Conversely, if B is not connected, then there exists z_0, z_1 and z_2 such that $z_0, z_1 \in B$, $z_2 \notin B$ and $z_0 < z_2 < z_1$. Assume $f(z) \in \Sigma^*_{p0}(\alpha,\beta,\gamma,z_0)$ and $g(z) \in \Sigma^*_{p0}(\alpha,\beta,\gamma,z_1)$ are not both equal to $1/z$. Then, for fixed z_2 and $0 \leq \lambda \leq 1$, we have from (6.1),

$$t(\lambda) = t(z_2,\lambda) = 1 + \lambda\sum_{n=1}^{\infty} a_n\left(z_2^{n+1} - z_0^{n+1}\right) + (1-\lambda)\sum_{n=1}^{\infty} b_n\left(z_2^{n+1} - z_1^{n+1}\right).$$

Since $t(z_2,0) < 1$ and $t(z_2,1) > 1$, there must exist a λ_0, $0 < \lambda_0 < 1$ such that $t(z_2,\lambda_0) = 1$ or $z_2 h(z_2) = 1$, where $h(z) = \lambda_0 f(z) + (1-\lambda_0)g(z)$. Thus $h(z) \in \Sigma^*_{p0}(\alpha,\beta,\gamma,z_2)$. From Lemma 1 we have $h(z) \notin \Sigma^*_{p0}(\alpha,\beta,\gamma,B)$, since $z_2 \notin B$ and $h(z) \neq z$. This implies the family $\Sigma^*_{p0}(\alpha,\beta,\gamma,B)$ is not convex which is a contradiction.

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ON A UNIVALENCE CRITERION

BY

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In this note we obtain a sufficient condition of univalence for the functions regular in the unit disk $U = \{z : |z| < 1\}$.

In the following, we shall use a result due to Pommerenke.

Theorem A. *[1] Let $r_0 \in (0,1]$ and let $L(z,t) = a_1(t)z + \ldots$, $a_1(t) \neq 0$, be regular in $U_{r_0} = \{z : |z| < r_0\}$ for all $t \in I = [0,\infty)$ and locally absolutely continuous in I, locally uniform with respect to U_{r_0}. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t},\ z \in U_{r_0}$$

where $p(z,t)$ is regular in U and satisfies $\operatorname{Re} p(z,t) > 0, z \in U, t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $L(z,t)/a_1(t)$ forms a normal family in U_{r_0}, then for each $t \in I, L(z,t)$ can be continued regularly in U and gives univalent function.

Theorem. *Let α be a complex number and $m > 0$ such that*

$$\left|\alpha - \frac{m+1}{2}\right| < \frac{m+1}{2} \tag{1}$$

If $f(z) = z + a_2 z^2 + \ldots$ is a regular function such that $f(z)/z \neq 0$ in the disk U and there exists a regular function g in U. $g(z) = z + b_2 z^2 + \ldots$, such that

(i) $$g(z)/z \neq 0, g'(z) \neq 0, z \in U$$

$$(ii) \qquad \left| \left[\frac{f(z)}{g(z)}\right]^{\alpha-1} \frac{f'(z)}{g'(z)} - \frac{m+1}{2} \right| < \frac{m+1}{2}, \forall z \in U$$

$$(iii) \qquad \left| \; |z|^{m+1} \left\{ \left[\frac{f(z)}{g(z)}\right]^{\alpha-1} \frac{f'(z)}{g'(z)} - \frac{m+1}{2} \right\} + \right.$$

$$\left. +(1-|z|^{m+1}) \left[(\alpha-1)\frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)} + 1 - \frac{m+1}{2} \right] \right| \le \frac{m+1}{2}$$

for all $z \in U$, then the function f is univalent in U.

Proof. From (ii), it results that $f(z) \cdot f'(z)/z \neq 0$ in $U - \{0\}$ and hence, for the function

$$\left(\frac{f(z)}{g(z)}\right)^{\alpha-1}$$

we can choose the regular branch in U, equal to 1 at the point $z = 0$.

Let $a, b > 0$ and $m = b/a$. Since $[g(z)/f(z)]^{\alpha-1} = 1 + c_1 z + \dots$, is regular in U and $f(z)/z \neq 0$ in U, it results that there exists $r_0, 0 < r_0 < 1$, such that the function

$$(2) \qquad h(z,t) = 1 + \alpha(e^{bt} - e^{-at}) \left[\frac{g(e^{-at}z)}{f(e^{-at}z)}\right]^{\alpha-1} \frac{zg'(e^{-at}z)}{f(e^{-at}z)}$$

is regular in $U_{r_0} = \{z : |z| < r_0\}$, for all $t \ge 0$ and hence, the function

$$(3) \qquad L(z,t) = f(e^{-at}z)[h(z,t)]^{1/\alpha} = a_1(t)z + \dots$$

is regular in U_{r_0}, for all $t \ge 0$ (in (2) and (3) we choose the branch equal to 1 at the origin).

The function $a_1(t)$ from (3) is given by:

$$(4) \qquad a_1(t) = \left[\alpha + (1-\alpha)\, e^{-(a+b)t}\right]^{1/\alpha} e^{\frac{a+b-\alpha a}{\alpha}t}$$

From (1), it results that $\mathrm{Re}\,\alpha > 0$, and hence $\mathrm{Re}\,\left[\alpha + (1-\alpha)e^{-(a+b)t}\right] > 0$ and $a_1(t) \neq 0$, for all $t \ge 0$.

From (1), we deduce that

$$\mathrm{Re}\,\frac{a+b-\alpha a}{\alpha} = a\,\mathrm{Re}\,\frac{m+1-\alpha}{\alpha} = \frac{a}{|\alpha|^2}\,\mathrm{Re}\,[-|\alpha|^2 + \alpha(m+1)] > 0,$$

and we conclude that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

It easy to prove that $L(z,t)/a_1(t)$ forms a normal family of regular functions in $U_{r_1} = \{z : |z| < r_1\}, 0 < r_1 < r_0$ and $L(z,t)$ is locally uniformly with respect to U_{r_0}, locally absolutely continuous in I.

Let $p(z,t)$ be the function defined in $U_{r_1} \times [0,\infty)$ by

$$(5) \qquad p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}.$$

In order to prove that the function p is regular and with positive real part in U, for all $t \ge 0$, it is sufficient to prove that the function:

$$(6) \qquad w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}$$

is regular in U for $t \ge 0$ and

$$(7) \qquad |w(z,t)| < 1$$

for all $z \in U$ and $t \ge 0$.

From (2), (3), (5) and (6) we have

$$(8) \qquad w(z,t) = \frac{(1+a)Q(z,t) + 1 - b}{(1-a)Q(z,t) + 1 + b}$$

where

$$(9) \qquad Q(z,t) = e^{-(a+b)t} \left\{ \left[\frac{f(e^{-at}z)}{g(e^{-at}z)}\right]^{\alpha-1} \frac{f'(e^{-at}z)}{g'(e^{-at}z)} - 1 \right\} + \left[1 - e^{-(a+b)t}\right] \left[(\alpha-1)\frac{e^{-at}zg'(e^{-at}z)}{g(e^{-at}z)} + \frac{e^{-at}zg''(e^{-at}z)}{g'(e^{-at}z)} \right].$$

Let's note $X = \mathrm{Re}\,Q(z,t)$, $Y = \mathrm{Im}\,Q(z,t)$.

We obtain

$$| w(z,t) |^2 = \frac{| (1+a)(X+iY)+1-b |^2}{| (1-a)(X+iY)+1+b |^2} =$$

$$= \frac{[(1+a)X+1-b]^2+[(1+a)Y]^2}{[(1-a)X+1+b]^2+[(1-a)Y]^2}$$

and, hence, we conclude that the inequality (7) has the form

$$X^2+Y^2+\frac{a-b}{a}X-\frac{b}{a}<0 \text{ or}$$

$$\left|Q(z,t)+1-\frac{m+1}{2}\right|<\frac{m+1}{2},$$

where $m=b/a$, equivalent to the inequality

$$\left|\frac{2}{m+1}[Q(z,t)+1]-1\right|<1.$$

Let's note

$$H(z,t)=\frac{2}{m+1}[Q(z,t)+1]-1. \tag{10}$$

The inequality (7) is equivalent to the inequality

$$|H(z,t)|<1.$$

The function $H(z,t)$ is regular in U for all $z\in U$ and $t\geq 0$.
Let us prove that $|H(z,t)|<1, \forall z\in U$ and $t>0$.
From the condition (ii) of the Theorem, for $t=0$ we obtain $|H(z,0)|<1$ for all $z\in U$. Since the function H is regular in $\overline{U}$ for $t>0$, it results that

$$|H(z,t)|<\max_{|z|=1}|H(z,t)|=|H(e^{i\theta},t)| \tag{11}$$

where θ is a real number, $z\in U, t\geq 0$.
If $u=e^{-at+i\theta}$, then $|u|=e^{-at}$ and $e^{-(a+b)t}=|u|^{m+1}$, where $m=b/a$, replacing in (9) and (10), from (11) we obtain

$$\begin{aligned}|H(z,t)|<\frac{2}{m+1}\Bigg| & |u|^{m+1}\left\{\left[\frac{f(u)}{g(u)}\right]^{\alpha-1}\frac{f'(u)}{g'(u)}-\frac{m+1}{2}\right\}+ \\ & +(1-|u|^{m+1})\left[(\alpha-1)\frac{ug'(u)}{g(u)}+\frac{ug''(u)}{g'(u)}+1-\frac{m+1}{2}\right]\Bigg|\end{aligned} \tag{12}$$

for all $z\in U$ and $t>0$.

Since $u\in U$, for $t>0$, from the conditions (iii) and (12) we obtain that $|H(z,t)|<1, \forall z\in U$ and $t>0$, hence, $|w(z,t)|<1$, and the function w is regular in U, for all $z\in U, t\geq 0$.

It results that $L(z,t)$ is a Loewner chain and $f(z)=L(z,0)$ for all $z\in U$, and we obtain that f is univalent in U.

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