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EXISTENCE RESULTS FOR THE ZEROS OF OPEN MAPS

BY

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1. Introduction. Let X be a topological space and Y a normed space. Given a mapping $f : X \longrightarrow Y$, any solution of the operator equation

$$(OE) \qquad f(x) = 0$$

will be referred to as a zero of f.

Sufficient conditions guaranteeing the existence of such a solution may be of a very different nature; we quote in this direction the contractive approach (D u g u n d j i and G r a n a s [9, ch.1, Section 1]) the topological degree one (D e i m l i n g [8, ch.2, Section 10]), the accretivity procedure (B r o w d e r [4], K i r k [14]) or the maximality one (A l t m a n [1], T u r i n i c i [21]). But, in the following, we shall be interested in a different approach of topological nature; details will be given in Section 4. Note that the list of corollaries for the main results we get, also includes the well known d'Alembert–Gauss theorem. All preliminary material for the statements in question is to be found in Section 2 (devoted to the study of some minimal norm properties). Finally, in Section 5, an application of these facts is given to a solvability result involving functions of two complex variables. This, among others, alows a deduction – in a different way – of the reciprocal mean value property due to P o m p e i u [18]. Some other aspects involving these facts will be discussed elsewhere.

2. Minimal norm properties. Let $(X, \|\cdot\|)$ be a Banach space and E some (nonempty) part of X. Consider the minimization problem

$$(P_E) \qquad \min\{\|x\|;\ x \in E\}.$$

In what follows, two (somewhat opposite) concepts related to (P_E) will be introduced and discussed.

A) We say that the ambient set E is *norm-descending*, provided

(B_1) *no element of E distinct from zero can be a solution of (P_E).*

This property is strongly related to the norm in question. to describe some circumstances under which it is valid, we need some conventions. Call the point x in E, *algebraically interior* to E, provided

$$(B_2)\quad \begin{cases} \text{for each line } L \text{ passing through } x,\ L\cap E \text{ includes a (linear)} \\ \text{segment containing } x \text{ in its (relative) interior.} \end{cases}$$

The set of all such points will be denoted

$$a\,\mathrm{int}\,(E) = \text{the algebraic interior of } E.$$

Now, it is clear that an assumption like

(A_1) *each point a of E distinct from zero is algebraically interior to E,*

implies the property in question (whatever the ambient norm may be). For, in such a case,

$$(2.1)\quad \begin{cases} \text{the linear segment } [\lambda a, \mu a] \text{ is included in } E \text{ for some } \lambda, \mu, \\ \text{with } 0<\lambda<1<\mu, \end{cases}$$

and the conclusion follows. Note that, a particular case of (A_1) is

(A_2) *each point a of E distinct from zero is topologically interior to E.*

It is to be underlined here that, under different regularity assumptions about the (nonempty) set of E of X, the distinction between these two conditions is not very significant. Precisely, we have (cf. B a r b u and P r e c u - p a n u [4, ch.2, Section 1])

Proposition 1. *The following are valid*

(i) *under the regularity assumption*

(A_3) *E is convex*

it follows that, either $\mathrm{int}\,(E)$ *is empty or else,* $\mathrm{int}\,(E) = a\,\mathrm{int}\,(E)$.

(ii) *under the assumptions (A_3) and*

(A_4) *E is closed*

it follows that, either $a\,\mathrm{int}\,(E)$ is empty or else, $a\,\mathrm{int}\,(E) = \mathrm{int}\,(E)$.

Proof. (Sketch) The first part is essentially based on the properties of the Minkowski functional. (See the quoted reference for details.) And the second one is deductible from the Baire category theorem.

B) We say that the ambient set E has the *minimal norm property* when

(B_3) *the problem (P_E) has at least one solution in E.*

Note that, whenever

(A_5) *E is compact,*

this property is fulfilled. A natural question to be posed is of what happens when this condition is replaced by the weaker one

(A_6) *E is closed and bounded.*

The answer is negative, in general. But, under some additional conditions, this happens. Precisely, we have (cf. C i o r ă n e s c u [5, ch.2, Section 3]).

Proposition 2. *Suppose (A_3) holds, as well as*

(A_7) *E is weakly compact.*

Then, necessarily, E has the minimal norm property. In particular, when

(A_8) *X is a reflexive space,*

condition (A_7) is fulfilled under (A_6).

Let us now return to (A_4). As already noted, a positive answer to the problem we deal with is not in general available – even if (A_8) were admitted – in terms of the initial norm over X. However, with an equivalent norm this is possible:

Proposition 3. *Let E be a (nonempty) closed part of the Banach space $(X, \|\cdot\|)$. Then, there exists an equivalent norm $\|\cdot\|_1$ over X such that E be endowed with the minimal norm property (with respect to $\|\cdot\|_1$).*

The proof of this statement due to Z a b r e i k o and K r a s n o - s e l s k i i [24] (see also D a n e š [7]) is essentially based on what is called now "the drop theorem". For other aspects of the problem we refer to the papers by G e o r g i e v [11] and P e n o t [17].

C) The above introduced concepts are the basic tools in a solvability type result over normed structures, largely used in the sequel. Let in the following X be a normed space.

Theorem 1. *Suppose that the nonempty part E of X with*

(A_9) *E is norm descending*

is such that, there exists some part G of X with

(A_{10}) *$E \cup G$ has the minimal norm property*

(A_{11}) *for each $x \in G$ there exists $u \in E \cup G$ with $\|u\| < \|x\|$.*

Then, necessarily, $0 \in E$.

Proof. By (A_{10}), there must be some point a in $E \cup G$ with

$$\|a\| \le \|x\|, \text{ for each } x \text{ in } E \cup G.$$

This element cannot be in G, because of (A_{11}); hence $a \in E$. If $a \neq 0$, then a contradiction is obtained via (A_9). Consequently, $a = 0$ and the proof is complete.

As explicitly stated, the completeness of the ambient norm of X is not necessary; but (cf. the developments above) the concrete situations under which (A_9) and/or (A_{10}) is fulfilled require such a regularity hypothesis. For a number of different approaches to this question we refer to the papers by A l t m a n [1] and T u r i n i c i [21].

3. Open maps. Let X and Y be a couple of topological spaces and $f : X \longrightarrow Y$, a function. We say that f is *open* provided

(D_1) *the image $f(D)$ of each open part D of X is an open part of Y.*

It is our aim in what follows to collect a number of basic openess results to be used further. The natural framework of these is the normed one; but, we must say that the topological setting is also appropriate.

A first way of attacking this problem is the one based on tangency concepts. Let Z be a (real) Banach space. Fix some part C of Z and some c in $\mathrm{cl}\,(C)$. Denote by $K_C(c)$ the set of all $z \in Z$ with the property

$$(D_2)\quad \begin{cases} z = \lim_n \lambda_n^{-1}(c_n - c), \text{ for some sequence } (\lambda_n) \text{ in }]0,\infty[\\ \text{converging to zero and some sequence } (c_n) \text{ in } C \text{ with } c_n \longrightarrow c. \end{cases}$$

This will be referred to as the *Bouligand-Severi tangent cone* of C at c.

For more information about these objects we refer to the survey paper by W a r d [23].

Now, let X, Y be (real) Banach spaces and $f : X \longrightarrow Y$, a function. For each $a \in X$, we may construct a multifunction $K_f(a)$ from X to Y by the formula

$$K_f(a)(x) = \{y \in Y;\ (x, y) \in K_{\mathrm{gr}\,(f)}(a, f(a))\}.$$

Denote also, for each $\rho > 0$

$$X(a,\rho) = \{x \in X;\ \|a - x\| < \rho\},\ X[a,\rho] = \{x \in X;\ \|a - x\| \le \rho\}.$$

These will be referred to as the *open/closed sphere* centered at a, with radius ρ. (The notation for the corresponding spheres in Y is similar.)

Proposition 4. *Let the functionf $: X \longrightarrow Y$ and the (nonempty) open part D of X be such that*

(C_1) $\mathrm{gr}\,(f) \cap (\mathrm{cl}\,(D) \times Y)$ *is closed*

$$(C_2)\quad \begin{cases} \text{for each } a \in D \text{ there exists an open neighbourhood } U \text{ of it} \\ \text{with } \mathrm{cl}\,(U) \subseteq D \text{ and a } \lambda > 0 \text{ with } Y(0,\lambda) \subseteq \mathrm{cl}\, K_f(x)(X(0,1)), \\ \text{for all } x \in U. \end{cases}$$

Then, $f(D)$ is open (in Y).

Proof. Take some point $a \in D$ and let U, λ be as above. Put

$$Z(U) = \bigcap_{x \in U} \{f(x) + \mathrm{cl}\, K_f(x)(X)\}.$$

For the arbitrary fixed $x \in U$ we have the homogeneity property

$$K_f(x)(\gamma u) = \gamma K_f(x)(u),\quad \gamma > 0,\ u \in X.$$

So, taking (C_2) into account

$$(3.1)\qquad \mathrm{cl}\, K_f(x)(X) = Y, \text{ for all such } x.$$

This firstly gives

$$Z(U) = \bigcap_{x \in U} \{f(x) + Y\} = Y; \tag{3.2}$$

and secondly, again combining with (C_2),

$$\begin{aligned} &Y(0,\lambda) \cap K_f(x)(X) \subseteq Y(0,\lambda) \cap \mathrm{cl}\, K_f(x)(X) = \\ &Y(0,\lambda) \subseteq \mathrm{cl}\, K_f(x)(X(0,1)), \qquad x \in U. \end{aligned} \tag{3.3}$$

Hence, by the result in U r s e s c u [22], we have a conclusion like

$$\begin{cases} Y(f(x), \lambda\varepsilon) \subseteq f(X(x,\varepsilon)), \text{ for each } x \in U \text{ and each } \varepsilon > 0 \\ \text{with } X(x,\varepsilon) \subseteq U. \end{cases} \tag{3.4}$$

We may now complete the argument as follows. Let $b \in f(D)$ be arbitrary fixed; hence

$$b = f(a), \text{ for some } a \in D.$$

Let the neighbourhood U of a with $\mathrm{cl}(U) \subseteq D$ and the number $\lambda > 0$ be as in (C_2); and let $\varepsilon > 0$ be such that $X(a,\varepsilon) \subseteq U$. We have, by (3.4),

$$f(D) \supseteq f(X(a,\varepsilon)) \supseteq Y(f(a), \lambda\varepsilon) = Y(b,\varepsilon),$$

wherefrom $b \in \mathrm{int}\, f(D)$.

As b was arbitrary in $f(D)$, the proof is complete.

This result may be viewed as a partial extension of the one due to C r a m e r and R a y [6] expressed in terms of *Gâteaux derivatives*

$$x \vdash Df(x) \quad (\text{from } X \text{ to } L(X,Y)).$$

(Since the used conditions are very similar to the above ones, we do not give further details.)

A common particular case of these is the statement in this area due to G r a v e s [12], expressed in terms of *Fréchet derivatives*

$$x \vdash df(x) \quad (\text{from } X \text{ to } L(X,Y)).$$

Precisely, we have

Proposition 5. *Let the functionf $: X \longrightarrow Y$ and the (nonempty) open part D of X be such that*

(C_3) *f is of class C^1 on D*

(C_4) *$df(x)$ is surjective, for each $x \in D$.*

Then, $f(D)$ is open (in Y).

Now, it is also possible to deduce such a property via accretivity methods. Let X be a Banach space, and $\varphi : R_+ \longrightarrow R_+$, a function. We say that the map $f : X \longrightarrow X$ is *φ-accretive* over a subset D of X, when

$$(D_3) \quad \begin{cases} \text{for each } x,y \in D,\ x \neq y, \text{ there exists } j = j(x,y) \in J(x-y) \\ \text{such that } < f(x) - f(y), j > \geq \varphi(\|x-y\|)\|x-y\|. \end{cases}$$

(Here, $< \cdot,\cdot >$ is the *pairing* between X and X^* (the topological dual of X) and J is the *normalized duality mapping* of the couple (X, X^*).) Now, we have (cf. M o r a l e s [16])

Proposition 6. *Let the function $f : X \longrightarrow Y$ and the (nonempty) open part D of X be such that*

(C_5) *f is continuous on D*

$$(C_6) \quad \begin{cases} f \text{ is } \varphi-\text{accretive on } D, \text{ for some function} \\ \varphi : R_+ \longrightarrow R_+ \text{ with } \liminf_{r \longrightarrow \infty} \varphi(r) > 0. \end{cases}$$

Then, $f(D)$ is open (in X).

This result extends some contributions in this area due to K i r k [14]. For other aspects we refer to R a y and W a l k e r [19].

Further, another way of deducing opness properties is the one based on analyticity concepts. Let $m \geq 1$ be a natural number. Take some function $f = (f_1, ..., f_m)$ from C^m to itself. We call it *analytic* over the (nonempty) part D of C^m, provided

$$(D_4) \qquad \frac{Df}{Dz}(a) = \left(\frac{\partial f_i}{\partial z_j}(a)\right) \text{ exists, for each } a = (a_1, ..., a_m) \text{ in } D.$$

The following statement is to be noted (cf. H e r v é [13, ch.4, Section 5]):

Proposition 7. *Let the function $f : C^m \longrightarrow C^m$ and the (nonempty) open part D of C^m be such that*

(C_7) *f is analytic over D*

(C_8) *each $a \in D$ is an isolated point of $f^{-1}(f(a))$*

Then, $f(D)$ is open (in C^m).

Remark. When $m = 1$, conclusion above is retainable if (C_8) is replaced by the weaker condition

$$(C_9) \qquad f \text{ is non-constant over } D.$$

See, in this direction, Stoilow [20, ch.2, Section 3]. But, when $m > 1$, this substitution cannot produce, in general, the conclusion above, as simple examples show.

Finally, it is possible to deduce such a property via topological methods. Let $n \geq 1$ be a natural number and $f = (f_1, ..., f_n)$ a function from R^n to itself. We have (cf. Deimling [8, ch.1, Section 4]):

Proposition 8. *Suppose that the function $f : R^n \longrightarrow R^n$ and the (nonempty) open part D of R^n are such that*

(C_{10}) *f is continuous on D*
(C_{11}) *f is locally injective on D.*

Then, $f(D)$ is open (in R^n).

Let again X, Y be a couple of topological spaces and $f : X \longrightarrow Y$, a function. We say that it is *closed*, when

(D_5) *the image $f(F)$ of each closed part F of X is closed part of Y.*

It is our aim in this last part of the present section to make a few remarks about this property. Let g be a multifunction from X to Y. Call this object *upper semicontinuous* (u.s.c.) at some point $a \in \text{dom}(g)$, provided

$$(D_6) \qquad \begin{cases} \text{for each neighbourhood } V \text{ of } G(a) \text{ there exists some} \\ \text{neighbourhood } U \text{ of } a \text{ with } g(U) \subseteq V. \end{cases}$$

(See Engelking [10, ch.1, Section 4] for some basic facts about this notion.) The following result is to be noted (cf. Kuratowski [15, ch. 1, Section 14]):

Proposition 9. *Let the function $f : X \longrightarrow Y$ and the (nonempty) closed part F of X be such that*

(C_{12}) *f is closed (in Y)*

$$(C_{13}) \quad \begin{cases} \text{the inverse multifunction } f^{-1} \text{ from } Y \text{ to } X \\ \text{is u.s.c. at each point of } f(X) \setminus f(F). \end{cases}$$

Then $f(F)$ is closed (in Y).

Proof. Let y be some point of $f(X) \setminus f(F)$. Hence

$$(\Phi \neq) f^{-1}(y) \subseteq X \setminus F \quad \text{(an open part of } X).$$

By (C_{13}), there must be some neighbourhood V of y with

$f^{-1}(V) \subseteq X \setminus F$; wherefrom,
$V \subseteq f(X) \setminus f(F)$ (by the definition of f^{-1}).

As y was arbitrarily chosen with such a property, we derive

$$f(X) \setminus f(F) \quad \text{is open (in } Y).$$

Combining with (C_{12}) gives

$$Y \setminus f(F) = (Y \setminus f(X)) \cup (f(X) \setminus f(F)) \quad \text{is open (in } Y)$$

and the proof is complete.

Further, let X, Y be a couple of Banach spaces and $\varphi : R_+ \longrightarrow R_+$ a function. We say that the map $f : X \longrightarrow Y$ is *φ-expansive* over a subset F of X, provided

$$(D_7) \qquad \|f(x) - f(y)\| \geq \varphi(\|x - y\|), \quad x, y \in F, \ x \neq y.$$

Proposition 10. *Let the function $f : X \longrightarrow Y$ and the (nonempty) closed part F of X be such that*

$(C_1)'$ $\text{gr}(f) \cap (F \times Y)$ *is closed*

$$(C_{14}) \quad \begin{cases} f \text{ is } \varphi\text{-expansive on } F, \text{ for some function} \\ \varphi : R_+ \longrightarrow R_+ \text{ with } \liminf_{r \to a} \varphi(r) > 0, \ 0 < a \leq \infty. \end{cases}$$

Then, $f(F)$ is closed (in Y).

Proof. Let y be some adherence point of $f(F)$; that is

$$f(x_n) \longrightarrow y, \quad \text{for some sequence } (x_n) \text{ in } F.$$

In particular, $(f(x_n))$ is a Cauchy sequence (in Y). On the other hand, by the choice of φ

$$(3.5) \qquad \begin{cases} \text{for each } \varepsilon > 0, \text{ there exists } \partial = \partial(\varepsilon) > 0 \text{ such that} \\ \varphi(r) < \partial \Longrightarrow r < \varepsilon. \end{cases}$$

(For, otherwise, there would be some $\varepsilon > 0$ and some sequence (r_n) in R_+ with

$$\varphi(r_n) \longrightarrow 0 \quad \text{and} \quad r_n \geq \varepsilon, \quad \text{for all} \quad n.$$

So that (passing to a subsequence if necessary) there exists a in $]0, \infty]$ with

$$\liminf_{r \to a} \varphi(r) = 0, \quad \text{contradiction}).$$

And this immediately gives (x_n) is also a Cauchy sequence (in X); so (by the closedness of F)

$$x_n \longrightarrow x, \quad \text{for some} \quad x \in F.$$

But then $(C_1)'$ yields

$$y = f(x) \quad (\text{that is, } y \in f(F)).$$

Hence the conclusion.

In particular, a sufficient condition for (D_7) is (D_3). Some related aspects may be found in B r o w d e r [4].

4. Main results. We are now in position to give an appropriate answer to the question of the introductory part. Let X be a topological space and Y a normed space. Denote, for each $A \subseteq X$,

$$\operatorname{qbd}(A) = \operatorname{cl}(A) \setminus A$$

and call it the *quasi-boundary* of A. Our first main result is

Theorem 2. *Let the function $f : X \longrightarrow Y$ and the (nonempty) part D of X be such that*

(H_1) *$f(D)$ is algebraically open (in Y)*

(H_2) *$f(\operatorname{cl}(D))$ has the minimal norm property*

(H_3) $\begin{cases} \text{for each } x \in \operatorname{qbd}(D) \text{ there exists } u \in \operatorname{cl}(D) \\ \text{with } \|f(u)\| < \|f(x)\|. \end{cases}$

Then, f has at least one zero in D.

Proof. Denote for simplicity

$$E = f(D), \quad G = f(\operatorname{qbd}(D)).$$

By the immediate relation

$$E \cup G = f(D \cup \operatorname{qbd}(D)) = f(\operatorname{cl}(D))$$

it follows (from $(H_1) - (H_3)$ and the developments in Section 2) that conditions $(A_9) - (A_{11})$ are fulfilled for (E, G) (in the normed space Y). Hence, Theorem 1 applies to these data. And, from this, conclusion is clear.

Now, a useful particular case of this statments is related to the obvious relation

$$\operatorname{qbd}(D) = \operatorname{bd}(D), \quad \text{when} \quad D \quad \text{is open} \quad (\text{in } X).$$

Precisely, our second main result is

Theorem 3. *Let the function $f : X \longrightarrow Y$ and the (nonempty) open part D of X be such that (H_2) is fulfilled, as well as*

(H_4) *$f(D)$ is open (in Y)*

(H_5) *$\inf\{\|f(x)\|;\ x \in \operatorname{bd}(D)\} > \|f(x)\|$, for some $u \in D$.*

Then, $f(x) = 0$ for at least one x in D.

The obtained results have, until now, a theoretical relevance only. So, in order to confer them a practical relevance too, we must combine these with the developments of the preceding sections. Let X and Y be a couple of Banach spaces. As a direct consequence of Theorem 2 one has (via Proposition 2):

Corollary 1. *Let the function $f : X \longrightarrow Y$ and the (nonempty) part D of X be such that $(H_1) - (H_3)$ are valid, as well as either*

(H_6) *$f(\operatorname{cl}(D))$ is convex and weakly compact*

when Y is not reflexive, or

$(H_6)'$ *$f(\operatorname{cl}(D))$ is convex and closed*

when Y is reflexive. Then, f has at least one zero in D.

Another consequence of Theorem 2 is to be formulated under the lines of Proposition 3. Precisely, we have

Corollary 2. *Let the function $f : X \longrightarrow Y$ and the nonempty part D of X with the property (H_1) and*

(H_7) *$f(\operatorname{cl}(D))$ is closed* *be such that, for at least one equivalent norm $\|\cdot\|_1$ over Y taken as in Proposition 3 (for $E = f(\mathrm{cl}\,(D))$ it is the case that*

$$(H_3)' \quad \begin{cases} \text{for each } x \in \mathrm{qbd}\,(D) \text{ there exists } u \in \mathrm{cl}\,(D) \\ \text{with } \|f(u)\|_1 < \|f(x)\|_1. \end{cases}$$

Then, f has at least a zero in D.

Now, in a similar way, one may state particular versions of Theorem 3 based on Propositions 5 and 6; the corresponding statement of these being evident, we omit the details. It would be more profitable for us giving some results in this area based on Proposition 7. So, let $m \geq 1$ be a natural number. The following consequence of Theorem 3 is to be noted.

Corollary 3. *Let the function $f : C^m \longrightarrow C^m$ and the (nonempty) open relatively compact part D of C^m be such that $(C_7) + (C_8) + (H_5)$ are valid, as well as*

(H_8) *f is continuous on $\mathrm{cl}\,(D)$.*

Then, $f(x) = 0$ for at least one $x \in D$.

As already precised, in the case $m = 1$ the conclusion of this statement is retainable with (C_8) replaced by the weaker condition (C_9). Concerning the specific assumption (H_5), the coercivity approach is often useful. We shall illustrate this by the following counterpart of the result above.

Corollary 4. *Let the function $f : C^m \longrightarrow C^m$ be such that*

(H_9) *f is continuous and open*

(H_{10}) $\lim\limits_{\|z\| \to \infty} \|f(z)\| = \infty.$

Then, $f(x) = 0$ for at least one $z \in C^m$.

Proof. By (H_{10}), there must be some $\rho > 0$ such that

$$\|f(z)\| > \|f(0)\| + 1, \quad \text{whenever} \quad \|z\| \geq \rho.$$

So, if we put

$$D = C^m(0, \rho) \quad (\text{hence } \mathrm{cl}\,(D) = C^m[0, \rho])$$

condition (H_5) is fulfilled. Moreover, by (H_9), conditions (H_2)+(H_4) are also fulfilled, by this D. Hence, Theorem 3 applies to our data and conclusion follows.

It is to be remarked that the regularity properties of (H_9) are independent each other; see E n g e l k i n g [10, ch.1, Section 4] for details.

Sufficient conditions for the openess portion of this condition were precised in the statement above. Note that, in case $m = 1$ these reduce to

(H_{11}) *f is analytic and nonconstant on C^m;*

and, the obtained result may be viewed as an extension of the classical d'Alembert–Gauss theorem.

Finally, another consequence of Theorem 3 is to be formulated under the lines of Proposition 8. Precisely, let $n \geq 1$ be a natural number. We have:

Corollary 5. *Let the function $f : R^n \longrightarrow R^n$ and the (nonempty) open relatively compact part D of R^n be such that $(C_{11}) + (C_5) + (H_8)$ are valid. Then, $f(x) = 0$ for at least one x in D.*

Now, a counterpart of Corollary 4 is directly available in this new setting. We shall give here an interesting particular version of it which reads as follows.

Corollary 6. *Let the function $f : R^n \longrightarrow R^n$ be such that*

(H_{12}) *f is continuous and bounded*
(H_{13}) *$g = f - i$ is open (i = the identity).*

Then, $f(x) = x$ for at least one $x \in R^n$.

Sufficient conditions for (H_{13}) were precised by the developments above. The obtained fixed point statement may be compared with the classical Brouwer's one.

5. An application. The corollaries we just stated may be viewed as an indication of the potential usefulness of the main result subsumed to Theorems 2 and 3. For, as already precised, these include some well known facts involving functions of complex and real variables. It is our aim in the following to show that these may be also useful in the treatment of converse mean value properties.

Let C stand for the set of complex numbers. Fix some (nonempty) open relatively compact part D of C and take a function $g : C \times C \longrightarrow C$. From a number of reasons to be explained further, it will be useful for us having a non-diagonal zero for g in $\mathrm{cl}\,(D) \times \mathrm{cl}\,(D)$; that is

(NE) $\qquad g(u, v) = 0, \quad \text{for some } u, v \in \mathrm{cl}\,(D), \ u \neq v.$

In this case as a direct application of the developments in the preceding section (related to complex variable functions), the following solvability result about (NE), is available.

Theorem 4. *Suppose that the function* $g : C\times C \longrightarrow C$ *and the point* $a \in D$ *are such that*

(K_1) $z \vdash |g(z,a)|$ *is lower semicontinuous on* $\mathrm{bd}\,(D)$

(K_2) $\left\{\begin{array}{l} v \vdash |g(z,v)| \textit{ is lower semicontinuous at } v=a, \\ \textit{uniformly with respect to } z\in \mathrm{bd}\,(D)\end{array}\right.$

(K_3) $\left\{\begin{array}{l} z \vdash g(z,z) \textit{ is continuous at } z=a, \textit{ (with } g(a,a)=0) \\ \textit{and not identical zero in each neighbourhood of } a\end{array}\right.$

(K_4) $\left\{\begin{array}{l} z \vdash g(z,v) \textit{ is non} - \textit{constant analytic on } D \textit{ and continuous} \\ \textit{on } \mathrm{cl}\,(D), \textit{for each } v \textit{ in } D\setminus\{a\} \textit{ sufficiently close } a.\end{array}\right.$

Then, either

$$g(z,a)=0 \quad \textit{for some} \quad z\in \mathrm{bd}\,(D) \tag{5.1}$$

or else (in the opposite case)

$$g(d,b)=0, \quad \textit{for some} \quad d,b\in D,\ d\neq b. \tag{5.2}$$

(Hence, the question above has a positive answer.)

Proof. Suppose that the former of these alternative is not true; i.e.,

(K_5) $g(z,a)\neq 0$, *for all* $z\in \mathrm{bd}\,(D)$.

Denote for simplicity

$$\mu=\frac{1}{3}\inf\{|g(z,a)|;\ z\in \mathrm{bd}\,(D)\}.$$

Clearly, $\mu>0$; this results by (K_1), the compactness of $\mathrm{cl}\,(D)$ and (K_5). On the other hand, by (K_2), there exists some $\partial>0$ such that $C(a,\partial)\subseteq D$ and

$$|g(u,w)|>|g(u,a)|-\mu,\ u\in \mathrm{bd}\,(D),\ w\in C(a,\partial).$$

Further, by the definition of μ,

$$|g(u,a)|\geq 2\mu>\mu, \quad u\in \mathrm{bd}\,(D),\ w\in C(a,\partial). \tag{5.3}$$

Finally, by (K_3), there exists some point b in $C(a,\partial)$, distinct from a, with

$$0<|g(b,b)|<\mu.$$

Hence (taking (5.3) into account)

$$0<|g(b,b)|<\inf\{|g(u,b)|;\ u\in \mathrm{bd}\,(D)\}. \tag{5.4}$$

We may now complete the argument as follows.
Let the function $f : C\longrightarrow C$ be introduced as

$$f(z)=g(z,b),\ z\in C.$$

By (K_4) and (5.4) (the second part), Corollary 3 applies (with $m=1$) to the couple (f,D). So, by its conclusion,

$$g(d,b)=0, \quad \text{for at least one} \quad d\in D.$$

Moreover, $d\neq b$, as results from (5.4) (the first part). Hence (5.2) must be true, as claimed.

Now, a basic particular case of (K_1) is that characterized as

$(K_1)'$ $z\vdash g(z,a)$ is continuous on $\mathrm{bd}\,(D)$;

note at this moment that both $(K_1)'$ and (K_4) are deductible from the global hypothesis

(K_6) $\left\{\begin{array}{l} z \vdash g(z,v) \textit{ is non} - -\textit{constant analytic on } D \\ \textit{and continuous on } \mathrm{cl}\,(D), \textit{ for each } v \textit{ in } D.\end{array}\right.$

Further, a basic particular case of (K_2) is the one expressed by

$(K_2)'$ $z\vdash g(z,v)$ *is continuous at* $v=a$, *uniformly over* $z\in \mathrm{bd}\,(D)$.

It remains now to discuss the regularity condition (K_3). The following particular case of these developments may be viewed as an argument for its consistency. Let $f : C\longrightarrow C$ be a function. Denote

$$Q_f(u,v)=\frac{f(u)-f(v)}{u-v},\quad u,v\in C,\ u\neq v.$$

This will be referred to as the *difference quotient* of f at $\{u,v\}$.

Corollary 7. *Let the function* $f : C\longrightarrow C$ *and the (nonempty) open part* D *of* C *be such that*

(K_7) f *is analytic on* D.

Then, for each $a\in D$ *it is the case that*

$$f'(a)=Q_f(u,v), \quad \textit{for some} \quad u=u(a),\ v=v(a) \textit{ in } D \textit{ with } u\neq v. \tag{5.5}$$

Proof. Let $a \in D$ and take some $\sigma > 0$ with $C[a,\sigma] \subseteq D$. For each ρ in $]0,\sigma]$ put

$$D_\rho = C(a,\rho) \quad (\text{hence } \operatorname{cl}(D_\rho) = C[a,\rho]).$$

Two alternatives are now open before us

Case 1. There exists ρ in $]0,\sigma]$ such that

(K_8) $f'(z) = f'(a)$, *for all* $z \in D_\rho$.

Then, f is affine over D_ρ and (5.5) becomes an identity (on the same subset).

Case 2. The opposite situation holds; i.e.,

(K_9) *for each* ρ *in* $]0,\sigma]$ *there exists* z_ρ *in* D_ρ *with* $f'(z_\rho) \neq f'(a)$.

Let the function $g : C \times C \longrightarrow C$ be given as

$$\begin{aligned} g(u,v) &= Q_f(u,v) - f'(a), && u,v \in D,\ u \neq v \\ &= f'(v) - f'(a), && u,v \in D \\ &= \text{arbitrary}, && \text{in rest.} \end{aligned}$$

We claim that all conditions of Theorem 4 are fulfilled for the triplet (g, D_σ, a). In fact, $(K_2)'$ is immediate, by the continuity properties of the difference quotient Q_f; and, from this, (K_2) is clear (cf. the remark above). Further, take some v in D and put

$$h_v(z) = g(z,v), \quad z \in C.$$

The analyticity of h_v over $D_\sigma \setminus \{v\}$ is clear, by definition; and, its corresponding at v property results at once from the relation

$$h_v'(v) = \frac{1}{2} f''(v)$$

deduced by a Taylor development procedure. The continuity of h_v over $\operatorname{cl}(D_\sigma)$ is again clear, by the continuity properties of the difference quotient Q_f we already evoked. And, an assertion like

(K_{10}) h_v *is constant over* D_σ.

gives (by the very definition of this function)

$$f \ \text{ is affine over } \ D_\sigma$$

in contradiction to (K_9). Hence, (K_6) is true; and, from this, $(K_1)'$ and (K_4) follow. Finally, the regularity condition (K_3) results at once from (K_7) and (K_9). Summing up, the claim about (g, D_σ, a) is proved. But, in this case, relations (5.1)+(5.2) of Theorem 4 give at once the conclusion we need. This ends the argument.

By convention, (5.5) will be referred to as a *reciprocal mean value property* for f over the (neonempty) open subset D. This result was established for the first time by P o m p e i u [18]; but the methods he used are different from ours. For other aspects we refer to the survey paper of B a n t a ş, D r u ţ u and T u r i n i c i [2] as well as the references therein.

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THE NULL CONTROLLABILITY OF THE POPULATION DYNAMICS WITH DIFFUSION AND INHOSPITABLE BOUNDARY

BY

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1. Introduction and problem formulation. Consider a biological population which is free to move in a bounded and open subset of $\boldsymbol{R}^N$ ($N \geq 1$), with sufficiently smooth boundary $\partial\Omega$. Suppose that the boundary of the environment is extremely inhospitable. If we denote by $y(a,x,t)$ the density of population of age $a \in [0, A]$, in the position $x \in \overline{\Omega}$ and at the moment $t \in \boldsymbol{R}^+$, then the dynamics of the population is described by the following system:

$$(1.1)\quad \begin{cases} y_t + y_a + \mu(a)y - \Delta_x y = u(a,x,t) & \text{on } (0,A)\times\Omega\times(0,\infty) \\ y(a,x,t) = 0 & \text{on } (0,A)\times\partial\Omega\times(0,\infty) \\ y(0,x,t) = \int_0^A b(a)y(a,x,t)da & \text{on } \Omega\times(0,\infty) \\ y(a,x,0) = y_0(a,x) & \text{on } (0,A)\times\Omega,\ \ A\in(0,\infty), \end{cases}$$

The biological significances of the terms in (1.1) are as follows:

i) μ is the rate of mortality;
ii) b is the rate of fertility;
iii) y_0 is the initial density of population;
iv) u is a possible infusion or harvest of population in Ω.

Equation $(1.1)_2$ is the mathematical formulation of the completely inhospitability of the boundary and $(1.1)_3$ (the so called "renewal equation") describes the birth.

For a more detailed description of the biological significance of the model we refer to [1], [6] and [7].