media, isotropic and non isotropic elsasticity, plasticity, continuous media with microstructure and other. After the death of Mendel Haimovici some doctorate thesis was finished under the guidance of Professor Constantin I. Borş. After dissertations were in their final stage at the time of his death.

On the other hand, his devotion to science has substantially contributed to the consolidation of the School of Mechanics of Iaşi and, indubitable, his exemplary passion for didactic activity gave rise to an emulation spirit for its collaborators and young students.

All these make us add a sign of gratitude, a symbol of our deep appreciation to the memory of the Romanian mathematician Constantin I. Borş.

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ON MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS AND WITH TWO FIXED POINTS

BY

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1. Introduction. Let Σ_p denote the class of functions of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_0 > 0;\ a_n \geq 0) \tag{1.1}$$

which are analytic and univalent in $U^* = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there.

A function $f(z) \in \Sigma_p$ is in the class $\Sigma_p^*(\alpha, \beta, \gamma)$ if it satisfies the condition

$$\left| \frac{z^2 f'(z) + a_0}{(2\gamma - 1) z^2 f'(x) + (2\alpha\gamma - 1) a_0} \right| < \beta \quad (z \in U^*) \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$ and γ $(\frac{1}{2} \leq \gamma \leq 1)$.

We note that for $a_0 = 1$, the class $\Sigma_p^*(\alpha, \beta, \gamma) = \Sigma_p(\alpha, \beta, \gamma)$ was studied by Cho, Lee and Owa [1].

For a given real number z_0 $(0 < z_0 < 1)$ let Σ_{p_i} $(i = 0, 1)$ be the subclasses of Σ_p satisfying $z_0 f(z_0) = 1$ and $-z_0^2 f'(z_0) = 1$, respectively. Let

$$\Sigma_{p_i}^*(\alpha, \beta, \gamma, z_0) = \Sigma_p^*(\alpha, \beta, \gamma) \cap \Sigma_{p_i} \quad (i = 0, 1). \tag{1.3}$$

In this paper we obtain coefficient estimates, distortion theorems, closure theorems and radius of convexity of order δ $(0 \leq \delta < 1)$ for the classes $\Sigma_{p_i}^*(\alpha, \beta, \gamma, z_0)$ $(i = 0, 1)$. Further we determine a necessary and sufficient condition that a subset B of the real interval $]0, 1[$ should satisfy the property that $\bigcup_{z_r \in B} \Sigma_{p_0}^*(\alpha, \beta, \gamma, z_r)$ and $\bigcup_{z_r \in B} \Sigma_{p_1}^*(\alpha, \beta, \gamma, z_r)$ each forms a convex family. Techniques used are similar to those of S i l v e r m a n [2] and U r a l e g a d d i and G a n i g i [3].

2. Coefficient estimates.

Theorem 1. *Let the function $f(z)$ be defined by (1.1). Then $f(z)$ is in the class $\Sigma_p^*(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta)a_n \le 2\beta\gamma(1-\alpha)a_0. \tag{2.1}$$

The result is sharp for the function

$$f(z)=\frac{a_0}{z}+\frac{2\beta\gamma(1-\alpha)a_0}{n(1+2\beta\gamma-\beta)}z^n \quad (n\ge 1). \tag{2.2}$$

Proof. The proof follows exactly on the lines of the proof of Theorem 2.3 in [1]; hence the details are omitted.

Theorem 2. *Let the function $f(z)$ be defined by (1.1). Then $f(z)\in \Sigma_{po}^*(\alpha,\beta,\gamma,z_0)$ if and only if*

$$\sum_{n=1}^{\infty}\left[\frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}+z_0^{n+1}\right]a_n \le 1. \tag{2.3}$$

Proof. Since $f(z)\in\Sigma_{po}^*(\alpha,\beta,\gamma,z_0)$ we have $z_0f(z_0)=1=a_0+$ $+\sum_{n=1}^{\infty}a_0z_0^{n+1}$ $(a_0>0;\ a_n\ge 0)$ which gives

$$a_0=1-\sum_{n=1}^{\infty}a_nz_0^{n+1}. \tag{2.4}$$

Substituting the value of a_0 in Theorem 1, we have the theorem.

Corollary 1. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_{po}^*(\alpha,\beta,\gamma,z_0)$. Then*

$$a_n\le\frac{2\beta\gamma(1-\alpha)}{[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]} \quad (n\ge 1). \tag{2.5}$$

The equality in (2.5) is attained for the function $f(z)$ given by

$$f(z)=\frac{n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z^{n+1}}{z[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]} \quad (n\ge 1). \tag{2.6}$$

Theorem 3. *Let the function $f(z)$ be defined by (1.1). Then $f(z)\in \Sigma_{p_1}^*(\alpha,\beta,\gamma,z_0)$ if and only if*

$$\sum_{n=1}^{\infty} n\left[\frac{(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}-z_0^{n+1}\right]a_n\le 1. \tag{2.7}$$

Proof. Since $-z_0^2f'(z_0)=1$, we have

$$a_0=1+\sum_{n=1}^{\infty}na_nz_0^{n+1}. \tag{2.8}$$

Eliminating a_0 from (2.1) and (2.8) we get the required result.

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_{p_1}^*(\alpha,\beta,\gamma,z_0)$. Then*

$$a_n\le\frac{2\beta\gamma(1-\alpha)}{n[(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]} \quad (n\ge 1). \tag{2.9}$$

The equality in (2.9) is attained for the function $f(z)$ given by

$$f(z)=\frac{n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z^{n+1}}{zn[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^{n+1}]} \quad (n\ge 1). \tag{2.10}$$

3. Distortion theorems.

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_{po}^*(\alpha,\beta,\gamma,z_0)$. Then*

$$|f(z)|\ge\frac{(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)r^2}{r[(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^2]} \tag{3.1}$$

for $0<|z|=r<1$. The result is sharp.

Proof. It follows from Theorem 2 that

$$\sum_{n=1}^{\infty}a_n\le\frac{2\beta\gamma(1-\alpha)}{[(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^2]}. \tag{3.2}$$

Further from (2.4), we have

$$a_0 = 1 - \sum_{n=1}^{\infty} a_n z_0^{n+1} \geq \frac{(1+2\beta\gamma-\beta)}{[(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^2]}. \tag{3.3}$$

Hence we have

$$|f(z)| \geq \frac{a_0}{r} - r\sum_{n=1}^{\infty} a_n \geq \frac{(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)r^2}{r[(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^2]} \tag{3.4}$$

by using (3.2) and (3.3). Further the result is sharp for the function $f(z)$ given by

$$f(z) = \frac{(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)r^2}{z[(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^2]}. \tag{3.5}$$

Theorem 5. *Let the function $f(z)$ be defined by (1.1) be in the class $\Sigma^*_{p_1}(\alpha,\beta,\gamma,z_0)$. Then*

$$|f(z)| \leq \frac{(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)r^2}{r[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^2]} \tag{3.6}$$

for $0<|z|=r<1$. The result is sharp.

Proof. It follows from Theorem 3 that

$$\sum_{n=1}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)}{[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^2]} \quad \text{and} \tag{3.7}$$

$$\sum_{n=1}^{\infty} na_n \leq \frac{2\beta\gamma(1-\alpha)}{[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^2]} \tag{3.8}$$

Further from (2.8), we have

$$a_0 = 1 + \sum_{n=1}^{\infty} na_n z_0^{n+1} \leq \frac{(1+2\beta\gamma-\beta)}{[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^2]}. \tag{3.9}$$

Hence we have

$$|f(z)| \leq \frac{a_0}{r} + r\sum_{n=1}^{\infty} a_n \leq \frac{(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)r^2}{r[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^2]} \tag{3.10}$$

by using (3.7) and (3.9). Further the result is sharp for the function $f(z)$ given by

$$f(z) = \frac{(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z^2}{z[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^2]}. \tag{3.11}$$

4. Closure theorems. Let the functions $f_j(z)$ be defined, for $j = 1, 2, \dots, m$, by

$$f_j(z) = \frac{a_{0,j}}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \quad (a_{0,j}>0;\ a_{n,j}\geq 0) \quad \text{for } z \in U^*. \tag{4.1}$$

We shall prove the following results for the closure of functions in the classes $\Sigma^*_{p_i}(\alpha,\beta,\gamma,z_0)$ $(i=0,1)$.

Theorem 6. *Let the functions $f_j(z)$ $(j=1,2,\dots,m)$ defined by (4.1) be in the class $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{j=1}^{m} d_j f_j(z) \quad (d_j \geq 0) \tag{4.2}$$

is also in the same class $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$, where

$$\sum_{j=1}^{m} d_j = 1. \tag{4.3}$$

Proof. According to the definition of $h(z)$ we can write

$$h(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad \text{where} \tag{4.4}$$

$$b_0 = \sum_{j=1}^{m} d_j a_{0,j} \quad \text{and} \quad b_n = \sum_{j=1}^{m} d_j a_{n,j} \quad (n=1,2,\dots,m). \tag{4.5}$$

Since $f_j(z)$ are in $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$ $(j=1,2,\dots,m)$, by means of Theorem 2, we have

$$\sum_{n=1}^{\infty} \left[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}\right] a_{n,j} \leq 2\beta\gamma(1-\alpha) \tag{4.6}$$

for every $j = 1, 2, ..., m$. Therefore we have

$$\sum_{n=1}^{\infty} \left[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}\right]\left(\sum_{j=1}^{m} d_j a_{n,j}\right) =$$
$$= \sum_{j=1}^{m} d_j \left(\sum_{n=1}^{\infty}\left[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}\right] a_{n,j}\right) \leq$$
$$\leq \left(\sum_{j=1}^{m} d_j\right) 2\beta\gamma(1-\alpha) = 2\beta\gamma(1-\alpha)$$

which shows that $h(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$. Thus we have the theorem.

Theorem 7. *Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (4.1) be in the class $\Sigma^*_{p_1}(\alpha,\beta,\gamma,z_0)$ for every $j = 1, 2, ..., m$. Then the function $h(z)$ defined by (4.2) is also in the same class $\Sigma^*_{p_1}(\alpha,\beta,\gamma,z_0)$, under the condition (4.3).*

The proof of the theorem is obtained by using the same technique as in the proof of Theorem 6 with Theorem 3.

Theorem 8. *The class $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$ is closed under convex linear combination.*

Proof. Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (4.1) be in the class $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \leq \lambda \leq 1) \tag{4.7}$$

is also in the class $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$. Since

$$H(z) = \frac{\lambda a_{0,1} + (1-\lambda)a_{0,2}}{z} + \sum_{n=1}^{\infty}\{\lambda a_{n,1} + (1-\lambda)a_{n,2}\} z^n, \tag{4.8}$$

with the aid of Theorem 2, we have

$$\sum_{n=1}^{\infty}\left[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}\right]\{\lambda a_{n,1} + (1-\lambda)a_{n,2}\} \leq \leq 2\beta\gamma(1-\alpha) \tag{4.9}$$

which implies that $H(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$. In a similar manner, by using Theorem 3, we can prove the following Theorem:

Theorem 9. *The class $\Sigma^*_{p_1}(\alpha,\beta,\gamma,z_0)$ is closed under convex linear combination.*

Theorem 10. *Define*

$$f_0(z) = \frac{1}{z} \quad \text{and} \tag{4.10}$$

$$f_n(z) = \frac{n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z^{n+1}}{z[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]} \quad (n \geq 1). \tag{4.11}$$

Then $f(z)$ is in the class $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$ if and only if it can be expressed in the form

$$f_n(z) = \sum_{n=0}^{\infty}\lambda_n f_n(z) \textit{where } \lambda_n \geq 0 \quad (n \geq 0) \quad \textit{and} \tag{4.12}$$

$$\sum_{n=0}^{\infty}\lambda_n = 1. \tag{4.13}$$

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\lambda_n f_n(z) = \lambda_0 f_0(z) + \sum_{n=1}^{\infty}\lambda_n f_n(z) = \\ &= \frac{\lambda_0}{z} + \sum_{n=1}^{\infty}\frac{n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z^{n+1}}{z[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]}\lambda_n = \\ &= \frac{1}{z}\left(\lambda_0 + \sum_{n=1}^{\infty}\frac{n(1+2\beta\gamma-\beta)\lambda_n}{[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]}\right) + \\ &+ \sum_{n=1}^{\infty}\frac{2\beta\gamma(1-\alpha)\lambda_n}{[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]}z^n. \end{aligned} \tag{4.14}$$

Then it follows that

$$\sum_{n=1}^{\infty}\frac{[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]}{2\beta\gamma(1-\alpha)}\,\frac{2\beta\gamma(1-\alpha)\lambda_n}{[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]} =$$
$$= \sum_{n=1}^{\infty}\lambda_n = 1-\lambda_0 \leq 1.$$

Also by definition we have $z_0 f_n(z_0) = 1$. Therefore

$$z_0 f(z_0) = \sum_{n=1}^{\infty}\lambda_n z_0 f_n(z_0) = \sum_{n=0}^{\infty}\lambda_n = 1.$$

This implies $f(z) \in \Sigma_{p_0}$. So by Theorem 2, $f(z) \in \Sigma^*_{p_0}(\alpha, \beta, \gamma, z_0)$.

Conversely, assume that the function $f(z)$ defined by (1.1) belongs to the class $\Sigma^*_{p_0}(\alpha, \beta, \gamma, z_0)$. Then

$$a_n \le \frac{2\beta\gamma(1-\alpha)}{[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]} \quad (n \ge 1). \tag{4.15}$$

Setting

$$\lambda_n = \frac{[n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z_0^{n+1}]}{2\beta\gamma(1-\alpha)} a_n \quad (n \ge 1), \text{ and} \tag{4.16}$$

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n. \tag{4.17}$$

Hence, we can see that $f(z)$ can be expressed in the form (4.12). This completes the proof of Theorem 10.

In a similar manner, we can prove the following Theorem:

Theorem 11. *Define*

$$f_0(z) = \frac{1}{z} \quad \text{and} \tag{4.18}$$

$$f_n(z) = \frac{n(1+2\beta\gamma-\beta)+2\beta\gamma(1-\alpha)z^{n+1}}{zn[(1+2\beta\gamma-\beta)-2\beta\gamma(1-\alpha)z_0^{n+1}]} \quad (n \ge 1). \tag{4.19}$$

Then $f(z)$ is in the class $\Sigma^*_{p_1}(\alpha, \beta, \gamma, z_0)$ if and only if it can be expressed in the form (4.12) *where $\lambda_n \ge 0$ $(n \le 0)$ and* (4.13).

5. Radius of convexity. In this section we determine the radius of convexity of order δ $(0 \le \delta < 1)$ for the classes $\Sigma^*_{p_i}(\alpha, \beta, \gamma, z_0)$ $(i = 0, 1)$.

Theorem 12. *Let the function $f(z)$ defined by* (1.1) *be in the class $\Sigma^*_{p_0}(\alpha, \beta, \gamma, z_0)$ or $\Sigma^*_{p_1}(\alpha, \beta, \gamma, z_0)$, then $f(z)$ is convex of order δ $(0 \le \delta < 1)$ in $0 < |z| < r^*(\alpha, \beta, \gamma, \delta)$, where*

$$r^*(\alpha, \beta, \gamma, \delta) = \inf_n \left\{ \frac{(1-\delta)(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)(n+2-\delta)} \right\}^{\frac{1}{n+1}} \quad (n \ge 1). \tag{5.1}$$

The result is sharp.

Proof. It is sufficient to show that $\left|1 + \dfrac{zf''(z)}{f'(z)}\right| \le 1 - \delta$ $(0 \le \delta < 1)$ for $0 < |z| < r^*(\alpha, \beta, \gamma, \delta)$. We have

$$\left|\frac{f'(z) + (zf'(z))'}{f'(z)}\right| \le \frac{\sum_{n=1}^{\infty} n(n+1)a_n|z|^{n+1}}{a_0 - \sum_{n=1}^{\infty} na_n|z|^{n+1}}.$$

Thus $\left|\dfrac{f'(z) + (zf'(z))'}{f'(z)}\right| \le 1 - \delta$ if

$$\sum_{n=1}^{\infty} n(n+2-\delta)a_n|z|^{n+1} \le (1-\delta)a_0. \tag{5.2}$$

When $f(z) \in \Sigma^*_{p_0}(\alpha, \beta, \gamma, z_0)$, using (2.4) we find that the inequality (5.2) is equivalent to

$$\sum_{n=1}^{\infty} \left[n(n+2-\delta)|z|^{n+1} + (1-\delta)z_0^{n+1}\right] a_n \le (1-\delta). \tag{5.3}$$

But Theorem 2 ensures

$$\sum_{n=1}^{\infty} \left\{(1-\delta)\left[\frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} + z_0^{n+1}\right]\right\} a_n \le (1-\delta). \tag{5.4}$$

Hence (5.3) holds if

$$\left[n(n+2-\delta)|z|^{n+1} + (1-\delta)z_0^{n+1}\right] a_n \le$$

$$\le \left\{(1-\delta)\left[\frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} + z_0^{n+1}\right]\right\} a_n, \quad n \ge 1$$

or if

$$|z| \le \left\{\frac{(1-\delta)(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)(n+2-\delta)}\right\}^{\frac{1}{n+1}} \quad (n \ge 1).$$

Thus $f(z)$ is convex function or order δ $(0 \le \delta < 1)$ in $0 < |z| < r^*(\alpha, \beta, \gamma, \delta)$.

In other case when $f(z) \in \Sigma^*_{p_1}(\alpha, \beta, \gamma, z_0)$, using (2.8) we find that the inequality (5.2) is equivalent to

$$\sum_{n=1}^{\infty} n\left[(n+2-\delta)|z|^{n+1} - (1-\delta)z_0^{n+1}\right] a_n \le (1-\delta). \tag{5.5}$$

Therefore, in view of Theorem 3, the inequality (5.5) hold if

$$n\left[(n+2-\delta)|z|^{n+1} - (1-\delta)z_0^{n+1}\right] a_n \le$$

$$\le (1-\delta)n\left[\frac{(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} - z_0^{n+1}\right] a_n, \quad n \ge 1$$

or if

$$|z| \leq \left\{ \frac{(1-\delta)(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)(n+2-\delta)} \right\}^{\frac{1}{n+1}} \quad (n \geq 1).$$

This completes the proof of the theorem.

Sharpness for the class $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$ follows by taking the function $f(z)$ given by (2.6), whereas, for the class $\Sigma^*_{p_1}(\alpha,\beta,\gamma,z_0)$, sharpness follows if we take the function $f(z)$ given by (2.10).

Remark. The conclusion of Theorem 12 is independent of z_0.

6. Convex family. Let B be a nonempty subset of the real interval $]0,1[$. We define the family $\Sigma^*_{p_0}(\alpha,\beta,\gamma,B)$ by

$$\Sigma^*_{p_0}(\alpha,\beta,\gamma,B) = \bigcup_{z_r \in B} \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_r).$$

If B has only one element, then $\Sigma^*_{p_0}(\alpha,\beta,\gamma,B)$ is known to be a convex family by Theorems 6 and 8. It is interest to invetigate this class for other subsets B. We shall make use of the following :

Lemma 1. *If $f(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0) \cap \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_1)$, where z_0 and z_1 are distinct positive numbers, then $f(z) = 1/z$.*

Proof. Let $f(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0) \cap \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_1)$ and let $f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n$ $(a_0 > 0; a_n \geq 0)$. Then $a_0 = 1 - \sum_{n=1}^{\infty} a_n z_0^{n+1} = 1 - \sum_{n=1}^{\infty} a_n z_1^{n+1}$. Since $a_n \geq 0$, $z_0 > 0$ and $z_1 > 0$, this implies $a_n \equiv 0$ for each $n \geq 1$ and $f(z) = 1/z$. Hence the lemma.

Remark. If the condition that the fixed points z_0, z_1 be positive is relaxed the lemma needs not be true. For example if $f(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$ and $f(z)$ is odd then $f(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,-z_0)$.

Theorem 13. *If B is contained in the interval $]0,1[$, then $\Sigma^*_{p_0}(\alpha,\beta,\gamma,B)$ is a convex family if and only if B is connected.*

Proof. Suppose B is connected and $z_0, z_1 \in B$ with $z_0 \leq z_1$.

To prove $\Sigma^*_{p_0}(\alpha,\beta,\gamma,B)$ is a convex family it suffices to show, for $f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$, $g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_1)$ and $0 \leq \lambda \leq 1$, that there exists a z_2 $(z_0 \leq z_2 \leq z_1)$ such that $h(z) = \lambda f(z) + (1-\lambda) g(z)$ is in $\Sigma^*_{p_0}(\alpha,\beta,\gamma,z_2)$. Since $f(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$ and $g(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_1)$, we have $a_0 = 1 - \sum_{n=1}^{\infty} a_n z_0^{n+1}$ and $b_0 = 1 - \sum_{n=1}^{\infty} b_n z_1^{n+1}$.

Therefore we have

$$\begin{aligned} t(z) = zh(z) &= \lambda a_0 + (1-\lambda) b_0 + \lambda \sum_{n=1}^{\infty} a_n z^{n+1} + \\ (6.1) \qquad &+ (1-\lambda) \sum_{n=1}^{\infty} b_n z^{n+1} = 1 + \sum_{n=1}^{\infty} a_n \left(z^{n+1} - z_0^{n+1}\right) + \\ &+ (1-\lambda) \sum_{n=1}^{\infty} b_n \left(z^{n+1} - z_1^{n+1}\right). \end{aligned}$$

$t(z)$ being real when z is real with $t(z_0) \leq 1$ and $t(z_1) \geq 1$, there exists $z_2 \in [z_0, z_1]$ such that $t(z_2) = 1$. This implies $z_2 h(z_2) = 1$ for some $z_2, z_0 \leq z_2 \leq z_1$, that is $h(z) \in \Sigma_{p_0}$.

Now, from (6.1) and $z_2 h(z_2) = 1$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \left[n(1+2\beta\gamma-\beta) + 2\beta\gamma(1-\alpha) z_2^{n+1}\right] \{\lambda a_n + (1-\lambda) b_n\} = \\ &= \lambda \sum_{n=1}^{\infty} \left[n(1+2\beta\gamma-\beta) + 2\beta\gamma(1-\alpha) z_2^{n+1}\right] a_n + \\ &+ (1-\lambda) \sum_{n=1}^{\infty} \left[n(1+2\beta\gamma-\beta) + 2\beta\gamma(1-\alpha) z_1^{n+1}\right] b_n + \\ &+ 2\beta\gamma(1-\alpha)\lambda \sum_{n=1}^{\infty} \left\{z_2^{n+1} - z_0^{n+1}\right\} a_n + \\ &+ 2\beta\gamma(1-\alpha)(1-\lambda) \sum_{n=1}^{\infty} \left\{z_2^{n+1} - z_0^{n+1}\right\} b_n = \\ &= \lambda \sum_{n=1}^{\infty} \left[n(1+2\beta\gamma-\beta) + 2\beta\gamma(1-\alpha) z_0^{n+1}\right] a_n + \\ &+ (1-\lambda) \sum_{n=1}^{\infty} \left[n(1+2\beta\gamma-\beta) + 2\beta\gamma(1-\alpha) z_1^{n+1}\right] b_n \leq \\ &\leq 2\beta\gamma(1-\alpha)\lambda + 2\beta\gamma(1-\alpha)(1-\lambda) = 2\beta\gamma(1-\alpha) \end{aligned}$$

by Theorem 2, since $f(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_0)$ and $g(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_1)$. Hence we have $h(z) \in \Sigma^*_{p_0}(\alpha,\beta,\gamma,z_2)$, by Theorem 2. Since z_0, z_1 and λ are arbitrary, the family $\Sigma^*_{p_0}(\alpha,\beta,\gamma,B)$ is convex.

Conversely, if B is not connected, then there exists z_0, z_1 and z_2 such that $z_0, z_1 \in B$, $z_2 \notin B$ and $z_0 < z_2 < z_1$. Assume $f(z) \in \Sigma^*_{po}(\alpha,\beta,\gamma,z_0)$ and $g(z) \in \Sigma^*_{po}(\alpha,\beta,\gamma,z_1)$ are not both equal to $1/z$. Then, for fixed z_2 and $0 \le \lambda \le 1$, we have from (6.1),

$$t(\lambda) = t(z_2,\lambda) = 1 + \lambda \sum_{n=1}^{\infty} a_n \left(z_2^{n+1} - z_0^{n+1}\right) + (1-\lambda)\sum_{n=1}^{\infty} b_n \left(z_2^{n+1} - z_1^{n+1}\right).$$

Since $t(z_2,0) < 1$ and $t(z_2,1) > 1$, there must exist a λ_0, $0 < \lambda_0 < 1$ such that $t(z_2,\lambda_0) = 1$ or $z_2 h(z_2) = 1$, where $h(z) = \lambda_0 f(z) + (1-\lambda_0)g(z)$. Thus $h(z) \in \Sigma^*_{po}(\alpha,\beta,\gamma,z_2)$. From Lemma 1 we have $h(z) \notin \Sigma^*_{po}(\alpha,\beta,\gamma,B)$, since $z_2 \notin B$ and $h(z) \ne z$. This implies the family $\Sigma^*_{po}(\alpha,\beta,\gamma,B)$ is not convex which is a contradiction.

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ON A UNIVALENCE CRITERION

BY

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In this note we obtain a sufficient condition of univalence for the functions regular in the unit disk $U = \{z : |z| < 1\}$.

In the following, we shall use a result due to Pommerenke.

Theorem A. *[1] Let $r_0 \in (0,1]$ and let $L(z,t) = a_1(t)z + \ldots$, $a_1(t) \ne 0$, be regular in $U_{r_0} = \{z : |z| < r_0\}$ for all $t \in I = [0,\infty)$ and locally absolutely continuous in I, locally uniform with respect to U_{r_0}. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, z \in U_{r_0}$$

where $p(z,t)$ is regular in U and satisfies $\operatorname{Re} p(z,t) > 0, z \in U, t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $L(z,t)/a_1(t)$ forms a normal family in U_{r_0}, then for each $t \in I, L(z,t)$ can be continued regularly in U and gives univalent function.

Theorem. *Let α be a complex number and $m > 0$ such that*

(1) $$\left|\alpha - \frac{m+1}{2}\right| < \frac{m+1}{2}$$

If $f(z) = z + a_2 z^2 + \ldots$ is a regular function such that $f(z)/z \ne 0$ in the disk U and there exists a regular function g in $U, g(z) = z + b_2 z^2 + \ldots$, such that

(i) $$g(z)/z \ne 0, g'(z) \ne 0, z \in U$$