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ON ASYMPTOTIC BEHAVIOUR OF NEUTRAL INTEGRODIFFERENTIAL EQUATIONS

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Introduction. Oscillations of delay differential equations have been extensively discussed by numerous authors. However oscillatory behaviour of integrodifferential equations especially those with infinite delay has received little attention in the literature.

In population dynamic models of integrodifferential equations, oscillation of model systems has certain ecological and evolutionary regnificance ([1-11]).

In this paper, we examine a necessary and sufficient condition for all solutions of the neutral integrodifferential equations of the form

$$\frac{d}{dt}\left[x(t) - c\int_0^\infty H(s)x(t-s)ds\right] + a\int_0^\infty k(s)x(t-s)ds + bx(t) = 0 \tag{1.1}$$

to have at least one zero on $(-\infty, +\infty)$ under the assumptions:

$$\int_0^\infty k(s)ds = 1, \quad \int_0^\infty sk(s)ds < \infty, \quad K : [0,\infty) \longrightarrow [0,\infty) \tag{A}$$

and

$$\int_0^\infty H(s)ds < \infty, \quad \int_0^\infty sH(s)ds < \infty, \quad H : [0,\infty) \longrightarrow [0,\infty);$$

(B) a, b and c are real positive constants such that

$$c \int_0^\infty H(s)ds < 1.$$

Definition 1.1. *Solutions of (1.1) which have at least one zero on $R = (-\infty, +\infty)$ are said to have "zero crossings". On the other hand if there is a solution x of (1.1) such that either $x(t) > 0$ on R or $x(t) < 0$ on R, then such a solution is said to have no "zero crossing", (sometimes these solutions are said to stay away from zero). For literature related to stability characteristic of neutral integrodifferential equations we refer to K o l m a n o v s k i i and N o s o v [7].*

Our condition is based on the nature of the roots of the characteristic equation associated with (1.1) which is

$$\lambda\left(1 - c\int_0^\infty H(s)e^{-\lambda s}ds\right) + a\int_0^\infty k(s)e^{-\lambda s}ds + b = 0 \tag{1.2}$$

In our study we follow the analysis of G o p a l s a m y [2,3] and generalizing the results in [3] to the equation (1.1).

2. Zero–Crossings. We shall derive a necessary and sufficient condition for all nontrivial solutions of (1.1) to have "zero crossings" (i.e. have at least one zero on R). Our condition is based on the nature of the roots of (1.2).

We need the following lemma

Lemma 2.1. *Suppose $K : [0, \infty) \longrightarrow [0, \infty)$, $K \neq 0$ on some suinterval of $[0, \infty)$; let a, b be positive constants and $0 \leq c < 1$. If (1.2) has not real roots, then there exists a positive number M such that,*

$$a\int_0^\infty k(s)e^{-\lambda s} + b > \lambda\left(1 - c\int_0^\infty H(s)e^{-\lambda s}ds\right) + M \tag{2.1}$$

where $\lambda \in R$, $\int_0^\infty cH(s)ds < 1$, $\int_0^\infty K(s)ds = 1$.

Proof. Define F as follows:

$$F(\lambda) = \lambda\left(1 - c\int_0^\infty H(s)e^{-\lambda s}ds\right) + a\int_0^\infty k(s)e^{-\lambda s}ds + b. \tag{2.2}$$

We note that the equation $F(\lambda) = 0$ will be supplemented with nonnegative initial value on $(-\infty, 0]$ such that $x(0) > 0$. Now, from (2.2) we obtain:

$$\left.\begin{array}{l} F(0) = a\int_0^\infty k(s)ds + b > 0 \\ \lim\limits_{\lambda \longrightarrow \pm\infty} F(\lambda) = \infty \\ F(\lambda) = 0 \quad \text{has no real root} \end{array}\right\} \Longrightarrow \inf_{\lambda \in R} F(\lambda) > 0$$

If $\inf_{\lambda \in R} F(\lambda) = 0$, then there exists a sequence $\{\lambda_n\} \subset R$, $|\lambda_n| < \infty$ such that $F(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since the sequence $\{\lambda_n\}$ is bounded, there exists a convergent subsequence, say $\{\lambda_{n_k}\}$, such that

$$\lambda_{n_k} \longrightarrow \lambda^* \text{ and } F(\lambda_{n_k}) \longrightarrow 0 \text{ as } n_k \longrightarrow \infty.$$

Since F is continuous in λ, it will follow that

$$F(\lambda_{n_k}) \longrightarrow F(\lambda^*) = 0 \text{ as } n_k \longrightarrow \infty,$$

and hence λ^* is a real root of F which is contradiction. Thus there exists a positive number M such that,

$$F(\lambda) = \lambda\left(1 - c\int_0^\infty H(s)e^{-\lambda s}ds\right) + a\int_0^\infty k(s)e^{-\lambda s}ds + b > M, \ \lambda \in R, \tag{2.3}$$

from which the result follows.

Theorem 2.2. *Let $a, b \in (0, \infty), c \in [0, 1), H : [0, \infty) \longrightarrow [0, \infty)$ be such that $c\int_0^\infty H(s)ds < 1$ and that $k(s), H(s)$ are eventually nonincreasing, $H, k \not\equiv 0$ on $[0, \infty)$. A necessary and sufficient condition for all nontrivial solution of (1.1) to have "zero crossing" is that the characteristic equation (1.2) has no real root.*

Proof. The necessity of the condition is easily seen; for instance if $F(\lambda) = 0$ has a real root, say $\mu \in (-\infty, \infty)$, then (1.1) has a solution of the form,

$$x(t) = Ae^{\mu t}, \quad A \in (-\infty, \infty), \tag{2.4}$$

which has not zero crossings. Thus the necessity of the condition follows.

The sufficiency part of the condition is proved as follows: we shall assume that (1.1) has a solution y which is strictly positive on $(-\infty,\infty)$ and then show that this will lead to a certain contradiction. The technique is similar to those used in the case of neutral differential equation with infinite delays.

Let y be a positive solution of (1.1) on $(-\infty,\infty)$. Define a sequence $\{z_n\}$ as follows:

$$(2.5)\qquad \begin{cases} z_0(t)=y(t)-c\int_0^\infty H(s)y(t-s)ds,\\ z_1(t)=z_0(t)-c\int_0^\infty H(s)z_0(t-s)ds,\\ \vdots\\ z_{n+1}(t)=z_n(t)-c\int_0^\infty H(s)z_n(t-s)ds.\end{cases}$$

It can be verified that (2.5) and (1.1) imply

$$(2.6)\qquad \begin{cases} \dot z_1(t)=-a\int_0^\infty k(s)z_0(t-s)ds-bz_0(t),\\ \dot z_2(t)=-a\int_0^\infty k(s)z_1(t-s)ds-bz_1(t),\\ \vdots\\ \dot z_n(t)=-a\int_0^\infty k(s)z_{n-1}(t-s)ds-bz_{n-1}(t),\ n=1,2\dots,\end{cases}$$

where $\dot z_0(t)=-a\int_0^\infty k(s)y(t-s)ds-by(t)$.

Since y is a positive solution of (1.1), we have from (1.1) that $y(t)-c\int_0^\infty H(s)y(t-s)ds$ is decreasing as t increases. Thus if $y(t)-c\int_0^\infty H(s)y(t-s)ds$ becomes zero or negative for some $t=t_0$, then for all $t>t_0$ we will have, $y(t)-c\int_0^\infty H(s)y(t-s)ds>0$.

As a consequence there exists a $\delta>0$ such that, $y(t)-c\int_0^\infty H(s)y(t-s)ds<-\delta$, for $t-t_1=t_0+1$, and therefore for $c\int_0^\infty H(s)ds<1$, we have, $y(t)<$

$$-\delta+c\int_0^\infty H(s)y(t-s)ds<-\delta+c\int_0^\infty H(s)\left[-\delta+c\int_0^\infty H(\eta)y(t-2\eta)d\eta\right]ds<$$

$$-\left(\frac{\delta}{1-c\int_0^\infty H(s)ds}\right)+y(t_0)e^{n\ln\left(c\int_0^\infty H(s)ds\right)},\quad t=ns+t_0,\ \text{if } t \text{ is large enough,}$$

then it will follow from $c\int_0^\infty H(s)ds<1$ and the above that,

$y(t)\le \frac{\delta}{1-c\int_0^\infty H(s)ds}$, which is impossible. Thus we conclude that,

$$y(t)-c\int_0^\infty H(s)y(t-s)ds>0,\ \text{for } t\in(-\infty,\infty).$$

Now, we note from,

$$(2.7)\qquad \dot z_{n+1}(t)=-a\int_0^\infty k(s)z_n(t-s)ds-bz_n(t),$$

that

$$\dot z_n(t)-c\int_0^\infty H(s)z_n(t-s)ds=-a\int_0^\infty k(s)z_n(t-s)ds-bz_n(t)$$

or

$$(2.8)\qquad \dot z_n(t)-c\int_0^\infty H(s)z_n(t-s)ds+a\int_0^\infty k(s)z_n(t-s)ds+bz_n(t)=0,\ n=0,1,2,\dots$$

Consider now the set γ_n of real numbers defined by

$$(2.9)\qquad \gamma_n=\{\lambda\ge 0|\ \dot z_n(t)+\lambda z_n(t)\le 0 \text{ eventually for } t>0\}.$$

Now we show that the existence of a positive solution of (1.1) implies that the set γ_n has the following contradiction

(i) γ_n is bounded ,
(ii) for $\lambda\in\gamma_n \Longrightarrow \lambda+M\in\gamma_n$.

Clearly $\lambda = 0 \in \gamma, \gamma_n$ is nonempty, $\gamma_n \in [0, \infty)$.

To establish (i), we have to show the existence of an upper bound of γ_n. Now integrating both sides of (2.7) on $[t-\alpha, t]$:

$$z_n(t) - z_n(t-\alpha) + a \int_{t-\alpha}^{t} \left(\int_0^\infty k(s) z_{n-1}(u-s) ds \right) du + b \int_{t-\alpha}^{t} z_{n-1}(s) ds = 0,$$

implying that

$$\int_{t-\alpha}^{t} \left[a \int_0^\infty k(s) z_{n-1}(u-s) ds + b z_{n-1}(u) \right] du = z_n(t-\alpha) - z_n(t) \leq z_n(t-\alpha),$$

$$\int_{t-\alpha}^{t} \left[a \int_0^\infty k(s) z_{n-1}(u-s) ds + b z_{n-1}(u) \right] du = z_n(t-\alpha)$$

or

$$a\alpha \int_0^\infty z_{n-1}(t-s) k(s) ds + b\alpha z_{n-1}(t) \leq z_n(t-\alpha),$$

for all $t > 0$ large enough.

Hence

$$a \int_0^\infty z_{n-1}(t-s) k(s) ds \leq \frac{1}{\alpha} z_n(t-\alpha) - b z_{n-1}(t). \tag{2.10}$$

Now we have $z_n(t-\alpha)$ by choose positive numbers α, β such that $\int_\alpha^\beta k(s) ds > 0$, $k \neq 0$ on some interval $[0, \infty)$, we have from (2.8)

$$0 = \dot{z}_n(t) - c \int_0^\infty H(s) \dot{z}_n(t-s) ds + a \int_0^\infty k(s) z_n(t-s) ds + b z_n(t)$$

$$\geq \dot{z}_n(t) - c \int_0^\infty H(s) \dot{z}_n(t-s) ds + a \int_\alpha^\beta k(s) z_n(t-s) ds + b z_n(t),$$

or

$$\dot{z}_n(t) + a \int_\alpha^\beta k(s) z_n(t-\alpha) ds + b z_n(t) \leq \dot{z}_n(t) + a \int_\alpha^\beta k(s) z_n(t-s) ds + b z_n(t) \leq 0.$$

Since z_n is decreasing, we have,

$$\dot{z}_n(t) + a \int_\alpha^\beta k(s) z_n(t-\alpha) ds + b z_n(t) \leq 0.$$

If we take $a \int_\alpha^\beta k(s) ds = A$, then

$$\dot{z}_n(t) + A z_n(t-\alpha) + b z_n(t) \leq 0. \tag{2.11}$$

Integrating both sides of the inequality (2.11) on $[t - \frac{\alpha}{2}, t]$ we obtain

$$z_n(t) - z_n\left(t - \frac{\alpha}{2}\right) + A \int_{t-\frac{\alpha}{2}}^{t} z_n(s-\alpha) ds + b \int_{t-\frac{\alpha}{2}}^{t} z_n(s) ds \leq 0.$$

Hence

$$z_n(t) - z_n\left(t - \frac{\alpha}{2}\right) + A z_n(t-\alpha)\frac{\alpha}{2} + b\frac{\alpha}{2} z_n(t) \leq 0, \tag{2.12}$$

or

$$\frac{\alpha}{2} A z_n(t-\alpha) \leq \frac{\alpha}{2} A z_n(t-\alpha) + \frac{\alpha}{2} b z_n(t) \leq z_n\left(t - \frac{\alpha}{2}\right) - z_n(t) \leq z_n\left(t - \frac{\alpha}{2}\right)$$

$$\Longrightarrow \frac{\alpha}{2} A z_n(t-\alpha) \leq z_n\left(t - \frac{\alpha}{2}\right).$$

Similarly, if we integrate (2.11) on $[t, t + \frac{\alpha}{2}]$ we obtain,

$$A \frac{\alpha}{2} z_n\left(t - \frac{\alpha}{2}\right) \leq z_n(t) \tag{2.13}$$

From (2.12), (2.13), we obtain,

$$z_n(t-\alpha) \leq \frac{4}{(A\alpha)^2} z_n(t).$$

Substituting in (2.10), we get

$$a\int_0^\infty z_{n-1}(t-s)k(s)ds \le \frac{1}{\alpha}z_n(t-\alpha) - bz_{n-1}(t),$$

or

$$a\int_0^\infty z_{n-1}(t-s)k(s)ds + bz_{n-1}(t-\alpha) \le \frac{1}{\alpha}\frac{4}{(A\alpha)^2}z_n(t). \tag{2.14}$$

Then from (2.7) and (2.14) we have,

$$0 = \dot z_n(t) + a\int_0^\infty k(s)z_{n-1}(t-s)ds + bz_{n-1}(t)$$

$$< \dot z_n(t) + \frac{1}{\alpha}\frac{4}{(A\alpha)^2}z_n(t),$$

but $\frac{4}{\alpha(A\alpha)^2} = \lambda > 0$, i.e., which show that $\lambda \notin \gamma_n$. γ_n is bounded. Then (i) is true. Now we prove part (ii). Let $\{\phi_n\}$ be the sequence defined by

$$\phi_n(t) = e^{\lambda t}z_n(t),\ \lambda \in \gamma_n, \tag{2.15}$$

Then we have,

$$\dot\phi_n(t) = e^{\lambda t}[\dot z_n(t) + \lambda z_n(t)] \le 0. \tag{2.16}$$

Since

$$\dot z(t) + \lambda z_n(t) \le 0,$$

then

$$\dot\phi_n(t) \le 0,$$

i.e., ϕ_n is non increasing. We have from (2.7), (2.15), (2.16) and lemma 2.1:

$$\dot z_{n+1}(t) + (\lambda+M)z_{n+1}(t) =$$

$$= -a\int_0^\infty k(s)z_n(t-s)ds - bz_n(t) + (\lambda+M)\left[z_n(t) - c\int_0^\infty H(s)z_n(t-s)ds\right] =$$

$$= -a\int_0^\infty k(s)e^{-\lambda(t-s)}\phi_n(t-s)ds - be^{-\lambda t}\phi_n(t) + (\lambda+M)\left[e^{-\lambda t}\phi_n(t) - \right.$$

$$\left. - c\int_0^\infty H(s)e^{-\lambda(t-s)}\phi_n(t-s)ds\right] \le e^{-\lambda t}[-a\int_0^\infty k(s)e^{\lambda s}\phi_n(t)ds - b\phi_n(t) +$$

$$+(\lambda+M)\phi_n(t) - (\lambda+M)c\int_0^\infty H(s)e^{\lambda s}\phi_n(t)ds] =$$

$$= e^{-\lambda t}\phi_n(t)\left[-a\int_0^\infty k(s)e^{\lambda s}ds - b + (\lambda+M) - (\lambda+M)c\int_0^\infty e^{\lambda s}H(s)ds\right]$$

≤ 0. Hence

$$\dot z_{n+1}(t) + (\lambda+M)z_{n+1}(t) < e^{-\lambda t}\phi_n(t)\cdot$$

$$\cdot\left[-a\int_0^\infty k(s)e^{\lambda s}ds - b + (\lambda+M)\left(1 - c\int_0^\infty H(s)e^{\lambda s}ds\right)\right] \le 0.$$

Therefore $(\lambda+M) \in \gamma_n$ which establishes (ii). Since (i), (ii) together cannot hold for γ_n, we have a contradiction. Thus (1.1) cannot have a positive solution on $(-\infty,\infty)$; since (1.1) is linear, in a similar way it cannot have a negative solution on $(-\infty,\infty)$. This completes the proof.

Corollary 2.3. *Assume that $a, b, c, k(s), H(S)$ are as in Theorem 1.2. If,*

$$a\int_0^\infty sk(s)ds > \frac{1}{e}\min\left[\left(1 - c\int_0^\infty H(s)ds\right), \left(1 - 2bce\int_0^\infty sH(s)ds\right)\right] \tag{2.17}$$

then all nontrivial solutions of (1.1) have zero crossing.

Proof. Suppose that (1.1) has a solution without zero crossing on $(-\infty,\infty)$. Then by Theorem 1.2, the characteristic equation of (1.1) has a real root; that is,

$$F(\lambda) = \lambda(1 - c\int_0^\infty H(s)e^{-\lambda s}ds) + a\int_0^\infty k(s)e^{-\lambda s}ds + b = 0. \tag{2.18}$$

has a real root. Since,

$$F(0) = a\int_0^\infty k(s)ds + b > 0,\ F(\lambda) > 0 \text{ for } \lambda \geq 0, \tag{2.19}$$

the real root has to be negative. Let $\lambda = -\mu$, $\mu > 0$ be such a root; then μ satisfies,

$$\mu\left(1 - c\int_0^\infty H(s)e^{\mu s}ds\right) = a\int_0^\infty k(s)e^{\mu s}d(s) + b, \tag{2.20}$$

or

$$\mu\left(1 - c\int_0^\infty H(s)ds\right) \geq \mu\left(1 - c\int_0^\infty H(s)e^{\mu s}ds\right) = \mu a\int_0^\infty k(s)s\frac{e^{\mu s}}{\mu s}ds + b,$$

or

$$1 - c\int_0^\infty H(s)ds \geq ea\int_0^\infty sk(s)ds,$$

or

$$a\int_0^\infty sk(s)ds \leq \frac{1}{e}\left(1 - c\int_0^\infty H(s)ds\right).$$

But this contradicts (2.17), so

$$a\int_0^\infty sk(s)ds > \frac{1}{e}\left(1 - c\int_0^\infty H(s)ds\right),$$

We also have from (2.20) that,

$$\mu = \mu c\int_0^\infty H(s)e^{\mu s}ds + a\int_0^\infty k(s)e^{\mu s}ds + b. \tag{2.21}$$

Thus

$$\mu > b \text{ and } \mu > \mu c\int_0^\infty H(s)e^{\mu s}ds.$$

From (2.21) we get,

$$\mu > bc\int_0^\infty H(s)e^{\mu s}ds + b + a\int_0^\infty k(s)e^{\mu s}ds.$$

Thus

$$1 > bc\int_0^\infty sH(s)\frac{e^{\mu s}}{\mu s}ds + \frac{b}{\mu} + a\int_0^\infty sk(s)\frac{e^{\mu s}}{\mu s}ds$$

$$> bec\int_0^\infty sH(s)ds + ae\int_0^\infty sk(s)ds + \frac{bc}{\mu}\int_0^\infty H(s)e^{\mu s}ds.$$

Then

$$1 > bce\int_0^\infty sH(s)ds + ae\int_0^\infty sk(s)ds + bc\int_0^\infty sH(s)\frac{e^{\mu s}}{\mu s}ds$$

$$> 2bce\int_0^\infty sH(s)ds + ae\int_0^\infty sk(s)ds,$$

or

$$a\int_0^\infty sk(s)ds \leq \frac{1}{e}\left(1 - 2bce\int_0^\infty sH(s)ds\right),$$

but this contradicts (2.17), and the conclusion is

$$a\int_0^\infty sk(s)ds > \frac{1}{e}(1 - 2bce\int_0^\infty sH(s)ds)$$

hold, and this completes the proof.

3. An application to level crossing. We shall now consider briefly the nonlinear logistic integrodifferential equation

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{1}{c}\left(a\int_0^\infty k(s)N(t-s)ds + bN(t)\right)\right], \tag{3.1}$$

where $a, b, c, r \in (0, \infty)$ and

$$\begin{cases} N(s) = \phi(s) \geq 0, \; s \in (-\infty, 0]; \; \phi(0) > 0; \\ k : [0, \infty) \longrightarrow [0, \infty); \; \int\limits_0^\infty k(s)ds = 1 \end{cases} \tag{3.2}$$

We derive sufficient conditions for all positive solutions of (3.1) to cross the positive equilibruim level in the sense that there exists a $t^* \in R$ for which

$$N(t^*) = \frac{c}{a+b}; \quad a > 0, \; b > 0. \tag{3.3}$$

Solutions satisfying (3.3) can be called equilibruim or "level-crossing" for short. The integrodifferential equation (3.1) represents a generalization of the familiar delay equation,

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{1}{c}(ax(t-\tau) + bx(t))\right],$$

with a finite discrete delay τ.
In the case $b = 0$, equation (3.1) represents a generalization of the familiar delay logistic equation,

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{1}{c}x(t-\tau)\right].$$

The following result shows that if the linear variational system corresponding to the positive equilibrium of (3.1) has zero crossings, then the nonlinear system (3.1) has level crossings.

Theorem 3.1. *Let $r, a, b, c, \in (0, \infty)$; suppose k is not identically zero on some subinterval of $[0, \infty)$ and that k is eventually nonincreasing.*
If

$$\frac{r}{a+b}\left[a\int\limits_0^\infty k(s)e^{\lambda s}ds + b\right] > \lambda \; \text{for} \; \lambda \in (0, \infty), \tag{3.4}$$

then all positive solutions of

$$\dot{N}(t) = rN(t)\left[1 - \frac{1}{c}\left(a\int\limits_0^\infty k(s)N(t-s)ds + bN(t)\right)\right] \tag{3.1}$$

have level crossings.

Proof. First we note that every solution of (3.1) and (3.2) satisfies,

$$N(t) > 0 \text{ for } t > 0. \tag{3.5}$$

We let

$$N(t) = \frac{c}{a+b}[1 + u(t)] \tag{3.6}$$

in (3.1) and derive that u is governed by

$$\dot{u}(t) = r(1 + u(t))\left[1 - \frac{a}{a+b}\int\limits_0^\infty k(s)(1 + u(t-s))ds - \frac{b}{a+b} - \frac{bu(u)}{a+b}\right].$$

Since $\int\limits_0^\infty k(s)ds = 1$, then,

$$\dot{u}(t) = \frac{-r}{a+b}(1 + u(t))\left[a\int\limits_0^\infty k(s)u(t-s)ds + bu(t)\right]. \tag{3.7}$$

It is easily seen from (3.5) and (3.6) that the problem of level crossing of N about $\frac{c}{a+b}$ is equivalent to that of "zero crossing" of u, where u satisfies (3.7).

As in section 2, one can show the existence of a positive number M such that,

$$\frac{r}{a+b}\left(\int\limits_0^\infty ak(s)e^{\lambda s}ds + b\right) \geq \lambda + M \text{ for } \lambda \in R. \tag{3.8}$$

Now suppose that (3.7) has a solution without zero crossing; for instance let $u(t) > 0$ for $t \in R$. It follows from (3.7) that u is decreasing on R and therefore,

$$u(t) \longrightarrow l \geq 0 \;\; \text{as } t \longrightarrow \infty.$$

One can show that $l = 0$, since otherwise u can become negative eventually for large t, contradicting the positively of u on R. Similarly one can show that if (3.7) has a solution u such that $u(t) < 0$ on R, then u will be nondecreasing since $(1 + u(t) > 0)$ on R, implying that,

$$u(t) \longrightarrow l \leq 0 \text{ as } t \longrightarrow \infty,$$

and one can show that $l = 0$.

We now rewrite (3.7) in the form

$$\frac{du(t)}{dt} + p(t)\left[\int_0^\infty ak(s)u(t-s)ds + bu(t)\right] = 0, \tag{3.9}$$

where

$$\begin{cases} P(t) = \frac{r}{a+b}(1+u(t)) > 0 \text{ on } R, \\ P(t) \longrightarrow \frac{r}{a+b} \text{ as } t \longrightarrow \infty, \end{cases} \tag{3.10}$$

whenever $u(t)$ denotes a solution of (3.9) without zero crossing. Suppose (3.7) has a solution u satisfying $u(t) > 0$ on R.
Define a set γ_u as follows:

$$\gamma_u = \{\lambda \geq 0 | \; \dot{u}(t) + \lambda u(t) \leq 0\} \tag{3.11}$$

eventually for all large t. It is easily seen that $\lambda = 0 \in \gamma_u$ and that γ_u is a subinterval of $[0, \infty)$.
The rest of the proof is accomplished by showing the set γ_u has contradictory properties P_1 and P_2:
$P_1 : \gamma_u$ is bounded
$P_2 : \lambda \in \gamma_u \Rightarrow \lambda + M \in \gamma_u$ where M is as in (3.8).
Define ϕ as follows:

$$\phi(t) = e^{\lambda t}u(t), \; \lambda \in \gamma_u \tag{3.12}$$

We note from (3.21) and (3.11) that

$$\dot{\phi}(t) = e^{\lambda t}[\dot{u}(t) + \lambda u(t)] \leq 0, \text{ eventually} \tag{3.13}$$

and so ϕ is nonincreasing eventually and ϕ is positive. We have from (3.9),

$$\dot{u}(t) + (\lambda + M)u(t) = -p(t)[\int_0^\infty ak(s)u(t-s)ds + bu(t)] + (\lambda + M)u(t)$$

$$= -p(t)\int_0^\infty ak(s)e^{-\lambda(t-s)}\phi(t-s)ds - p(t)be^{-\lambda t}\phi(t) + (\lambda + M)e^{-\lambda t}\phi(t)$$

$$= e^{-\lambda t}\left[-p(t)\int_0^\infty ak(s)e^{\lambda s}\phi(t-s)ds - p(t)b\phi(t) + (\lambda + M)\phi(t)\right]$$

$$\leq e^{-\lambda t}\left[-p(t)\phi(t)a\int_0^\infty k(s)e^{\lambda s}ds - bp(t)\phi(t) + (\lambda + M)\phi(t)\right]$$

$$= \phi(t)e^{-\lambda t}\left[-p(t)a\int_0^\infty k(s)e^{\lambda s}ds + (\lambda + M) - bp(t)\right]$$

$$= \phi(t)e^{-\lambda t}\left[-p(t)\left(a\int_0^\infty k(s)e^{\lambda s}ds + b\right) + (\lambda + M)\right]$$

$$\leq e^{-\lambda t}\phi(t)\left[-(\liminf_{t\to\infty} p(t)\left(a\int_0^\infty k(s)e^{\lambda s}ds + b\right) + (\lambda + M)\right]$$

$$\leq e^{-\lambda t}\phi(t)\left[\frac{-r}{a+b}\left(a\int_0^\infty k(s)e^{\lambda s}ds + b\right) + (\lambda + M)\right] \leq 0,$$

showing that $\lambda + M \in \gamma_u$ and hence the validity of P_2 follows, which completes the proof.

Remark. Our results, in this paper namely Corollary 2.3 and Theorem 3.1 are generalizations to the corresponding results in [2] & [3], which can be obtained from ours by taking $b = 0$.

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CONE VALUED LYAPUNOV FUNCTIONS AND LIPSCHITZ STABILITY OF CERTAIN NONLINEAR PERTURBED SYSTEMS

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1. Introduction. The method of Lyapunov functions offers a very flexible and effective mechanism in the study of qualitative properties of nonlinear differential systems [8, 9, 10, 11, 13, 14]. However, the limitation of the method led to the use of arbitrary cones rather than the standard cone R^n_+ which is utilized in the method of vector Lyapunov funcions [7]. Besides, it is now well known that making use of cone valued Lyapunov functions is beneficial in applications [1, 6, 7, 12].

The concept of Lipschitz stability in differential equations was introduced by Dannan and Elaydi [4, 5]. In this paper, we study Lipschitz stability using the technique of cone valued Lyapunov functions. In section 3, we obtain a necessary and sufficient conditions for ϕ_0-uniform Lipschitz stability while in section 4, our result is applied to obtain sufficient conditions for the Lipschitz stability of certain nonlinear perturbed systems. In the special case when the cone is the usual cone in R^n, we deduce new results in terms of vector Lyapunov functions.

2. Notations and definitions. Consider the differential system

$$x' = f(t,x), \qquad x(t_0) = x_{o'} \tag{1}$$

where $f \in C(R_+ \times R^n, R^n)$, $R_+ = [0,\infty)$, $f(t,0) = 0$ and $x(t) = x(t,t_0,x_0)$ is the solution of (1) with $x(t_0,t_0,x_0) = x_0$, and $t_0 \geq 0$.

Let $K \subseteq R^n$ be a nonempty cone, that is, K is closed, convex with $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$ with interior $K^o \neq \emptyset$.