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# ON SOME PROPERTIES OF ALMOST COSYMPLECTIC MANIFOLDS

BY

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**0. Introduction.** In this paper we study some properties of almost cosymplectic manifolds. §1 is devoted to the basic definitions and some preliminary properties of almost cosymplectic manifolds. In §2 ~ §5 we get some interesting results on almost cosymplectic manifolds (cf.[1],[6], [9]). By using some results from §1 to §5, we present §6 and §7 which are the main purposes of this paper, that is, in §6 we determine the curvature tensor of an almost cosymplectic manifold with $\overline{\xi}$ belonging to the $k$-nullity distribution except proper almost cosymplectic manifolds with $k = 0$ and with constant $\overline{\phi}$-sectional curvature. Then we get a generalization of Olszak's theorem ([8], Theorem 6.1). And in §7 we study invariant submanifolds of codimension 2 of almost cosymplectic manifolds, and get, by the way of S m y t h ([10]), a theorem which corresponds to L u d d e n 's theorem ([7], Theorem 4.4).

**1. Almost cosymplectic manifolds.** Let $(M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold, that is, $M$ is a differentiable manifold and $(\phi, \xi, \eta, g)$ an almost contact metric structure on $M$, formed by tensor fiedls $\phi, \xi, \eta$ of type (1,1), (1,0) and (0,1) respectively, and a Riemannian metric $g$ such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1,$$

$$\eta(X) = g(X, \xi), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

On such a manifold we may always define 2-form $\Phi$ by $\Phi(X, Y) = g(\phi X, Y)$. Then $(M, \phi, \xi, \eta, g)$ is said to be an almost cosymplectic manifold if the forms $\Phi$ and $\eta$ are closed, i.e., $d\Phi = 0$, $d\eta = 0$, where $d$ is the operator of exterior differentiation. In particular, if the almost contact structure of an almost cosymplectic manifold is normal, then it is said to be a cosymplectic manifold. As it is known, an almost contact metric structure is cosymplectic if and only if $\nabla\phi$ vanishes, where $\nabla$ is the covariant differentiation with respect to $g$ (see [5], cf.[8]). Hereafter, we simply describe the almost cosymplectic manifold and the cosymplectic manifold by acs-manifold and cs-manifold respectively. On acs-manifolds we define an operator $h$ by $h = -\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then $h$ is symmetric (see [5]). Moreover, in an acs-manifold $M$ we have $\nabla_\xi\xi = 0$, $\nabla_\xi\phi = 0$, $\nabla_X\xi = \phi hX$, $h\xi = 0$, $\phi h + h\phi = 0, Trh = 0$ (where $Trh$ is the trace of $h$), $Ric(\xi,\xi) = g(Q\xi,\xi) = -Trh^2$ and

$$\frac{1}{2}(R(\xi,X)\xi - \phi R(\xi,\phi X)\xi) = h^2X, \tag{1.1}$$

where $R, Ric$ and $Q$ are the Riemannian curvature tensor, the Ricci curvature and the Ricci operator on $M$ respectively (see [5],[8]). Using these equations, in an acs-manifold $M$ we easily get the following:

$$(\nabla_\xi h)X = h^2\phi X - \phi R(\xi,X)\xi. \tag{1.2}$$

In cs-manifolds, we have $R(\phi X,\phi Y)Z = R(X,Y)Z$, $R(X,Y)\xi = 0$, $R(X,Y)\phi Z = \phi R(X,Y)Z$, $\phi Q = Q\phi$, $Q\xi = 0$ ([4]).

We need, in the sequel, the following theorem.

**Theorem A ([8]).** *Acs-manifolds of non-zero constant curvature do not exist in dimension $(2n+1) \geq 5$.*

**2. Locally symmetric acs-manifolds.** At first we prove the following lemma.

**Lemma 2.1.** *If an acs-manifold $M$ is locally symmetric, then $\nabla_\xi h = 0$.*

**Proof.** Differentiating (1.1) and (1.2) covariantly along $\xi$, we obtain $(\nabla_\xi h)^2 = 0$. Since $\nabla_\xi h$ is symmetric, we have $\nabla_\xi h = 0$.

Then we get the following proposition.

**Proposition 2.1.** *Let $M$ be a $(2n+1)$-dimensional $(n > 1)$ locally symmetric acs-manifold such that its sectional curvature $K(\xi,X)$ with respect to a plane section containing $\xi$ equals a constant $C$. Then $K(\xi,X) = 0$ on $M$.*

**Proof.** Since the acs-manifold $M$ is locally symmetric, we get, from Lemma 2.1. and (1.2), $R(\xi,X)\xi = h^2X$. By hypothesis we see that $h^2 = -C(I - \eta\otimes\xi)$, from which $R(\xi,X)\xi = C(-X + \eta(X)\xi)$. Now we assume that $C \neq 0$. Differentiating this equation covariantly along $Y$ and then replacing $Y$ by $\phi Y$, we have $R(hY,X)\xi + R(\xi,X)hY = C(g(hY,X)\xi + \eta(X)hY)$. And, replacing $Y$ by $hY$ in this equation, we get $R(Y,X)\xi + R(\xi,X)Y + 2C\eta(Y)X = C(g(X,Y)\xi + \eta(X)Y)$. Differentiating this gotten equation covariantly along $Z$ and replacing $Z$ by $\phi Z$, we find $R(Y,X)hZ + R(hZ,X)Y + 2Cg(hZ,Y)X = C(g(X,Y)hZ + g(hZ,X)Y)$. Replacing $Z$ by $hZ$, we obtain

$$R(Y,X)Z + R(Z,X)Y = C(g(X,Y)Z + g(Z,X)Y - 2g(Y,Z)X) \tag{2.1}$$

Permuting $X, Y, Z$ cyclically, we have the following two equations:

$$R(Z,Y)X + R(X,Y)Z = C(g(Y,Z)X + g(X,Y)Z - 2g(Z,X)Y) \tag{2.2}$$

$$R(X,Z)Y + R(Y,Z)X = C(g(Z,X)Y + g(Y,Z)X - 2g(X,Y)Z) \tag{2.3}$$

Substracting 2 times (2.2) from 4 times (2.1) and adding (2.3) to the resulting equation, we have $R(X,Y)Z = C(g(Y,Z)X - g(X,Z)Y)$. However, Theorem A, which is contradiction in dimension $(2n+1) \geq 5$. Therefore we have $K(\xi,X) = 0$ on $M$ in dimension $(2n+1) \geq 5$.

**3. Acs-manifolds with $\xi$ belonging to the $k$-nullity distribution.** The $k$-nullity distribution of a Riemannian manifold $(M,g)$, for a real number $k$, is a distribnution $N(k) : p \longrightarrow N_p(k) = \{Z \in T_p(M) \mid R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)$ for any $X, Y \in T_p(M)\}$. From now on, we suppose that $M^{2n+1}(\phi,\xi,\eta,g)$ is a $(2n+1)$-dimensional acs-manifold with $\xi$ belonging to the $k$-nullity distribution, i.e.,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y). \tag{3.1}$$

Then if $M^{2n+1}$ is a cs-manifold, we can see that $k = 0$ easily (the converse fails in general). From this, acs-manifolds with $\xi$ belonging to the non- zero $k$-nullity distribution are proper acs-manifolds (i.e., non cs-acs-manifolds).

**Lemma 3.1.** *Let $M^{2n+1}$ be an acs-manifold with $\xi$ belonging to the $k$-nullity distribution. Then we have*

$$Q\xi = (2nk)\xi, \qquad h^2 = k\phi^2 \ (k \leq 0), \tag{3.2}$$

(3.3) $$(\nabla_X\phi)Y = -g(hX,Y)\xi + \eta(Y)hX,$$

(3.4) $$(\nabla_X h)Y - (\nabla_Y h)X = k(2g(X,\phi Y)\xi + (\eta(X)\phi Y - \eta(Y)\phi X)),$$

*for all vector fields $X, Y$ on $M^{2n+1}$ ($h^2 = k\phi^2$ in (3.2.) holds good in any acs-manifold. However, (3.3) and (3.4) hold good except for proper acs-manifolds with $k = 0$, see the proof of Proposition 4.1 in [5]).*

**Proof.** For the proofs of (3.2) and (3.3), see the proof of Proposition in [5]. So we prove (3.4). At first we prove that on any acs-manifold

(3.5) $$g(R(\xi,X)Y,Z) = -g(X,(\nabla_Z\phi h)Y - (\nabla_Y\phi h)Z)$$

is valid. Because, we first get $R(Y,X)\xi = \nabla_Y(\phi h)X - \nabla_X(\phi h)Y$ easily, from which we obtain (3.5). By (3.3), (3.5) turns into $-g(X,\phi((\nabla_Z h)Y - (\nabla_Y h)Z)) = g(R(\xi,X)Y,Z)$. It follows that

(3.6) $$-(\nabla_Y h)Z + (\nabla_Z h)Y - g((\nabla_Z h)Y - (\nabla_Y h)Z,\xi)\xi = \phi R(Y,Z)\xi.$$

However, using ($\nabla_Z h$ is symmetric and (3.2), we find $g((\nabla_Z h)Y,\xi) = g(Y,(\nabla_Z h)\xi) = -kg(Y,\phi Z)$. Hence, using this, (3.1) and (3.6), we have $-(\nabla_Y h)Z + (\nabla_Z h)Y = -2kg(Y,\phi Z)\xi - k(\eta(Y)\phi X - \eta(Z)\phi Y)$. This completes the proof of the lemma.

**Lemma 3.2.** *Let $M$ be an acs-manifold with $\xi$ belonging to the $k$-nullity distribution and except proper acs-manifolds with $k = 0$. Then we have the followings:*

(3.7) $$R(X,Y)\phi Z - \phi R(X,Y)Z = k(\eta(Y)g(\phi X,Z) -$$
$$-\eta(X)g(\phi Y,Z))\xi + g(hX,Z)\phi hY - g(hY,Z)\phi hX +$$
$$+g(\phi hX,Z)hY - g(\phi hY,Z)hX + k\eta(Z)(\eta(X)\phi Y - \eta(Y)\phi X)$$

(3.8) $$g(\phi R(\phi X,\phi Y)Z,\phi W) = g(R(X,Y)Z,W) +$$
$$+k\eta(Y)\eta(W)g(Z,X) - k\eta(Y)\eta(Z)g(W,X) + k\eta(X)\eta(Z)g(W,Y) -$$
$$-k\eta(X)\eta(W)g(Z,Y) - g(hW,X)g(hZ,Y) + g(hZ,X)g(hW,Y) -$$
$$-g(\phi hW,X)g(hZ,\phi Y) + g(\phi hZ,X)g(hW,\phi Y).$$



(3.9) $$\phi R(\phi X,\phi Y)\phi Z + R(X,Y)Z = k(\eta(X)g(Z,Y)\xi -$$
$$-\eta(Y)g(X,Z)\xi + \eta(Y)\eta(Z)X - eta(X)\eta(Z)Y).$$

**Proof.** At first we prove (3.7). Assuming that $X, Y, Z$ are (local) vector fields such as $(\nabla X)_p = (\nabla Y)_p = (\nabla Z)_p = 0$ for a fixed point of $p$ of $M$, the Ricci identity for $\phi$ is given by $R(X,Y)\phi Z - \phi R(X,Y)Z = (\nabla_X\nabla_Y\phi)Z - (\nabla_Y\nabla_X\phi)Z - (\nabla_{[X,Y]}\phi)Z$. Thus we get at the point $p$, $R(X,Y)\phi Z - \phi R(X,Y)Z = \nabla_X(\nabla_Y\phi)Z - \nabla_Y(\nabla_X\phi)Z$. So, using Lemma 3.1 and this equation we easily get (3.7). Using (3.1) and (3.7) we also obtain (3.8) easily. Last we prove (3.9). Replacing $X$ and $Y$ by $\phi X$ and $\phi Y$ respectively in (3.7) and taking the inner propduct with $\phi W$, we obtain, by (3.8), our result.

**Theorem 3.1.** *Let $M$ be an acs-manifold with $\xi$ belonging to the $k$-nullity distribution and except proper acs-manifolds with $k = 0$. Then we have $Q\phi = \phi Q$.*

**Proof.** Let $\{\widetilde{E}_i = E_i, \widetilde{E}_{i+n} = \phi E_i, \xi, 1 \le i \le n\}$ be a $\phi$-basis. Putting $Y = Z = \widetilde{E}_i, \xi$ in (3.9) and summing over $i$, we get $\phi Q\phi X - \phi R(\phi X,\xi)\xi + QX - R(X,\xi)\xi = 2nk\eta(X)\xi$. On the other hand, from (3.1) we find $\phi Q\phi X + QX = 2nk\eta(X)\xi$. Operating $\phi$ to this equation and using (3.2), we obtain $Q\phi = \phi Q$.

Present author [5] proved the following proposition.

**Proposition A.** *Let $M$ be an acs-manifold. If $\xi$ belongs to the $k$-nullity distribution. Then $k \le 0$. If $k < 0$, then $M$ admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ defined by the eigenspaces of $h$, where $\lambda = \sqrt{-k}$.*

Here we show that neither $D(-\lambda) \oplus D(0)$ nor $D(\lambda) \oplus D(0)$ is integrable in acs-manifolds. At first we prove the following lemma.

**Lemma 3.3.** *Let $M$ be an acs-manifold with $\xi$ belonging to the $k$-nullity distribution ($k < 0$). Then $\nabla_\xi X \in D(\lambda)$ (or $\nabla_\xi X \in D(-\lambda)$) for any $X \in D(\lambda)$ (or $X \in D(-\lambda)$).*

**Proof.** Let $X \in D(\lambda)$. Using (3.1) and (3.2), we calculate

$$-kX = R(\xi,X)\xi = \nabla_\xi(\phi hX) - \phi h[\xi,X] =$$
$$= \lambda\phi\nabla_\xi X - \phi h\nabla_\xi X - kX.$$

So, $\phi h\nabla_\xi X = \lambda\phi\nabla_\xi X$. Operating $\phi$ and using $g(\nabla_\xi X,\xi = 0$, we get $h\nabla_\xi X = \lambda\nabla_\xi X$. This means that $\nabla_\xi X \in D(\lambda)$. Now supposing $Y \in D(\lambda)$, then $X = \phi Y \in D(\lambda)$ ans so $h\nabla_\xi X = \lambda\nabla_\xi X$ or $h\nabla_\xi Y = -\lambda\nabla_\xi Y$. This completes the proof of the lemma.

**Proposition 3.1.** *Let $M$ be an acs-manifold with $\xi$ belonging to the $k$-nullity distribution ($k < 0$). Then neither the distribution $D(-\lambda) \oplus D(0)$ nor $D(\lambda) \oplus D(0)$ is integrable in an acs-manifold.*

**Proof.** We suppose that $D(-\lambda) \oplus D(0)$ is integrable. Let $X \in D(-\lambda)$, i.e., $hX = -\lambda X$. Then, from (3.1) and Proposition A, we find $kX = R(X,\xi)\xi = -\nabla_\xi(\phi hX) - \phi h[X,\xi] = \lambda\phi\nabla_\xi X - \phi h[X,\xi]$. Thus, $\phi h[X,\xi] = \lambda\phi\nabla_\xi X - \lambda^2 X$. Operating $\phi$ and using $g(\nabla_\xi X, \xi) = 0$, we get

$$h[X,\xi] = \lambda\nabla_\xi X - \lambda^2\phi X. \tag{3.10}$$

Since $D(-\lambda) \oplus D(0)$ is integrable, we must have either $h[X,\xi] = 0$ or $h[X,\xi] = -\lambda[X,\xi]$. If we suppose that $h[X,\xi] = 0$, then we have, from (3.10),

$$\lambda\nabla_\xi X = \lambda^2\phi X \tag{3.11}$$

If $h[X,\xi] = -\lambda[X,\xi]$, then we get, by (3.10),

$$\lambda^2\phi X = -\lambda^2\phi X. \tag{3.12}$$

Next, we also suppose that $D(\lambda) \oplus D(0)$ is integrable. Let $X \in D(\lambda)$, i.e., $hX = \lambda X$. In the similar way with the former $X \in D(-\lambda)$ we get $\lambda\nabla_\xi X = -\lambda^2\phi X$ and $\lambda^2\phi X = -\lambda^2\phi X$. However, (3.11), (3.12) and these results lead us to the contradiction from Lemma 3.3 and $\lambda > 0$.

**4. Acs-manifolds with harmonic curvature.** The curvature of a Riemannian manifold is said to be harmonic if the divergence of its curvature tensor is zero. And it is well-known that a Riemannian manifold has harmonic curvature if and only if the Ricci operator $Q$ satisfies

$$(\nabla_X Q)Y = (\nabla_Y Q)X \tag{4.1}$$

for any vector fields $X$ and $Y$. Here we prove the following lemma.

**Lemma 4.1.** *Let $M$ be an acs-manifold with $\xi$ belonging to the non-zero $k$-nullity distribution. If the curvature is harmonic, then, $hQ = Qh$.*

**Proof.** Using (3.2), we have $g((\nabla_X Q)Y, \xi) = g(Y, \nabla_X(Q\xi)) - g(Y, Q\nabla_X\xi) = 2nkg(Y, \phi hX) - g(Y, Q\phi hX)$. It follows that $0 = g((\nabla_X Q)Y - (\nabla_Y Q)X, \xi) = -g(Y, (Q\phi h + hQ\phi)X)$. Thus we get $Q\phi h + hQ\phi = 0$. Using Theorem 3.1 and (3.2), we obtain $hQ = Qh$.

From this lemma we prove the following theorem.



**Theorem 4.1.** *Let $M^{2n+1}$ be an acs-manifold with $\xi$ belonging to the non-zero $k$-nullity distribution. Then the curvature on $M^{2n+1}$ is harmonic if and only if $M^{2n+1}$ is Einstein and the scalar curvature of $M^{2n+1}$ is $2n(2n+1)k$.*

**Proof.** At first we suppose that the curvature of $M^{2n+1}$ is harmonic. Let $X, Y \in D(\lambda)$. Using Lemma 4.1, we find $QX \in D(\lambda)$. Thus, by Lemma 3.3, we get $\nabla_\xi(QX) \in D(\lambda)$, from which $g(\nabla_\xi(QX), \phi Y) = 0$. Therefore, by Lemma 3.3 and the assumption, we get $g((\nabla_X Q)\xi, \phi Y) = 0$, so that, $2nkg(\nabla_X\xi, \phi Y) = g(Q\phi hX, \phi Y)$. Using Theorem 3.1 and Lemma 4.1, we obtain $g(hQX, Y) = 2nkg(hX, Y)$. This means $g(QX, Y) = 2nkg(X, Y)$. If $Z, W \in D(-\lambda), \phi Z, \phi W \in D(\lambda)$. Thus we get $g(Q\phi Z, \phi W) = 2nkg(\phi Z, \phi W)$. Using Theorem 3.1 and (3.2), we obtain $g(QZ, W) = 2nkg(Z, W)$. From (3.2) we see that $M^{2n+1}$ is Einstein and the scalar curvature of $M^{2n+1}$ is $2n(2n+1)k$. The converse is evident.

**5. Acs-manifolds with $R(\xi, X) \cdot C = 0$.** Next we consider an acs-manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ with $\xi$ belonging to the $k$- nullity distribution such that $R(\xi, X) \cdot C = 0$, where $C$ is the Weyl conformal curvature tensor and $R(\xi, X)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vector $X$. Denote the Weyl conformal curvature tensor on $M^{2n+1}(n > 1)$ by ($S$ is the scalar curvature)

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}(g(QY,Z)X - g(QX,Z)Y + \\ + g(Y,Z)QX - g(X,Z)QY) + \frac{S}{2n(2n-1)}(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{5.1}$$

Then we have the following theorem by the same way as [1].

**Theorem 5.1.** *Let $M^{2n+1}(n > 1)$ be an acs-manifold with $\xi$ belonging to the non-zero $k$-nullity distribution. If $R(\xi, X) \cdot C = 0$ is satisfied on $M^{2n+1}$, then $M^{2n+1}$ is $\eta$-Einstein.*

**Proof.** Using (5.1), we can easily obtain $\sum_i C(X, E_i)E_i = 0(\{E_i\}$ is an orthonormal frame). Moreover, using (3.1), (3.2) and this equation we get, from (5.1) and $R(\xi, E_i) \cdot C)(E_i, Z)W = 0$,

$$\begin{aligned} g(C(\xi,Y)Z,\xi) = \frac{1}{2n-1}\Big\{\left(\frac{S}{2n} - k\right)g(Y,Z) - g(QY,Z) + \\ + \left((2n+1)k - \frac{S}{2n}\right)\eta(Y)\eta(Z)\Big\} \end{aligned} \tag{5.2}$$

$$2nkg(C(\xi, Z)W, \xi) = 0. \tag{5.3}$$

Since $k \neq 0$, from (5.3) we find $g(C(\xi, Z)W, \xi) = 0$. Therefore we get, from (5.2), our result.

**6. Acs–manifolds of constant $\phi$–sectional curvature.** For any vector $X \in T_m(M)$ ($M$ is an acs-manifold) which is orthogonal to $\xi$, the sectional curvature $K(X, \phi X)$ is caled the $\phi$-sectional curvature of the plane $\{X, \phi X\}$ at point $m \in M$. If $K(X, \phi X)$ does not depend on the choice of $X \in T_m(M)$, $M$ is called that $M$ has the constant $\phi$- sectional curvature at $m$. Next we consider an acs-manifold with pointwise constant $\phi$–sectional curvature. Then we get the following theorem.

**Theorem 6.1.** *Let $M^{2n+1}$ be an acs-manifold with $\xi$ belonging to the $k$-nullity distribution and except proper acs-manifolds with $k = 0$. If $M^{2n+1}$ is of pointwise constant $\phi$–sectional curvature $H$, then the curvature tensor has form*

$$\begin{aligned}4R(X, Y)Z = {} & H(g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \\ & + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(\phi Y, Z)\phi X - \\ & - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) + 4k(\eta(Y)\eta(Z)X - \\ & - \eta(X)\eta(Z)Y + \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi) + \\ & + 2(g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX),\end{aligned} \tag{6.1}$$

*where $H$ is constant on $M$ if $n \neq 1$. Especially, when $M$ is a cs– manifold i.e., $k = 0$, (6.1) is reduced to the Blair's formula (see [7]).*

**Proof.** By the assumption we have $g(R(X, \phi X)X, \phi X) + H_m \| X \|^4 = 0$ at any point $m \in M$ and for any $X \in T_m(M), X \perp \xi$. It is clear that this condition implies

$$g(R(\phi X, \phi^2 X)\phi X, \phi^2 X) + H_m g(\phi X, \phi X)g(\phi X, \phi X) = 0 \tag{6.2}$$

at any point $m \in M$ and for any $X \in T_m(M)$. Set $P(X, Y, Z, W) = g(R(\phi X, \phi^2 Y)\phi Z, \phi^2 W) + H_m g(\phi X, \phi Z)g(\phi Y, \phi W)$. The tensor $P$ satisfies $P(X, Y, Z, W) = P(Z, W, X, Y)$. Therefore (6.2) is equivalent to

$$\begin{aligned}& P(X, Y, Z, W) + P(X, Y, W, Z) + P(Y, X, Z, W) + P(Y, X, W, Z) + \\ & + P(X, W, Y, Z) + P(X, W, Z, Y) + P(W, X, Y, Z) + P(W, X, Z, Y) + \\ & + P(X, Z, Y, W) + P(X, Z, W, Y) + P(Z, X, Y, W) + P(Z, X, W, Y) = 0.\end{aligned} \tag{6.3}$$



By using (3.7) and (3.1), (6.3) leads us to, after some lengthy computation,

$$\begin{aligned}& 2g(R(\phi X, Y)\phi Z, W) + 2g(R(\phi X, W)\phi Y, Z) + \\ & + 2g(R(\phi X, Z)\phi W, Y) + k(\eta(Y)\eta(Z)g(X, W) + \eta(Y)\eta(W)g(X, Z) + \\ & + \eta(Z)\eta(W)g(X, Y) - \eta(X)\eta(Y)g(Z, W) - \eta(X)\eta(Z)g(Y, W) - \\ & - \eta(X)\eta(W)g(Y, Z)) + (k - 2H)(\eta(X)\eta(Y)g(Z, W) + \eta(X)\eta(Z)g(Y, W) + \\ & + \eta(X)\eta(W)g(Y, Z) + \eta(Y)\eta(Z)g(X, W) + \eta(Y)\eta(W)g(X, Z) + \\ & + \eta(Z)\eta(W)g(X, Y)) + 2H(g(X, Y)g(Z, W) + g(X, Z)g(Y, W) + \\ & + g(X, W)g(Y, Z)) + 6(H - k)\eta(X)\eta(Y)\eta(Z)\eta(W) = 0.\end{aligned} \tag{6.4}$$

Using (3.7), (3.8) and (3.1), we obtain

$$\begin{aligned}& 3g(R(\phi X, Y)\phi Z, W) - g(R(Y, Z)X, W) + g(R(W, Y)X, Z) + \\ & + (k - H)(\eta(Y)\eta(Z)g(X, W) + \eta(Y)\eta(W)g(X, Z) + \eta(Z)\eta(W)g(X, Y) + \\ & + \eta(X)\eta(Y)g(Z, W) + \eta(X)\eta(Z)g(Y, W) + \eta(X)\eta(W)g(Y, Z)) + \\ & + 3k(\eta(Y)\eta(W)g(X, Z) - \eta(Z)\eta(W)g(X, Y) - \eta(X)\eta(Y)g(Z, W)) + \\ & + H(g(X, Y)g(Z, W) + g(X, Z)g(Y, W) + g(X, W)g(Y, Z)) + \\ & + 2g(hY, Z)g(hX, W) - g(hY, W)g(hX, Z) - g(hW, Z)g(hX, Y) + \\ & + g(\phi hW, Y)g(\phi hX, Z) + g(\phi hX, Y)g(\phi hZ, W) - \\ & - 2g(\phi hX, W)g(\phi hY, Z) + 3(H - k)\eta(X)\eta(Y)\eta(Z)\eta(W) = 0.\end{aligned} \tag{6.5}$$

Replacing $\phi X$ and $\phi Z$ instead of $X$ and $Z$ in (6.5) respectively, and using (3.1) and (3.8), we have

$$\begin{aligned}g(R(Y, \phi Z)\phi X, W) = {} & 3g(R(X, Y)Z, W) - g(R(Y, W)X, Z) + \\ & + H(g(\phi X, Y)g(\phi Z, W) - g(\phi X, W)g(\phi Y, Z) + g(X, Z)g(Y, W)) - \\ & - k(2\eta(Y)\eta(Z)g(X, W) + 2\eta(X)\eta(W)g(Y, Z) + \eta(Z)\eta(W)g(X, Y) + \\ & + \eta(X)\eta(Y)g(Z, W)) + (4k - H)\eta(Y)\eta(W)g(X, Z) +\end{aligned} \tag{6.6}$$

$$+(3k-H)\eta(X)\eta(Z)g(Y,W)-(k-H)\eta(X)\eta(Y)\eta(Z)\eta(W)-$$
$$-g(hY,Z)g(hX,W)+g(hY,W)g(hX,Z)-g(\phi hW,Y)g(\phi hX,Z)+$$
$$+g(\phi hX,W)g(\phi hY,Z.$$

Exchanging $Y$ for $W$ in (6.6), moreover exchanging $Z$ for $W$ in the gotten equation and using (3.7), we obtain (6.1). Next we show that $H$ is constant on $M$ if $n \neq 1$. From (6.1) we get the following easily:

$$Ric(Y,Z)=\left(\frac{(n+1)H}{2}+k\right)g(Y,Z)+ \tag{6.7}$$
$$+\left((2n-1)k-\frac{(n+1)H}{2}\right)\eta(Y)\eta(Z)$$

$$S=n(n+1)H+4k. \tag{6.8}$$

On the other hand, from the Bianchi identity, we get $2\sum_{i=1}^{2n+1}(\nabla_{E_i}Ric)(E_i,Z)-\nabla_Z S=0$ ($\{E_i\}$ is an orthonormal frame). Substituting (6.7) and (6.8) into this equation, we find $(n-1)\nabla_Z H+\eta(Z)\nabla_\xi H=0$, from which, putting $Z=\xi$, we have $\nabla_\xi H=o$. Therefore we get $(n-1)\nabla_Z H=0$, which completes the proof.

From (6.7) we get the following.

**Theorem 6.2.** *Let $M^{2n+1}$ be an acs–manifold with $\xi$ belonging to the $k$–nullity distribution and except proper acs–manifolds with $k=0$. If $M^{2n+1}$ has a constant $\phi$–sectional curvature, then $M^{2n+1}$ is $\eta$–Einstein.*

**Corollary 6.1.** *Let $M^{2n+1}$ be an acs–manifold with $\xi$ belonging to the $k$–nullity distribution, except proper acs–manifolds with $k=0$ and with a constant $\phi$–sectional curvature $H$. Then Einstein manifolds $M^{2n+1}$ with $k \neq \frac{(n+1)H}{2(2n-1)}$ do not exist.*

**Remark 6.1.** O l s z a k ([8], Theorem 6.1) proved Theorem A in §1. In an acs–manifold $M^{2n+1}$ of non–zero constant curvature, $\xi$ belongs to the non–zero $k$–nullity distribution and $M^{2n+1}$ has a constant $\phi$– sectional curvature $H$. Then $H=k\neq 0$. Therefore, if $n\neq 1$, $k\neq\frac{(n+1)H}{2(2n-1)}$. On the other hand, $M^{2n+1}$ is Einstein. So, Corollary 6.1 is a generalization of Olszak's Theorem 6.1.



**7. Invariant submanifolds of an acs–manifold with $\overline{\xi}$ belonging to the $k$–nullity distribution.** Let $M^{2n+1}$ be a $(2n+1)$– dimensional $C^\infty$–submanifold of a $(2r+1)$–dimensional acs–manifold $\overline{M}^{2r+1}(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$. We simply describe $\overline{M}^{2r+1}$ and $M^{2n+1}$ by $\overline{M}$ respectively. $M$ is said to be invariant if the structure vector field $\xi$ is tangent to $M$ everywhere on $M$ and $\phi X$ is tangent to $M$ for any vector field $X$ tangent to $M$ at every point of $M$. If $M$ is an invariant submanifold of $\overline{M}$, we can put $\overline{\phi}X=\phi X,\overline{\xi}=\xi,\overline{\eta}(X)=\eta(X)$. Then it is well-known that if $\overline{M}(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is an acs–manifold (resp. cs–manifold), then $M$ is also an acs–manifold (resp. cs–manifold) with respect to the induced structure $(\phi,\xi,\eta,g)$ ($g$ is the induced metric) (e.g., see [3]). Here if we define an operator $h=\frac{-1}{2}\mathcal{L}_\xi\phi$ on an invariant submanifold $M$ of an acs–manifold $\overline{M}$, then we have the results that $h$ is symmetric and $h$ anti–comutes with $\phi$ (i.e., $\phi h+h\phi=0$). And, by the definition of $\overline{h}$ we can see that $\overline{h}X$ is tangent to $M$ and $\overline{h}X=hX$ for any vector field $X\in T_m(M)$. So, the induced structure $(\phi,\xi,\eta,h,g)$ on an acs–manifold $M$ satisfies the identites in §1. The Gauss formula is given by $\overline{\nabla}_X Y=\nabla_X Y+\sum_A g(H_A X,Y)N_A$ and the Gauss equation is

$$\overline{g}(\overline{R}(X,Y)Z,W)=g(R(X,Y)Z,W)- \tag{7.2.}$$
$$-\sum_B g(H_BY,Z)g(H_BX,W)+\sum_B g(H_BX,Z)g(H_BY.W)$$

for any vector fields $X,Y,Z$ and $W$ on $M$, where $\nabla$ denote the induced Riemannian connection on $M$ and $H_B(B=1,2,\ldots,2(r-n))$ denote the second fundamental forms of $M$. We need the following theorem later and moreover we have Lemma 7.1.

**Theorem B ([2] and [3]).** *Let $M$ be an invariant submanifold of an acs– manifold $\overline{M}$. Then $M$ is minimal and $H_A\phi=-\phi H_A$, especially $H_A\xi=0$ $(A=1,\ldots,2(r-n))$.*

**Lemma 7.1.** *Let $M$ be an invariant submanifold of an acs–manifold $M$ with $\overline{\xi}$ belonging to the $\overline{k}$–nullity distribution and except proper acs–manifolds with $\overline{k}=0$. Then $H_Ah=hH_A$ $(A=1,2,\ldots,2(r-n))$ are valid.*

**Proof.** First, by the Gauss formula we calculate $\overline{\nabla}_X(\overline{h}Y)$ in two waies:

$$\overline{\nabla}_X(\overline{h}Y)=(\nabla_Xh)Y+h(\nabla_XY)+$$
$$+\sum_A g(H_AX,hY)N_A=(\overline{\nabla}_X\overline{h})Y+h(\nabla_XY)+$$

$$+\sum_B g(H_B X, Y)\bar{h}N_B$$

Thus, $(\nabla_X h)Y + \sum_A g(H_A X, hY)N_A = (\bar{\nabla}_X \bar{h})Y + \sum_B g(H_B X, Y)\bar{h}N_B$. Here, we interchange $X$ and $Y$ in this equation and subtract these two equations. On the other hand, by (7.2) and Theorem B, $\xi = \bar{\xi}$ belongs to the $\bar{k}$-nullity distribution on $M$. So, using this result and (3.4), we get our result.

From (6.1) and (7.2) we get the following theorem easily.

**Theorem 7.1.** *Let $\bar{M}^{2r+1}$ be an acs-manifold with $\bar{\xi}$ belonging to the $\bar{k}$-nullity distribution, except proper acs-manifolds with $\bar{k} = 0$ and with a constant $\bar{\phi}$-sectional curvature $\bar{H}$. If $M^{2n+1}$ is an invariant submanifold of $\bar{M}^{2r+1}$, then the curvature tensor of $M^{2n+1}$ is given by*

$$(7.3) \qquad g(R(X,Y)Z,W) = \frac{\bar{H}}{4}(g(X,W)g(Y,Z) - g(X,Z)g(Y,W) +$$
$$+\eta(X)\eta(Z)g(Y,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) -$$
$$-\eta(Y)\eta(Z)g(X,W) + g(\phi X,Z)g(\phi W,Y) - g(\phi X,W)g(\phi Z,Y) +$$
$$+2g(\phi X,Y)g(\phi W,Z)) + \bar{k}(\eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(Z)g(X,W) -$$
$$-\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(W)g(X,Z)) + \frac{1}{2}(g(hY,Z)g(hX,W) -$$
$$-g(hY,W)g(hX,Z) + g(\phi hW,Y)g(\phi hX,Z) - g(\phi hX,W)g(\phi hY,Z)) +$$
$$+\sum_B g(H_B Y,Z)g(H_B X,W) - \sum_B g(H_B X,Z)g(H_B Y,W)$$

Using (7.3) and Theorem B, we get, in the assumption of Theorem 7.1.,

$$(7.4) \qquad Ric(Y,Z) = \left(\frac{(n+1)}{2}\bar{H} + \bar{k}\right)g(Y,Z) +$$
$$+\left((2n-1)\bar{k} - \frac{(n+1)}{2}\bar{H}\right)\eta(Y)\eta(Z) - \sum_B g(H_B Y, H_B Z).$$

(7.5) $S = n(n+1)\bar{H} + 4\bar{k} - \sum_B TrH_B^2$ ($S$ is the scalar curvature on $M^{2n+1}$).

Let $\bar{M}^{2n+3}$ be an acs-manifold with $\bar{\xi}$ belonging to the $\bar{k}$-nullity distribution, except proper acs-manifolds with $\bar{k} = 0$ and with a constant $\bar{\phi}$-sectional curvature $\bar{H}$. Hereafter we consider an invariant submanifold $M^{2n+1}$ of codimension 2 of $\bar{M}^{2n+3}$. From now we simply denote $\bar{M}^{2n+3}$ and $M^{2n+1}$ as $\bar{M}$ and $M$ respectively. Let $V$ be a unit normal vector field on $M$ in $\bar{M}$. Since $M$ is invariant under the action of $\bar{\phi}$, we can set

$$(7.6) \qquad \bar{\nabla}_X Y = \nabla_X Y + h_1(X,Y)V + h_2(X,Y)\bar{\phi}V$$

for any vector fields $X$ and $Y$ on $M$. By $g(V,V) = 1$, we get $\bar{\nabla}_X V = -A(X) + t(X)\bar{\phi}V$, where $A(X)$ is the second fundamental form on $M$ and $t$ is a 1-form on $M$. Then, operating $\bar{\phi}$ to this equation and using (3.3), we find $h_1(X,Y) = g(AX,Y)$ and $h_2(X,Y) = g(\phi AX,Y)$. Here, we need the following lemma.

**Lemma 7.2.** *Let $X$ be any vector field on $M$. Then we have $\bar{R}ic(X,Y) = Ric(X,Y) - 2dt(X,\phi Y)$.*

**Proof.** At first we generally prove that on any $(2n+1)$-dimensional acs- manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $\xi$ belonging to the $k$-nullity distribution and except proper acs-manifolds with $k = 0$

$$(7.7) \qquad g(Q^*X,Y) = g(QX,Y) - kg(X,Y) - (2n-1)k\eta(X)\eta(Y)$$

is valid, where $g(Q^*X,Y) = -\frac{1}{2}\sum_{i=1}^{2n+1} g(R(X,\phi Y)E_i,\phi E_i)$ ($\{E_i\}$ is an orthonormal frame).

$g(Q^*X,Y) = -\frac{1}{2}\sum_{i=1}^{2n+1}\{g(R(\phi Y,E_i,X) - g(R(E_i,X)\phi Y,\phi E_i)\}$. On the other hand, using (3.1), (3.7), (3.2) and Theorem 3.1, we get (7.7). Next we compute $\bar{R}(X,Y)V$. From (3.3), (7.6) and Theorem B we have

$$(7.8) \qquad \bar{R}(X,Y)V = -\{(\nabla_X A)Y - (\nabla_Y A)X - t(X)\phi AY + t(Y)\phi AX\} +$$
$$+2\{dt(X,Y) - g(\phi A^2 Y,X)\}\bar{\phi}V.$$

Here, we also calculate $\bar{g}(\bar{Q}^*X,Y)$. $\bar{g}(\bar{Q}^*X,Y) = -\frac{1}{2}\sum_{i=1}^{2n+1}\bar{g}(\bar{R}(X,\phi Y)E_i,\phi E_i) - \bar{g}(\bar{R}(X,\phi Y)V,\bar{\phi}V)$. Using (7.2), Theorem B and (7.8), we have

$$(7.9) \qquad \bar{g}(\bar{Q}^*X,Y) = g(Q^*X,Y) - 2dt(X,\phi Y).$$

On the other hand, since $M$ is invariant, $\xi = \bar{\xi}$ belongs to the $\bar{k}$-nullity distribution on $M$. Thus, by (7.7) we obtain our result.

Here, we consider an invariant submanifold $M$ with harmonic curvature in $\bar{M}$. Since $M$ is invariant, $\xi = \bar{\xi}$ belongs to the non-zero $\bar{k}$– nullity distribution on $M$. Therefore, from Theorem 4.1 $M$ is Einstein and we have

$$(7.10) \qquad Ric(X,Y) = 2n\bar{k}g(X,Y)$$

for any vector fields $X$ and $Y$ on $M$. So, by (7.2), (7.4), (7.10) and Theorem B we see that

$$(7.11) \quad 2g(A^2X,Y) = \left(\frac{(n+1)}{2}\bar{H} - (2n-1)\bar{k}\right)(g(X,Y) - \eta(X)\eta(Y)).$$

If $A^2 = 0$ (that is, $M$ is totally geodesic), then $\bar{k} = \frac{(n+1)\bar{H}}{2(2n-1)}(n \neq 1), \bar{k} = \bar{H}(n = 1)$.
Except for this possibility, we can set $D_\lambda{}^+ = \{X \in T_x(M) | AX = \lambda X\}$, $D_\lambda{}^- = \{X \in T_x(M) | AX = -\lambda X\}$ for $x \in M$, $D_\lambda = D_\lambda{}^+ \oplus D_\lambda{}^-$, where $\lambda > 0$ and $D_0 = \{X \in T_x(M) | AX = 0\}$. Then these distributions are mutually orthogonal. And, from (7.11) we get

$$(7.12) \qquad 2\lambda^2 = \frac{(n+1)}{2}\bar{H} - (2n-1)\bar{k},$$

which shows that the characteristic roots of $A$ are constant on $M$. Moreover, using Theorem B, $\phi$ interchanges $D_\lambda{}^+$ and $D_\lambda{}^-$. This means that the dimensions of $D_\lambda{}^+$ and $D_\lambda{}^-$ are equal to $n$. Here we have the following three lemmas.

**Lemma 7.3.** *If $X, Y \in D_\lambda (\lambda \neq 0)$, then $\nabla_X Y \in D_\lambda$. In particular $D_\lambda$ is involutive.*

**Proof.** If $X, Y \in D_\lambda{}^+$ (or $D_\lambda{}^-$), then $g(Y, \xi) = 0$, so that $g(\nabla_X Y, \xi + g(Y, \phi hX) = 0$. On the other hand, from Lemma 7.1 we see that if $X \in D_\lambda{}^+$ (or $D_\lambda{}^-$), then $hX \in D_\lambda{}^+$ (or $D_\lambda{}^-$). Therefore we get $\nabla_X Y \in D_\lambda$. Next if $X \in D_\lambda{}^+$, $Y \in D_\lambda{}^-$, then $\phi Y \in D_\lambda{}^+$, so that $\nabla_X(\phi Y) = (\nabla_X \phi)Y + \phi(\nabla_X Y) \in D_\lambda$. Here, using (3.3) and Lemma 7.1, we get $\phi(\nabla_X Y) \in D_\lambda$. Thus $\nabla_X Y \in D_\lambda$. Similary we obtain $\nabla_X Y \in D_\lambda$ if $X \in D_\lambda{}^-$, $Y \in D_\lambda{}^+$.

**Lemma 7.4.** *If $X \in D_\lambda$, $Y \in D_\lambda{}^+ (\lambda \neq 0)$, then $(\nabla_X Y)_\lambda{}^- = \frac{1}{2}t(X)\phi Y$.*

**Proof.** First, from (7.6) the Codazzi equation is $\nabla_X(AY) - \nabla_Y(AX) - A([X,Y]) - t(X)\phi AY + t(Y)\phi AX = 0$. Suppose $X \in D_\lambda{}^-$, then this Codazzi equation becomes, by Theorem B, $\lambda\nabla_X Y + \lambda\nabla_Y X - A[X,Y] - \lambda t(X)\phi Y - \lambda t(Y)\phi X = 0$. It follows that $(\nabla_X Y)_\lambda{}^- = \frac{1}{2}t(X)\phi Y$ and $(\nabla_Y X)_\lambda{}^+ = \frac{1}{2}t(Y)\phi X$ when $X \in D_\lambda{}^-$ and $Y \in D_\lambda{}^+$, where the components of a vector field $X$ in the distribution $D_\lambda{}^+$ and $D_\lambda{}^- (\lambda \neq 0)$ generally denote by $X_\lambda{}^+$ and $X_\lambda{}^-$ respectively. When $X \in D_\lambda{}^+$ and $(Y) \in D_\lambda{}^+$, $\phi Y \in D_\lambda{}^-$. So, by the above latter equation, we have $\nabla_X(\phi Y) = \frac{1}{2}t(X)\phi^2 Y$. Thus $(\nabla_X \phi)Y + \phi(\nabla_X Y) = -\frac{1}{2}t(X)Y$. Operating $\phi$ to this equation, we get, by (3.3) and Lemma 7.3, $\nabla_X Y = \frac{1}{2}t(X)\phi Y$. Therefore $(\nabla_X Y)_\lambda{}^- = \frac{1}{2}t(X)\phi Y$, which completes the proof.



**Lemma 7.5.** *If $Z \in D_\lambda{}^+$ is a unit vector field and $\lambda \neq 0$, then $g(R(Z, \phi Z)\phi Z, Z) = dt(\phi Z, Z)$.*

**Proof.** If $X$, $Y \in D_\lambda$, then, using Lemmas 7.3 and 7.4 repeatedly, we obtain $\nabla_X \nabla_Y Z = \frac{1}{2}X(t(Y))\phi Z + \frac{1}{2}t(Y)(\nabla_X \phi)Z - \frac{1}{4}t(X)t(Y)Z + \frac{1}{2}t(Y)\phi(\nabla_X Z)_\lambda{}^+ + \frac{1}{2}t(X)\phi(\nabla_Y Z)_\lambda{}^+ + (\nabla_X(\nabla_Y Z)_\lambda{}^+)_\lambda{}^+$. Since $Z$ is a unit vector in $D_\lambda{}^+$, this equation yields, by (3.3), $g(\nabla_X \nabla_Y Z, \phi Z) = \frac{1}{2}X(t(Y))$. By virtue of Lemma 7.3 the distribution $D_\lambda$ is involutive, so that $[X,Y] \in D_\lambda$ and from Lemma 7.4 we obtain $g(\nabla_{[X,Y]}Z, \phi Z) = \frac{1}{2}t([X,Y])$. Thus $g(R(X,Y)Z, \phi Z) = dt(X,Y)$, from which $g(R(Z, \phi Z)\phi Z, Z) = dt(\phi Z, Z)$.
Here we get, from Lemma 7.5, the following theorem.

**Theorem 7.2.** *Let $\bar{M}^{2n+3}$ be an acs–manifold with $\xi$ belonging to the non–zero $\bar{k}$–nullity distribution and with a constant $\phi$– sectional curvature $\bar{H}$, and $M^{2n+1}$ be an invariant submanifold of codimension 2 in $\bar{M}^{2n+3}$. Then the curvature of $M^{2n+1}$ is harmonic if and only if $M^{2n+1}$ is totally geodesic and $\bar{k} = \frac{(n+1)\bar{H}}{2(2n-1)}(n \neq 1)$, $\bar{k} = \bar{H}(n = 1)$.*

**Proof.** First we suppose that the curvature of $M^{2n+1}$ is harmonic. If $Z \in D_\lambda{}^+$, $g(Z,Z) = 1$. We have, by Lemma 7.5, $g(R(Z, \phi Z)\phi Z, Z) = dt(\phi Z, Z)$. However, from (7.2) and Theorem B we find $g(R(Z, \phi Z)\phi Z, Z) = \bar{g}(\bar{R}(Z, \phi Z)\phi Z, Z) - 2g(A^2 Z, Z)$.
Therefore we obtain $dt(\phi Z, Z) = \bar{H} - 2\lambda^2$. On the other hand, from (6.7) we see that $\bar{R}ic(Z,Z) = \frac{(n+2)}{2}\bar{H} + \bar{k}$, so that, by Lemma 7.2, $Ric(Z,Z) = \frac{(n+2)}{2}\bar{H} + \bar{k} + 2dt(Z, \phi Z)$. However, we have $Ric(Z,Z) = \frac{(n+1)}{2}\bar{H} + \bar{k} - 2\lambda^2$ because of (7.4) and Theorem B. Therefore we get $\lambda^2 = \frac{1}{4}\bar{H}$. From this we have $\bar{H} \geq 0$ and when $\bar{H} = 0$, $\lambda = 0$, which is the contradiction to $\lambda \neq 0$. Thus $\bar{H} > 0$. Moreover (7.12) and $\lambda^2 = \frac{1}{4}\bar{H}$ leads us to $\frac{n}{2}\bar{H} - (2n-1)\bar{k} = 0$. By means of (3.2), $\bar{k} < 0$, and $\bar{H} > 0$, which is the contradiction. Therefore we see that $M^{2n+1}$ is totally geodesic and $\bar{k} = \frac{(n+1)\bar{H}}{2(2n-1)}(n \neq 1)$, $\bar{k} = \bar{H}(n = 1)$. By (7.4) the converse is evident.

**Remark 7.1.** Ludden ([7], Theorem 4.4) proved the following:

**Theorem C.** *Let $M^{2n+1}$ be an invariant submanifold of codimension 2 of a cs–manifold $M^{2n+3}$. Then $M^{2n+1}$ is totally geodesic if and only if $M^{(}2n+1)$ is of constant $\phi$–sectional curvature. In Theorem C, if $M^{2n+1}$ is of constant $\phi$–sectional curvature, then $M^{2n+1}$ is $\eta$–Einstein (see the proof of Theorem 4.4 in [7]). Thus the curvature of $M^{2n+1}$ is harmonic. So, Theorem 7.2, in the case where $M^{2n+1}$ is a proper acs–manifold, is a theorem which corresponds to Theorem C.*

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# SEMI-INVARIANT SUBMANIFOLDS OF A QUASI-K-SASAKIAN MANIFOLD

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**0. Introduction.** The notion of semi-invariant submanifold introduced by A. B e j a n c u - N. P a p a g h i u c [3] has been intensively studied by many people (see references in [2]). In [7] O u b i ñ a have obtained a clasification for almost contact structures, and in [5] D. C h i n e a obtained some results on invariant submanifolds of quasi-K-Sasakian manifolds.

It is the purpose of the present paper to consider and study the concept of semi-invariant submanifolds in case of a quasi-K-Sasakian manifold. First we present some general formulae and basic results from the theory of quasi -K- Sasaki manifolds in order to use them in the next sections. Next we obtain necessary and sufficient conditions for the integrability of distributions defined on a semi-invariant submanifold of a quasi-K-Sasakian manifold. Finally we a dealing with totally contact umbilical semi-invariant submanifolds of a quasi-K-Sasakian manifold. More precisely, we prove that a totally contact umbilical semi-invariant submanifold of $\tilde{M}$ is totally contact geodesic if $dim D^{\perp} > 1$ and $fN_f(X,Y) \in \Gamma(D \oplus \{\xi\})$ for any $X, Y \in \Gamma(D^{\perp})$.

**1. Preliminaries.** Let $\tilde{M}$ be a real 2n+1-dimensional differentiable manifold with $(f, \xi, \eta, g)$ an almost contact metric structure. In this case we have

$$(1)\qquad \begin{cases} f^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad \eta(X) = g(X,\xi) \\ \eta \circ f = 0; \quad f(\xi) = 0, \quad g(X, fY) + g(fX, Y) = 0, \end{cases}$$

for any vector fields X,Y on $\tilde{M}$, where I is the identity on the tangent bundle $T\tilde{M}$ of $\tilde{M}$.