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Received: 20.I.1995

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Tomul XLII, s.I.a, Matematică, 1996, f1.

CONE VALUED LYAPUNOV FUNCTIONS AND LIPSCHITZ STABILITY OF CERTAIN NONLINEAR PERTURBED SYSTEMS

BY

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1. Introduction. The method of Lyapunov functions offers a very flexible and effective mechanism in the study of qualitative properties of nonlinear differential systems [8, 9, 10, 11, 13, 14]. However, the limitation of the method led to the use of arbitrary cones rather than the standard cone R^n_+ which is utilized in the method of vector Lyapunov funcions [7]. Besides, it is now well known that making use of cone valued Lyapunov functions is beneficial in applications [1, 6, 7, 12].

The concept of Lipschitz stability in differential equations was introduced by Dannan and Elaydi [4, 5]. In this paper, we study Lipschitz stability using the technique of cone valued Lyapunov functions. In section 3, we obtain a necessary and sufficient conditions for ϕ_0–uniform Lipschitz stability while in section 4, our result is applied to obtain sufficient conditions for the Lipschitz stability of certain nonlinear perturbed systems. In the special case when the cone is the usual cone in R^n, we deduce new results in terms of vector Lyapunov functions.

2. Notations and definitions. Consider the differential system

$$x' = f(t,x), \qquad x(t_0) = x_o, \tag{1}$$

where $f \in C(R_+ \times R^n, R^n)$, $R_+ = [0,\infty)$, $f(t,0) = 0$ and $x(t) = x(t,t_0,x_0)$ is the solution of (1) with $x(t_0,t_0,x_0) = x_0$, and $t_0 \geq 0$.

Let $K \subseteq R^n$ be a nonempty cone, that is, K is closed, convex with $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$ with interior $K^o \neq \emptyset$.

For any $x, y \in R^n$, we let $x \geq y$ iff $y - x \in K$ and for any functions $u, v : R_+ \to R^n, u \geq v$ iff $u(t) \geq v(t)$ on R_+. Let $K^* = \{\varphi \in R^n : (\varphi, x) \geq 0$ for all $x \in K\}$ and $K_0^* = K^* \setminus \{0\}$.

Definition 2.1. *A function $g : R^n \to R^n$ is sait to be quasi monotone nondescreasing relative to the cone $K \subseteq R^n$ if there exists a $\varphi \in K_0^*$ such that $x \geq y$ and $(\varphi, y - x) = 0$ implies $(\varphi, g(y) - g(x)) \geq 0$.*

Consider the comparison system

$$u' = g(t, u), \qquad u(t_0) = u_0 \tag{2}$$

where $g \in C(R_+ \times K, R^n)$. Let $u(t) = u(t, t_0, u_0)$ be any solution of the system (2) such that $u(t_0, t_0, u_0) = u_0$.

Definition 2.2. *A generalized norm from R^n to R_+^s is a mapping $\|\cdot\|_G : R^n \to R_+^s$ defined by $\|x\|_G = (\alpha_1(x), \ldots.., \alpha_n(x))$ such that*

(i) $\|x\|_G \geq 0$ *(i.e $\alpha_i(x) \geq 0$ for $i = 1, 2, 3, \ldots.s.$),*

(ii) $\|x\|_G = 0$ *iff* $x = 0$,

(iii) $\|\lambda x\|_G = |\lambda| \, \|x\|_G$,

(iv) $\|x + y\|_G \leq \|x\|_G + \|y\|_G$.

If $R^n = R_+^s = R_+^n$, then we have a generalized norm $\|u\|_{\widehat{G}}$ for $u \in R_+^n$ defined by $\|u\|_{\widehat{G}} = (|u_1|, \ldots., |u_n|) = (u_1, \ldots\ldots, u_n)$. Define $S_\rho^* = \{x \in R^n : \|x\|_{\widehat{G}} < \rho, \rho \in K\}$ and $S^*(\rho) = \{u \in K : \|u\|_{\widehat{G}} < \rho, \rho \in K\}$.

Definition 2.3. *The differential system (1) is said to be φ_0-uniformly Lipschitz stable if $\exists\ M \geq 1, \delta > 0$ and $\varphi_0 \in K_0^*$ such that $(\varphi_0, x(t, t_0, x_0)) \leq M(\varphi_0, x - o)$ for $(\varphi_0, x_0) < \delta$ and $t \geq t_0$.*

3. Converse theorem. In this section, we give necessary and sufficient conditions for wich we have φ_0-uniform Lipschitz stanility of the system (1) by the method of cone valued Lyapunov functions.

Theorem 3.1. *Assume that $f \in C(R_+ \times K, R^n)$, $f(t, 0) = 0$ and for each $t \in R_+, \|f(t, x) - f(t, y)\|_{\widehat{G}} \underset{R}{\leq} L(t)\|x - y\|_{\widehat{G}}$. Then the zero solution of (1) is φ_0-uniformly Lipschitz stable if and only if there exists a cone valued Lyapunov function $V(t, x)$ such that*

(I) $V \in C(R_+ \times S^*(\rho), K$, $V(t, 0) = 0$ and locally Lipschitzian in x relative to K for each $t \in R_+$,

(II) $(\varphi_0, \|x\|_{\widehat{G}} \leq (\varphi_0, V(t, x)) \leq (\varphi_0, L\|x\|_{\widehat{G}})$, for $\varphi_0 \in K_0^*$ and some constant $L \geq 1$,

(III) $D^+V(t, x) \underset{R}{\leq} 0$.

Proof. We show that the zero solution of (1) is φ_0-uniformly Lipschitz stable if (I), (II), and (III) hold. Clearly, we have $(\varphi_0, \|x\|_{\widehat{G}}) \leq (\varphi_0, V(t, x(t, t_0, x_0)) \leq (\varphi_0, V(t_0, x_0)) \leq (\varphi_0, L\|x_0\|_{\widehat{G}}) \leq L(\varphi_0, \|x_0\|_{\widehat{G}})$, which is the φ_0-uniform Lipschitz stability of then zero solution of (1). Conversely if the zero solution of (1) is φ_0-uniformly Lipschitz stable, define a cone valued function V as follows:

$$V(t, x) = \sup_{\tau \geq 0}\{\|x(t + \tau, t, x)\|_{\widehat{G}}(1 + e^{-\tau - t})\}.$$

$V(t, 0) = 0, \tau = 0$ and by uniqueness of solutions, $\|x\|_{\widehat{G}} \underset{R}{\leq} V(t, x)$. Hence we can find $\varphi_0 \in K_0^*$ such that $(\varphi_0, \|x\|_{\widehat{G}}) \leq (\varphi_0, V(t, x))$.
Besides,

$$(\varphi_0, V(t, x)) = \sup_{\tau \geq 0}\{(1 + e^{-t-\tau})(\varphi_0, \|x(t + \tau, t, x)\|_{\widehat{G}})\}$$

$$\leq \sup_{\tau \geq 0}\{(\varphi_0, M\|x_0\|_{\widehat{G}})(1 + e^{-\tau - t})\} \leq (\varphi_0, 2M\|x_0\|_{\widehat{G}}) = (\varphi_0, L\|x_0\|_{\widehat{G}}).$$

Since f is locally Lipschitzian in x relative to K and uniformly in t, there exists $L = L(M, \delta)$ such that $\|x(t + \tau, t, x) - y(t + \tau, t, y)\|_{\widehat{G}} \underset{R}{\leq} e^{L\tau}\|x - y\|_{\widehat{G}}$, Lemma 4.7 in [1], for $\|x\|_{\widehat{G}} < \delta, \|y\|_{\widehat{G}} < \delta$ and $\tau \geq 0$. Now, $(\varphi_0\|x(t+\tau, t, x)\|_{\widehat{G}}) \leq (\varphi_0, M\|x\|_{\widehat{G}}) \leq (\varphi_0, M\delta)$ and $(\varphi_0, \|y(t+\tau, t, x)\|_{\widehat{G}}) \leq (\varphi_0, M\|y\|_{\widehat{G}}) \leq (\varphi_0, M\delta)$, hence if we set $M = e^T$, then $\sup_{\tau \geq 0}\{\|x(t + \tau, t, x)\|_{\widehat{G}}(1 + e^{-\tau - t})\} \leq \sup_{\tau \geq 0}\{\|x\|_{\widehat{G}}(e^T + e^{T-(t+\tau)})\}$. Then sup is attained if $0 \leq t + \tau \leq T$ in the case $t < T$ and if $\tau = 0$ in the case $t \geq T$. Therefore, for $\|x\|_{\widehat{G}} < \delta$ and $\|y\|_{\widehat{G}} < \delta$,

$$\|V(t, x) - V(t, y)\|_{\widehat{G}} \underset{R}{\leq} \sup_{\tau \geq 0}\{\|s(t + \tau, t, x) - y(t + \tau, t, y)\|_{\widehat{G}}(1 + e^{-t-\tau})\}$$

$$\underset{R}{\leq} \sup_{\tau}\{e^L\|x - y\|_{\widehat{G}}(1 + e^{-t-\tau})\},$$

where $0 \leq t + \tau \leq T$ if $t < T$ and $\tau = 0$ if $t \geq T$. Hence, $\|V(t, x) - V(t, y)\|_{\widehat{G}} \leq R\|x - y\|_{\widehat{G}}$. Also $\|V(t + h, x + y) - V(t, x)\|_{\widehat{G}} \underset{R}{\leq} \|V(t + h, x + y) - V(t + h, x))\|_{\widehat{G}} + \|V(t + h, x) - V(t + h, x(t + h, t, x))\|_{\widehat{G}} +$
$+ \|V(t + h, x(t + h, t, x)) - V(t, x)\|_{\widehat{G}}$.

$V(t,x)$ is locally Lipschitzian in x relative to K and x is continous, then the first two terms can be made small if y and h are small. As $h \to 0$, the third term tends to zero, hence $V(t,x)$ is continuous in all its arguments. Finally,

$$D^+V(t,x) = \limsup_{h\to 0} \frac{1}{h}\{V(t+h, x(t+h,t,x)) - V(t,x)\}$$

$$= \limsup_{h\to 0}\{\sup_{\tau\geq 0}(\|x(t+h+\tau,t,x)\|_{\hat{G}}(1+e^{-t-\tau-h}))$$

$$- \sup_{\tau\geq 0}(\|x(t+\tau,t,x)\|_{\hat{G}}(1+e^{-t-\tau})\}$$

$$= \limsup_{h\to 0}\{\sup_{\tau\geq h}\|x(t+\tau,t,x)\|_{\hat{G}}(1+e^{-t-\tau}) - \sup_{\tau\geq 0}\|x(t+\tau,t,x)\|_{\hat{G}}(1+e^{-t-\tau}0\} \leq_K 0$$

and the proof is complete.

Remark 3.2. If $K = R^n_+$ and $\varphi = (1,1,\ldots,1)$ then we have a special case of Theorem 3.1. In this case we have the method of vector Lyapunov functions for a necessary and sufficient condition for the uniform Lipschitz stability of the zero solution of (1), which is also a new result.

4. Application of Results. In this section we apply the result of the last section to the nonlinear perturbed system

$$y' = f(t,y) + H(t,y), \quad y(t_0) = y_0 \quad t \geq 0 \tag{3}$$

where $f, H \in C(R_+ \times S^*(\rho), R^n)$ and $f(t,0) = H(t,0) = 0$.

The following result gives sufficient conditions for the φ_0-uniform Lipschitz stability of the perturbed system (3).

Theorem 4.1. *Let the zero solution of (1) be φ_0-uniformly Lipschitz stable with Lipschitz constant M and there exists $g \in C(R_+ \times K, K)$ such that $g(t,0) = 0$ and $g(t,u)$ is quasimonotone nondecreasing in u relative to K for each $t \in R_+$ and such that*

$$\|H(t,y)\|_{\hat{G}} \leq_K Mg(t, \|y\|_{\hat{G}}).$$

If the zero solution of

$$u' = Mg(t,u),\ u(t_0) = u_0 \geq 0, \tag{4}$$

is φ_0-uniformly Lipschitz stable, then the zero solution of (3) is also φ_0-uniformly Lipschitz stable.

Proof. The zero solution of (4) is φ_0-uniformly Lipschitz stable, so by Theorem 3.1, there exists a cone valued Lyapunov function $V(t,u)$ with properties (I), (II) and (III). Therefore,

$$D^+_{(3)}V(t,y) \leq_K Mg(t, \|y\|_{\hat{G}}).$$

Let $r(t,t_0,u_0)$ be the maximal solution of (4) existing for $t \geq t_0$, then if $(\varphi_0, \|y_0\|_{\hat{G}}) < \frac{\delta}{LM}$, and $y(t,t_0,y_0)$ is any solution of (3), Theorem 1.3.4. in [11] imply

$$V(t, y(t,t_0,y_0)) \leq_K r(t,t_0,u_0), \quad t \geq t_0, \tag{5}$$

if we set $u_0 = V(t,y_0)$. Hence property (II) of Theorem 3.1 and (5) imply $(\varphi_0, \|y(t,t_0,y_0)\|_{\hat{G}}) \leq (\varphi_0. V(t, y(t,t_0,y_0)) \leq (\varphi_0, r(t,t_0,u_0))$. Since (4) is φ_0-uniformly Lipschitz stable, then $(\varphi_0, r(t,t_0,u_0)) \leq L(\varphi_0,u_0)$ if $(\varphi_0, \|y_0\|_{\hat{G}}) < \frac{\delta}{LM}$, and

$$(\varphi_0, \|y(t,t_0,y_0)\|_{\hat{G}}) \leq L(\varphi_0,u_0) \leq L(\varphi_0, M\|y\|_{\hat{G}}) = M_1(\varphi_0, \|y\|_{\hat{G}}), t \geq t_0,$$

where $M_1 = LM$. The proof is complete.

Remark 4.2. The case $K = R^n_+$ and $\varphi_0 = (1,1,\ldots,1)$ reduces to a special case of Theorem 4.1 and it is the method of vector Lyapunov functions for uniform Lipschitz stability wich is also new.

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Received: 15.III.1995

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TWO–POINT TAYLOR SERIES FOR THE NUMERICAL SOLUTION OF TWO–POINT BOUNDARY VALUE PROBLEMS

BY

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1. Introduction. In this paper we consider the two-point boundary value problems:

$$L(y) = f(x, y), \qquad x \in R \tag{1.1}$$

$$B(y) = \beta, \qquad x \in \partial R \tag{1.2}$$

where L denotes a general differential operator involving spatial derivatives of y, B represents the appropriate number of boundary conditions and R is domain with boundary ∂R.

Problems of the form (1.1) and (1.2) occur in various aspects of engineering, physics and mathematics [5,9]. Various methods, including finite difference methods [5,7,8,1,2] and spline methods [4] to mention a few, have been used to tackle this class of problems.

The purpose of this paper is to obtain a numerical solution using the two-point Taylor series. The method presented will reduce the work involved in the numerical solution of the boundary value problem to solving a system of linear simultaneous equations. The series, involving a set of parameters, is made to satisfy both the differential equation and the associated boundary conditions. By using the same number of points uniformly spread in the range of integration as the number of parameters to be determined, equal number of equations as the parameters to be determined can be obtained. These equations can then be solved iteratively.

2. Description of the method. Let a and b be two distinct points. If $f(x) \in C^{2m}[a, b]$, then $f(x)$ can be approximated by the polynomial