5. D a n n a n, F.M. and E l a y d i, S. – *Lipschitz stability of nonlinear systems of differential equations II, Lyapunov functions*, J. Math. Anal. Appl. 143 (1989) 517–529.
6. K o k s a l, S. and L a k s h m i k a n t h a m, V. – *Higher derivates of Lyapunov functions*. Nonlinear Analysis, (to appear).
7. L a k s h m i k a n t h a m, V. and L e e l a, S. – *Cone value Lyapunov functions*. Nonlinear Analysis, 1 (1977) 215–22.
8. L a k s h m i k a n t h a m, V. and L e e l a, S. – *Differential and integral inequalities*, Vol I. Academic Press, New York. 1969.
9. L a k s h m i k a n t h a m, V., L e e l a, S. and M a r t y n y u k, A.A. – *Stability Analysis of nonlinear systems*, Marcel Dekker Inc. 1989.
10. L a k s h m i k a n t h a m, V., L e e l a, S. and M a r t y n y u k, A.A. – *Practical stability of nonlinear systems*, World Scientific Publications,. Singapore, 1990.
11. L a k s h m i k a n t h a m, V., M a t r o s o v, M. and S i v a s u n d a r a m, S. – *Vector Lyapunov functions and stability analysis of nonlinear systems*, Kluwer Dordrecht, 1991.
12. L a k s h m i k a n t h a m, V. and P a p a g i o r g i o u, N. S. – *Cone Valued Lyapunov functions and stability theory. Nonlinear Amalysis*, 22 (1994) no 3, 381–390.
13. R a j a l a k m y, S. and S i v a s u n d a r a m, S. – *Vector Lyapunov functions and the technique in perturbation theory*, J. Math. Anal. Appl. 164 (1992) 560–570.
14. S i l j a k, D.D. – *Large Scale Dynamical systems*, North Holland, New York, 1978.

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TWO–POINT TAYLOR SERIES FOR THE NUMERICAL SOLUTION OF TWO–POINT BOUNDARY VALUE PROBLEMS

BY

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1. Introduction. In this paper we consider the two-point boundary value problems:

$$L(y) = f(x, y), \qquad x \in R \tag{1.1}$$

$$B(y) = \beta, \qquad x \in \partial R \tag{1.2}$$

where L denotes a general differential operator involving spatial derivatives of y, B represents the appropriate number of boundary conditions and R is domain with boundary ∂R.

Problems of the form (1.1) and (1.2) occur in various aspects of engineering, physics and mathematics [5,9]. Various methods, including finite difference methods [5,7,8,1,2] and spline methods [4] to mention a few, have been used to tackle this class of problems.

The purpose of this paper is to obtain a numerical solution using the two-point Taylor series. The method presented will reduce the work involved in the numerical solution of the boundary value problem to solving a system of linear simultaneous equations. The series, involving a set of parameters, is made to satisfy both the differential equation and the associated boundary conditions. By using the same number of points uniformly spread in the range of integration as the number of parameters to be determined, equal number of equations as the parameters to be determined can be obtained. These equations can then be solved iteratively.

2. Description of the method. Let a and b be two distinct points. If $f(x) \in C^{2m}[a, b]$, then $f(x)$ can be approximated by the polynomial

$P_{2m-1}(x)$ given as

$$(2.1)\qquad P_{2m-1}(x) = (x-a)^m \sum_{k=0}^{m-1} \frac{B_k(x-b)^k}{k!} + (x-b)^m \sum_{k=0}^{m-1} \frac{A_k(x-a)^k}{k!}$$

where

$$(2.2)\qquad A_k = \frac{d^k}{dx^k}\left[\frac{f(x)}{(x-b)^m}\right]_{x=a}, \quad B_k = \frac{d^k}{dx^k}\left[\frac{f(x)}{(x-a)^m}\right]_{x=b}$$

and the remainder is given as

$$(2.3)\qquad f(x) - P_{2m-1}(x) = \frac{f^{(2m)}(\theta)}{(2m)!}(x-a)^m(x-b)^m$$

and $a < \theta < b$. This is the two-point Taylor interpolation as in [6]. For $m = 2$ and $h = b - a$, after some simplifications, we have

(2.4)

$$P_3(x) = \frac{(x-a)^2}{h^2} f(b) + \frac{(x-b)^2}{h^2} f(a) + \frac{(x-b)(x-a)^2}{h^3}[hf'(b) - 2f(b)] +$$

$$+ \frac{(x-a)(x-b)^2}{h^3}[hf'(a) + 2f(a)].$$

Also, we note that

$$(2.5)\qquad P_{2m-1}(a) = f(a), P'_{2m-1}(a) = f'(a), \ldots, P^{(m-1)}_{2m-1}(a) = f^{(m-1)}(a)$$

$$(2.6).\qquad P_{2m-1}(b) = f(b), P'_{2m-1}(b) = f'(b), \ldots, P^{(m-1)}_{2m-1}(b) = f^{(m-1)}(b)$$

It is assumed that an approximation to the solution $y(x)$ of (1.1) and (1.2) is an expression of the form in (2.1) which depends on a number of parameters $A_{k,s}$ and $B_{k,s}$. The polynomial $P_{2m-1}(x)$ is made to satisfy equations (1.1) and (1.2).

3. The Error Estimate. From equations (2.5) and (2.6), we anticipate a result of the form

$$(3.1)\qquad y(x) - P_{2n-1}(x) = C(x-a)^n(x-b)^n$$

where C is a constant to be determined. Define the function

$$(3.2)\qquad F(x) = y(x) - P_{2n-1}(x) - C(x-a)^n(x-b)^n.$$

Then

$$(3.3)\qquad F(a) = F'(a) = f''(a) = \ldots = F^{n-1}(a) = 0.$$

$$(3.4)\qquad F(b) = F'(b) = F''(b) = \ldots = F^{n-1}(b) = 0.$$

Since $F(a) = F(b) = 0$, applying Rolle's theorem shows that $F'(x)$ has three zeros, one between $x = a$ and $x = b$ and two other at the two end points. Successive applications of Rolle's theorem now show that $F''(x), F'''(x), \ldots, F^{(n-1)}(x)$ has $4, 5 \ldots n+1$ zeros in [a,b]. Continuing in this manner we see that $F^{(n)}(x), F^{(n+1)}(x), F^{(n+2)}(x), \ldots, F^{(2n-1)}(x)$ have $n, n-1, n-2, \ldots$ zeros in [a,b] and $F^{(2n)}(\theta)$ has at least one zero in [a,b] such that $a < \theta < b$. In this case

$$(3.5)\qquad F^{(2n)}(\theta) = y^{(2n)}(\theta) - C(2n)! = 0.$$

Thus

$$(3.6)\qquad C = \frac{y^{(2n)}(\theta)}{(2n)!}$$

Substituting back into equation (3.1) gives

$$y(x) - P_{2n-1}(x) = \frac{y^{(2n)}(\theta)}{(2n)!}(x-a)^n(x-b)^n.$$

4. Numerical examples. To illustrate the method and to demonstrate its accuracy we will consider the following examples:

1. Solve the differential equation

$$(4.1)\qquad y'' = (1+x^2)y,$$

subject to the boundary conditions

$$(4.2)\qquad y(0) = 1, \qquad y(1) = e^{\frac{1}{2}}.$$

The exact solution is $y(x) = e^{x^2/2}$. The function $y(x)$ is approximated by

$$y(x) \approx (x-a)^m y(b) + (x-b)^m y(a) + \tag{4.3}$$

$$+(x-a)^m \sum_{k=0}^{m-1} \frac{B_k (x-b)^k}{k!} + (x-b)^m \sum_{k=0}^{m-1} A_k \frac{(x-a)^k}{k!}$$

where $a = 0$ and $b = 1$. With $m = 2$, we have

$$y(x) = x^2 e^{\frac{1}{2}} B_0 + (x-1)^2 A_0 + x^2(x-1)B_1 + (x-1)^2 x A_1.$$

Substituting back into equation (4.1) gives

$$2 - (1+x^2)(x-1)^2 + (2-(1+x^2)x^2)e^{\frac{1}{2}} + (6x - 2 - (1+x^2)(x-1)x^2)B_1$$

$$+(6x - 4 - (1+x^2)(x-1)^2 x)A_1 = 0.$$

With $x = 0, \frac{1}{2}, \frac{2}{3}$ and $x = 1$, we have after some simplifications, $A_0 = B_0 = 1$ and

$$A_1 - 0.03802281 B_1 = 2.125127196$$

$$A_1 - 20.6923078 B_1 = 38.11838565;$$

giving

$$A_1 = -1.74265332 \text{ and } B_1 = 2.05886660.$$

It is observed that the accuracy improves as m increases. The results are shown in Table 1 below.

Table 1.

m	Max. Error
2	6.349×10^{-3}
3	6.372×10^{-4}
4	3.829×10^{-6}
6	3.350×10^{-7}

2. Solve the differential equation

$$y'' = y - 4xe^x \tag{4.4}$$

subject to the mixed boundary conditions

$$y'(0) - y(0) = 1, \; y'(1) + y(1) = -e. \tag{4.5}$$

The exact solution is $y(x) = x(1-x)e^x$. This type of equation requires special treatment because of the nature of the boundary conditions. The polynomial $P_{(2m-1)}(x)$ in as given in (4.3) must be made to satisfy both equations (4.4) and (4.5). It should be pointed out that we need not compute the constants A_k and B_k as given in (2.2), unless when the values of the derivatives are required at the boundary points. The results obtained for $m = 4$ and $m = 6$ are shown in Table 2.

Table 2.

m	Max. Error
4	6.912×10^{-6}
6	5.761×10^{-7}

3. Solve the differential equation

$$y'' = ((2-x)e^{2y} + 1/(1+x))/3, \tag{4.6}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = -\log_e 2. \tag{4.7}$$

The example is from [1] and the exact solution is $y(x) = -\log_e(1+x)$. This is a non-linear equation and so we need to update the value of $P_{2m-1}(x)$ after each circle of iteration to be able to compute e^{2y} in (4.6). The results obtained for $m = 4$ and $m = 6$ are shown in Table 3.

Table 3.

m	Max. Error
4	3.267×10^{-4}
6	1.455×10^{-7}

4. Solve the differential equation

$$y^{iv} = xy - (11 + ax + x^2 - x^3)\exp(x) \tag{4.8}$$

with boundary conditions

$$y(-1) = 0, \; y''(-1) = 2/e, \; y(1) = 0, \; y''(1) = -6e. \tag{4.9}$$

This fourth order differential equation is from [2,5] with the exact solution $y(x) = (1-x)e^x$. As in example 2, the expression (4.3) must satisfy both equations (4.8) and (4.9). The results for $m = 4$ and $m = 6$ are shown in Table 4.

Table 4.

m	Max. Error
4	2.15×10^{-3}
6	4.87×10^{-7}

In all the cases considered the values of the dependent variable y_k are computed at the grid points x_k, where

$$x_k = \frac{(b-a)k}{10}, \quad k = 1(1)11; \tag{4.10}$$

and $x = a$ and $x = b$ are the lower and upper boundary points respectively. The maximum error at the points in each case is as shown in the above tables.

5. Concluding remarks. It is observed that to get a fairly accurate results the value of the parameter m must be greater than the order of the differential equation by at least 2. The accuracy can be improved further by increasing the number of terms in the series, that is the value of m.

The method is very simple to use and it does not involve the computation of any derivative of the dependent variable.

REFERENCES

1. C h a w l a, M.M. – *An Eighth Order Tridiogonal Finite Difference Method for Nonlinear Two-point Boundary value Problems*, BIT 17(1977), 281-285.
2. C h a w l a, M.M. – *Finite Difference Methods For Two-point Boundary Value problems Involving High Order Differential Equations*, BIT 19(1979), 27-33.
3. C h a w l a, M.M. and K a t t s, C.P. – *Finite Difference Methods for A Class Of Two-point Bounadry Value Problems with Mixed Boundary Conditions*, J.Comp. and Appl.Maths.6 (1980), 3, 189- 196.
4. P a p a m i c h a e l, N. and W o r s e y, A.J. – *Boundary Value Problems Occuring in Plate Deflection theory*, J.Comp. and Appl. Maths. 7(1981), 3, 187-189.
5. R a v i P. A g a r w a l l and A k r i v i s, G. – *A Cubic Spline Method for the Solution of a linear Fourth-order Two-point Boundary Value Problem*, J. Comp and Appl. Maths. 8(1982), 3, 145-154.
6. P h i l i p J. D a v i s and P h i l i p R a b i n o w i t z – *Methods of Numerical Integration*, Academic press, New York/San Francisco/London 1975, pp.20.
7. H e n r i c i, P. – *Discrete Variable Methods in Ordinary Differential Equations*. John Wiley, New York 1962.
8. U s m a n i, R.A. – *On the Numerical Integration of a Boundary Value Problem Involving a Fourth order Linear Differential Equation*, BIT 17(1977), 227-234.
9. U s m a n i, R.A. – *An $O(h^6)$ finite difference Analogue for the Solution of Some Differential Equations occuring in Plate-Diflection theory*, J.Inst.Maths Applics. 20(1977), 331- 333.

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