mathematics and Its Applications, vol. 248, Kluwer Academic Publishers, Dordrecht, Boston, London, 1993, 276 p., ISBN 0-7923-2229-0.

Cartea este o introducere cuprinzătoare în teoria funcţiilor spline de una sau mai multe variabile. Sunt incluse noi dezvoltări mai ales în domeniul interpolării Birkhoff, al proprietăţilor extremale ale funcţiilor spline şi la interpolarea multidimensională. Cele treisprezece capitole ale cărţii sunt dedicate următoarelor direcţii: interpolarea polinomială, spaţiul funcţiilor spline, funcţii B-spline, interpolarea prin funcţii spline, funcţii spline naturale, funcţii spline perfecte, funcţii monospline, funcţii spline periodice, funcţii B-spline multidimensionale, diferenţe divizate de mai multe variabile, funcţii box-spline, interpolarea multidimensională. O parte din rezultate sunt prezentate fără demonstraţii ca exerciţii, având însă indicaţii de rezolvare. Cartea este adresată studenţilor şi cercetătorilor interesaţi în analiza numerică, în special în teoria aproximării.

C. Ilioi

DING-ZHU DU and JIE SUN (editors) - *Advances in Optimization and Approximation*, Nonconvex Optimization and Its Applications, vol. 1, Kluwer Academic Publishers, Dordrecht, Boston, London, 1994, 389 p., ISBN 0-7923-2785-3.

Volumul este o colecţie de lucrări recente din domeniul optimizării şi aproximării dedicate profesorului Minyi Yue de la Institutul de Matematici Aplicate de la Beijing, China. Lucrările acoperă o arie foarte largă de preocupări din optimizarea geometrică, algoritmi de proiecţie a gradientului, din programarea liniară finită şi semi-infinită, de asemenea unele aplicaţii din biologia moleculară. Accentul este pus pe aspectele algoritmice din optimizare, pe algoritmi de aproximare în problemele de optimizare globală şi