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# SOME CONVERGENCES OF CONVEX FUNCTIONS AND ITS $\varepsilon$–SUBDIFFERENTIALS

BY

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**1. Introduction.** A productive way in convex analysis which is frequently used in the last time consists in the topologizated the family $C(X)$ of all convex closed subset of $X$ ($X$ is a normal space) with different topologies and the study of the various operators on $\mathcal{C}(X)$. Recently, some papers established one link between the convergence of proper lower semicontinuous convex functions and the convergence of its subdifferentials. In this context, Attouch and Beer [2] and A.Verona and M.E.Verona [11], correlate the slice topology of the functions with Kuratowski– Painlevé convergence of the subgradients in Banach spaces, respectively in arbitrary normed spaces. In this paper we have interested to settle the same problem in the case of the affine convergence of convex functios (section 3), as well as the scalar convergence of the subdifferentials in reflexive Banach spaces (section 4), these last results being reformulated in terms of directional derivative too.

**2. Preliminaries.** Let $X$ be a normed space with the norm $||\cdot||$, and on $X \times \boldsymbol{R}$ we consider the box norm $||(x,\alpha)|| = \max\{||x\}, |\alpha|\}$. By $X^*$ we denote the topological dual, namely the space of linear continuous functionals on $X$, with the usual norm.

Given one function $f : X \to (-\infty, \infty]$, by its *epigraph* we mean the set epi $f = \{(x,\alpha) \in X \times \boldsymbol{R};\ f(x) \leq \alpha\}$, and the *domain* of $f$ is *dom* $f = \{x \in X; f(x) \neq +\infty\}$. Also, the *hypograph* of $f$ is defined by *hypo* $f = \{(x,\alpha) \in X \times \boldsymbol{R}; f(x) \geq \alpha\}$. A function $f$ is called to be *convex* (respectively *lower semicontinuous*) if epi $f$ is a convex (respectively closed) subsets in $X \times \boldsymbol{R}$; $f$ is *proper* if epi $f$ is not empty. The family of proper lower semicontinuous convex functions will be denoted by $\Gamma(X)$. For $f$ we define its *conjugate*

$f^* : X^* \to [-\infty, \infty]$ by $f^*(x^*) = \sup\{x^*(x) - f(x);\ x \in X\}$. We recall that if $f$ is proper lower semicontinuous convex function, $f^*$ has the same properties.

A link between $f$ and $f^*$ is given by Young inequality:

$$x^*(x) \leq f^*(x^*) + f(x),\ \forall x \in X,\ \forall x^* \in X^*.$$

An element $x^* \in X^*$ is called a *subgradient* of $f$ at $x_0 \in \text{dom } f$ if

$$f(x_0) - f(x) \leq x^*(x_0 - x),\ \forall x \in X,$$

equivalently

$$f^*(x^*) + f(x_0) = x^*(x_0).$$

The *subdifferential* of $f$ at $x_0$ denoted by $\partial f(x_0)$, is defined like the set (possible empty) of all subgradients of $f$ at $x_0$.

For $\varepsilon \geq 0$, the $\varepsilon$–*subdifferential* of $f$ at $x_0$ denoted $\partial_\varepsilon f(x_0)$ is the set of those $x^* \in X^*$ which satisfies

$$f(x_0) - f(x) \leq x^*(x_0 - x) + \varepsilon \text{ ( or } f^*(x^*) + f(x_0) \leq x^*(x_0) + \varepsilon).$$

Let be $f \in \Gamma(X)$.

By the *domain of the subdifferential* we mean $\text{dom } \partial f = \{x \in \text{ dom } f; \partial f(x) \neq \emptyset\}$ and the *range* of $\partial f$ is $\text{rng } \partial f = \{x^* X^*; \exists x \in X \text{ s.t. } x^* \in \partial f(x)\}$.

If $\varepsilon > 0$, then $\partial_\varepsilon f(x_0) \neq \emptyset$, and for $\varepsilon \leq \lambda$ it occurs the inclusion $\partial_\varepsilon f(x_0) \subset \partial_\lambda f(x_0)$. With each convex function we associate its *directional derivative* at $x_0 \in \text{dom } f$ as follows:

$$f'_+(x_0; u) = \lim_{t \searrow 0} \frac{f(x_0 + tu) - f(x_0)}{t} = \inf_{t>0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

and the $\varepsilon - derivative$ of $f$ is

$$f'_\varepsilon(x_0; u) = \inf_{t>0} \frac{f(x_0 + tu) - f(x_0) + \varepsilon}{t}.$$

Evidently, $f'_+(x_0; u) \leq f'_\varepsilon(x_0; u)$ for every $u \in X$ and $\varepsilon \geq 0$, and $f'_+(x_0; u) = \inf_{\varepsilon>0} f'_\varepsilon(x_0; u)$ (see [4]).

We shall use also the following well known formula:

$$f'_\varepsilon(x_0; u) = \sup\{x^*(u); x^* \in \partial_\varepsilon f(x_0)\} \text{ for each } x_0 \in \text{ dom } f,\ u \in X,$$

respectively

$$f'_+(x_0;u) = \sup\{x^*(u); x^* \in \partial f(x_0) \text{ for each } x_0 \in \text{ dom } f,\ u \in X,$$

if $f$ is continuous in $x_0$.

**3. The results of affine convergence with respect to subdifferentials.** By identifying the functions with their epigraphs, the affine topology $\tau_A$ on $\Gamma(X)$ (which was introduced by G.Beer in [3]) is generated by the all sets of the form

$$\begin{cases} \Gamma(X) \cap V^-( \text{ where } V \text{ is open in } X \times \boldsymbol{R}) \\ \Gamma(X) \cap (\text{ hypo } a)^{\#} (\text{ where } a \text{ is an affine continuous functional on } X) \end{cases}$$

or, equivalent,

$$\begin{cases} \Gamma(X) \cap (W \times (-\infty, \alpha))^- = \{f \in \Gamma(X);\ epif \cap (W \times (-\infty, \alpha)) \neq \emptyset\} \\ \Gamma(X) \cap (\text{ hypo } a)^{\#} = \{f \in \Gamma(X); \inf\{f(x) - a(x); x \in X\} > 0\}. \end{cases}$$

(where $W$ is open in $X$). (We denote by $V^- = \{C \in \mathcal{C}(X); C \cap V \neq \emptyset\}$ and by $V^{\#} = \{C \in \mathcal{C}(X); \inf\{||c - v||; c \in C, v \in V\} > 0\}$.)

In the same paper, Beer establishes a comparasion with the well known Mosco topology, as soon as the linear topology, namely $\tau_M \subset \tau_A \subset \tau_L$ on $\Gamma(X)$.

For the affine convergence of the sequences of functions we have the following characterization:

**Lemma 3.1.** *Let be $f_n, f \in \Gamma(X)$. Then $f_n \overset{\tau_A}{\to} f$ if and only if:*

*a)* $\forall x \in X, \exists x_n \to x s.t. f(x) \geq \limsup_n f_n(x_n)$

*b)* $\forall x^* \in X^*, f^*(x^*) \geq \limsup_n f_n^*(x^*)$.

**Proof.** Writing the convergence of $f_n$ at $f$ with the first type neighbourhoods, we obtain $\forall x \in X, \forall \alpha > f(x), \forall W$ an open neighbourhood for $x, \exists n_{W,\alpha}$ such that $\forall n \geq n_{W,\alpha}, \exists x_n \in W$ with $f_n(x_n) < \alpha$ that is $a$).

Using for $\tau_A$ the neighbourhoods of second type, b) is trivial in the case $f^*(x^*) = +\infty$; for $f^*(x^*) < +\infty$ the property becomes

$$\forall x^* X^*, \forall \alpha \in \boldsymbol{R}, s.t. \inf\{f(x) - x^*(x) + \alpha, x \in X\} > 0$$

$\exists n_{x^*,\alpha} \in I\!N$ s.t. $\forall n \geq n_{x^*,\alpha}, \inf_{x \in X}(f_n(x) - x^*(x) + \alpha) > 0$, equivalent with $\forall x^* \in X^*, \forall \alpha \in \boldsymbol{R}$ s.t. $\alpha > f^*(x^*), \exists n_{x^*,\alpha}$ s.t. $\forall n \geq n_{x^*,\alpha} \Rightarrow \alpha \geq f_n^*(x^*)$,that is b).

**Lemma 3.2.** *Let $f_n, f \in \Gamma(X)$; we consider the following assertions:*

1. $f_n \overset{\tau_A}{\to} f$ *(namely* $a) + b$*))*
2. $\begin{cases} a') \ \forall x \in \operatorname{dom} \partial f, \exists x_n \to x \ s.t. f(x) \geq \limsup_n f_n(x_n) \\ b') \ \forall x^* \in \operatorname{rng} \partial f, f^*(x^*) \geq \limsup_n f_n^*(x^*) \end{cases}$
3. $\begin{cases} a'') \ \forall x \in \operatorname{dom} \partial f, \exists x_n \to x \ s.t. f(x) = \lim_n f_n(x_n) \\ b'') \ \forall x^* \in \operatorname{rng} \partial f, f^*(x^*) = \lim_n f_n^*(x^*) \end{cases}$

*We have 1)⇒2)⇔3); if $X$ is a Banach space, then those three assertions are equivalent.*

**Proof.** 1)⇒2) is trivial.

2)⇔3) Let $x \in \operatorname{dom} \partial f, x_n \to x$ from $a'$) and $x^* \in \partial f(x)$. From Young's inequality it follows $f_n(x_n) \geq x^*(x_n) - f_n^*(x^*)$, whence, taking limit inferior as $n \to \infty$, we found $\liminf_n f_n(x_n) \geq x^*(x) - f^*(x^*) = f(x)$, which along with $a'$), get us $f(x) = \lim_n f_n(x_n)$. Similarly, we obtain $b''$).

3)⇒1) in Banach spaces.

a) The conclusion being evidently for $f(x) = +\infty$, we assume $x \in \operatorname{dom} f$. Using a well known result of convex analysis (see, e.g., Theorem 3.17 form [7]), there exists $u_k \in \operatorname{dom} \partial f$ such that $f(x) = \lim_k f(u_k)$; but $\forall k, \exists u_n^k \overset{n}{\to} u_k$ with $f(u_k) = \lim_n f_n(u_n^k)$; this implies $\exists k(n) \nearrow \infty$ s.t. $f(x) = \lim_n f_n(u_n^{k(n)})$. So the required sequence is $x_n = u_n^{k(n)}$.

b) Let $x^* \in \operatorname{dom} f^*, x_k^* \in \operatorname{dom} \partial f^*$ with $x_k^* \to x^*$. Form the lower semicontinuity of the functions $f_n, \forall \varepsilon > 0$,

$$f^*(x_k^*) = \lim_n f_n^*(x_k^*) \geq \lim_n \liminf_k f_n^*(x_k^*) - \varepsilon \geq \limsup_n f_n^*(x^*) - \varepsilon$$

for $k$ sufficiently large.

Since $x_k^* \operatorname{dom} \partial f^*$ there exists $x_k \in \operatorname{dom} f$ such that $f^*(x_k^*) = x_k^*(x_k) - f(x_k) \leq x^*(x_k) + \varepsilon - f(x_k) \leq f^*(x^*) + \varepsilon$ for $k$ sufficiently large, which yields, from above inequality, $f_n^*(x^*) \leq f^*(x^*) + 2\varepsilon, \forall \varepsilon > 0$. This implies b).

**Remark 3.1.** In fact, from the proof, we see that both conditions a) and b) are equivalent with

$$\begin{cases} \forall x \in \operatorname{dom} f, \exists x_n \to x \ s.t. f(x) = \lim_n f_n(x_n) \\ \forall x^* \in X^*, f^*(x^*) \geq \limsup_n f_n^*(x^*). \end{cases}$$

**Remark 3.2.** The assertion $b''$) is clearly weaker that

$$b''') \ \forall x^* \in \operatorname{rng} \partial f, \ \exists x_n^* \to x^* \text{ with } f^*(x^*) = \lim_n f_n^*(x_n^*)$$

.

In [11], the authors show that $a'$) and $b'''$) concur with the slice convergence of functions $f_n$ to $f$; so the affine convergence is stronger that slice convergence.

If we call the convergence, given by $a'$) and $b'$), the *weak affine convergence* $\tau_{\omega A}$), then it has the same property.

Following Attouch and Beer, we are led to the definition below:

**Definition 3.1.** *We assume that $f_n, f \in \Gamma(X)$; we said that a sequence $(u_n, u^*) \in \operatorname{dom} f_n \times \operatorname{dom} f^*$ is seminormalizing if $u_n \to u \in \operatorname{dom} f$ with both $\limsup_n f_n(u_n) \leq f(u)$ and $\limsup_n f_n^*(u^*) \leq f^*(u^*)$.*

**Theorem 3.1.** *Let $f_n, f \in \Gamma(X)$. The following assertions are equivalent:*

1. $f_n \overset{\omega A}{\to} f$ *(namely $a'$) and $b'$))*
2. $\begin{cases} i) & \forall x \in X, \exists x_n \to x \text{ s.t.} \forall x^* \in \partial f(x), \exists \varepsilon_n \to 0 \\ & \text{with } x^* \in \partial_{\varepsilon_n} f_n(x_n), \text{ for any } n \\ ii) & \text{there exists a seminormalizing sequence.} \end{cases}$
3. $\begin{cases} i) & \forall x \in X, \exists x_n \to x \text{ s.t.} \forall x^* \in \partial f(x), \forall \varepsilon > 0 \text{ we have} \\ & x^* \in \partial_{\varepsilon} f_n(x_n) \text{ for } n \text{ sufficiently large} \\ ii) & \text{there exists a seminormalizing sequence.} \end{cases}$

**Proof.** 1)⇒ 2) Let $x \in \operatorname{dom} \partial f$, $x^* \in \partial f(x)$ and $x_n \to x$ given by $a'$). $x^*(x) = f(x) + f^*(x^*) = \lim_n [f_n(x_n) + f_n^*(x^*)] \geq \lim_n x^*(x_n) = x^*(x)$, whence there exists $\lim_n [f_n(x_n) + f_n^*(x^*) - x^*(x_n)] = 0$. Denoting $\varepsilon_n = d_{f_n}(x_n, x^*) = f_n^*(x^*) + f_n(x_n) - x^*(x_n)$, it has the property (see [11]) that is the smallest number $\varepsilon > 0$, such that $(x_n, x^*) \in \partial_{\varepsilon} f_n$, namely $x^* \in \partial_{\varepsilon_n} f_n(x_n)$.

2)⇒3) It yields from the monotonicity of the subdifferential operator.

3)⇒1) The idea of the proof is an adjustement of wich is used by Attouch and Beer in Theorem 4.2 [2].

If $x \in \operatorname{dom} \partial f, x^* \in \partial f(x)$, by hypothesis we have $f_n(x_n) \leq f_n(u_n) + x^*(x_n - u_n) + \varepsilon$, with $u_n$ form the definition of seminormalizing sequence.

Taking lim sup with respect to $n$ and relying by Rochafellar integration formula we find $\limsup_n f_n(x_n) \leq f(x)$.

The Rochafellar formula holds for $f^*$, so also $\limsup_n f_n^*(x^*) \leq f^*(x^*)$, $\forall x^* \in \operatorname{rng} \partial f$. ∎

We are inquired to establish conditions which assure just the inclusion between the subdifferentials; in this direction it occurs:

**Corollary 3.1.** *Let* $f_n, f \in \Gamma(X), f_n \overset{\omega A}{\to} f$.

*i) If* $\exists x_0 \in \operatorname{dom} f$ *such that* $f$ *is continuous in* $x_0$ *and* $x_n \to x_0$ *given by Theorem 3.1., then there exists* $\varepsilon_n > 0$ *with the property* $\partial f(x_0) \subset \partial_{\varepsilon_n} f_n(x_n)$ *if and only if* $\inf\{f_n(x) - f'_+(x_0; x - x_n);\ x \in X\} > -\infty$.

*ii) If* $f$ *is continuous on* $X$, *then the inclusion from i) is assured by the condition:*

$$\inf\{f_n(x) - f(x);\ x \in X\} > -\infty.$$

*The desired* $\varepsilon_n$ *is* $\varepsilon_n = \sup\{d_{f_n}(x_n; x^*);\ x^* \in \partial f(x_0)\}$.

The proof of this result is similarly to the proof of Corollary 3.2 above; after this enunciation we shall also discusse the naturality of the condition $\inf\{f_n(x) - f(x);\ x \in X\} > -\infty$ by lemmas.

Rearranging the proof of Theorem 3.1, for the affine convergence we obtain:

**Theorem 3.2.** *Let* $f_n, f \in \Gamma(X)$. *We have the equivalences:*

1. $f_n \overset{\tau_A}{\to} f$
2. $\begin{cases} i)\ \forall x \in X, \exists x_n \to x \text{ s.t } \forall \varepsilon, \delta > 0 \\ \quad (\delta \geq 0 \text{ for Banach spaces}),\ \forall x^* \in \partial_\delta f(x) \\ \quad \text{then } x^* \in \partial_{\varepsilon+\delta} f_n(x_n) \text{ for } n \text{ sufficiently large.} \\ ii)\ \text{there exists a seminormalizing sequence.} \end{cases}$

**Corollary 3.2.** *Let* $f_n, f \in \Gamma(X), f_n \overset{\tau_A}{\to} f,\ x_0 \in \operatorname{dom} f$ *and* $x_n \to x_0$ *from the characterization of affine convergence (see 2(i) in the th.3.1). then the following properties are fulfilled.*

*i)* $\forall \delta > 0, \exists \varepsilon_n(\delta) > 0$ *such that*

$$\delta_\delta f(x_0) \subset \partial_{\delta+\varepsilon_n} f_n(x_n) \tag{1}$$

*if and only if* $\inf\{f_n(x) - f'_\delta(x_0; x - x_n);\ x \in X\} > -\infty$. *Particularly, if* $\inf\{f_n(x + x_n - x_0) - f(x);\ x \in X\} > -\infty$, *then the inclusion (1) holds too, and* $\varepsilon_n$ *is independent of* $\delta$.

*ii) Suppose that* $f$ *is continuously on* $\operatorname{dom} f$ *and* $\inf\{f_n(x) - f(x);\ x \in X\} > -\infty$; *then* $\exists \varepsilon_n \geq 0$ *s.t.* $\forall \delta > 0$ *the inclusion (1) is satisfied.*

**Proof.** i) Let $x^* \in \partial_\delta f(x_0)$. Then $\forall \varepsilon > 0, \exists n_{\varepsilon,\delta}$ such that $\delta \geq f^*(x^*) + f(x_0) - x^*(x_0) \geq f_n^*(x^*) + f_n(x_n) - x^*(x_n) - \varepsilon$.

We denote $\alpha_n(x^*) = f_n^*(x^*) + f_n(x_n) - x^*(x_n)$. A necessary and sufficient condition for any $\delta > 0$ to exist an $\varepsilon_n(\delta) > 0$ such that $\alpha_n(x^*) \leq \delta + \varepsilon, \forall x^* \in \partial_\delta f(x_0)$, is that $\alpha_n(\delta) < \infty$, where $\alpha_n(\delta) = \sup\{\alpha_n(x^*);\ x^* \in \partial_\delta f(x_0)\}$ (then we get $\varepsilon_n(\delta) = \begin{cases} \alpha_n(\delta) \text{ for } \alpha_n(\delta) \geq 0 \\ 0 \text{ for } \alpha_n(\delta) < 0 \end{cases}$).

Now $\alpha_n(\delta) = \sup_{x^* \in \partial_\delta f(x_0)} \sup_{x \in X} \{x^*(x) - f_n(x) - x^*(x_n)\} + f_n(x_n) =$

$$\sup_{x \in X} \{ \sup_{x^* \in \partial_\delta f(x_0)} [x^*(x - x_n) - f_n(x)] + f_n(x_n) =$$

$$= \sup_{x \in X} [f'_\delta(x_0; x - x_n) - f_n(x)] + f_n(x_n)$$ (in case $\delta = 0$, like in Corollary 3.1, for this formula we need the continuity of $f$ at $x_0$).

So

$$(2) \quad \alpha_n(\delta) = \sup_{x \in X} \left[ \inf_{t>0} \frac{f(x_0 + t(x - x_n)) - f(x_0) + \delta}{t} - f_n(x) \right] + f_n(x_n).$$

Using the convexity of $f$, we obtain

$$\alpha_n(\delta) \leq \sup_{x \in X} \inf_{t>0} \left[ f(x + x_0 - x_n) - f(x_0) - f_n(x) + \frac{\delta}{t} \right] + f_n(x_n) =$$
$$= \sup_{x \in X} [f(x + x_0 - x_n) - f_n(x)] + f_n(x_n) - f(x_0) =$$
$$= \sup_{y \in X} [f(y) - f_n(y + x_n - x_0)] + f_n(x_n) - f(x_0) = \beta_n,$$

who is not depndently on $\delta$. We give $\varepsilon_n = \begin{cases} \beta_n, \text{ for } \alpha_n \geq 0 \\ o \text{ for } \alpha_n < 0 \end{cases}$.

ii) From (2), we choose $n$ large enogh such that

$$\alpha_n(\delta) \leq \sup_{x \in X} \left[ \inf_{t>0} \frac{f(x_0 + t(x - x_0)) + \gamma - f(x_0) + \delta}{t} - f_n(x) \right] + f_n(x_n)$$

and from convexity we find $\alpha_n(\delta) \leq \sup_{x \in X} [f(x) - f_n(x)] + f_n(x_n) - f(x_0)$. ∎

The inclusion (1) suggests us an analogy with the results of Hiriart–Urruty ([5]) relating to the characterization of $\varepsilon$–minima of difference of convex functions, namely $x_0$ is an $\varepsilon$–minimum of $h = g - f$ on $X$ iff $\partial_\delta f(x_0) \subset \partial_{\delta+\varepsilon} g(x_0)$ for any $\delta > 0$ (we recall that $x_0$ is an $\varepsilon$–minimum of $h$ on $X$ if $h$ is finite at $x_0$ and $h(x) \geq h(x_0) - \varepsilon$ for all $x \in X$).

For this comparasion first we state a link between the subdifferentials of $f$ at $x_0$ and $x_n$; after that we get a converse implication between the inclusion of subdifferentials and $\inf_{x \in X} [f_n(x) - f(x)]$.

**Lemma 3.3.** *Let $f \in \Gamma(X)$, $x_0 \in \operatorname{dom} f$. Suppose $f$ is continuous at $x_0$ and $x_n \to x_0$; then $\forall \delta > 0, \forall \varepsilon > 0, \exists n_{\varepsilon,\delta}$ such that $\forall n \geq n_{\varepsilon,\delta}$ we have $\delta_\delta f(x_0) \subset \partial_{\delta+\varepsilon} f(x_n)$ and $\partial_\delta f(x_n) \subset \partial_{\delta+\varepsilon} f(x_0)$.*

**Lemma 3.4.** *If $f, f_n \in \Gamma(X)$, $f$ continuous at $x_0$ such that $\forall \varepsilon, \delta > 0$, we have $\partial_\delta f(x_0) \subset \partial_{\varepsilon+\delta} f_n(x_n)$ for $n$ sufficiently large whenever $x_n \to x_0$, then $\inf\limits_{x\in X}[f_n(x) - f(x)] > -\infty$ (i.e. $dom f_n \subset \operatorname{dom} f$ for $n$ sufficiently large).*

**Proof.** By hypothesis and Lemma 3.3, for $n$ sufficiently large (who depends on $\delta$ and $\varepsilon$), it occurs the inclusion $\partial_{\delta/2} f(x_n) \subset \partial_\delta f(x_0) \subset \partial_{\varepsilon+\delta} f_n(x_n)$. Using Theorem 4.4 [5] it results that $x_n$ is a global $(\varepsilon + \frac{\delta}{2})$–minimum of the function $f_n - f$, i.e. $f_n(x) - f(x) \geq f_n(x_n) - f(x_n) - (\varepsilon + \frac{\delta}{2})$ for $n$ sufficiently large. Taking $\inf\limits_{x\in X}$, by the continuity of $f$ at $x_0$, we find $\inf\limits_{x\in X}[f_n(x) - f(x)] > -\infty$. ∎

In accordance with Corollary 3.2, in the case of affine convergence at a continuous function $f, x_n$ happens to be a weaker condition than the condition from Lemma 3.4 (to be an $\varepsilon + \frac{\delta}{2}$–minimum for $n$ sufficiently large), $\varepsilon > 0$ being fixed for every $n$.

**4. The scalar convergence of the subdifferentials in Banach spaces.** Now, we propose to see for certain convergence of the subdifferentials what is the property induced for the functions and we observe that for the subdifferentials we must consider a sufficiently weak convergence (in [2] and [11], the Kuratowski convergence, who is a weak convergence, is equivlent with the slice convergence of the functions, which – conformally to the clasiffication form [10]–is strong enough). This reason (and not only) shows us to deal with the scalar convergence of the subdifferentials.

**Definition 4.1.** *We say That the sequence $(A^n)$ of closed convex sets is scalar convergent at $A (A^n \xrightarrow{s} A)$ if $\sup\limits_{a\in A^n} x^*(a) \to \sup\limits_{a\in A} x^*(a), \forall x^* \in X^*$.*

We also use the following notations ([10]):

$$A^n \xrightarrow{s+} A \Leftrightarrow \limsup_n \sup_{a\in A^n} x^*(a) \leq \sup_{a\in A} x^*(a), \forall x^* \in X^*$$

$$A^n \xrightarrow{s-} A \Leftrightarrow \sup_{a\in A} x^*(a) \leq \liminf_n \sup_{a\in A^n} x^*(a), \forall x^* \in X^*$$

We have:

**Theorem 4.1.** *Let $X$ be a reflexive Banach space, $f_n, f \in \Gamma(X), x \in \operatorname{dom} f, x_n \in \operatorname{dom} f_n$.*

1. *If $\limsup\limits_n f_n(x_n) \leq f(x)$ and $\varepsilon, \varepsilon_n > 0, \varepsilon_n \to 0$ such that $\partial_{\varepsilon_n} f_n(x_n) \xrightarrow{s+} \partial_\varepsilon f(x)$, then $\forall v \in X, \forall v_n \in X$ such that $\limsup\limits_n f_n(v_n) \leq f(v)$.*

*2. Moreover, if* $f(x) \leq \limsup_n f_n(x_n)$ *and* $\partial_{\varepsilon_n} f_n(x_n) \overset{s^-}{\to} \partial_\varepsilon f(x)$, *too (i.e.* $f(x) = \limsup_n f_n(x_n)$ *and* $\partial_{\varepsilon_n} f_n(x_n) \overset{s}{\to} \partial_\varepsilon f(x)$*), then* $f(x) = \min_{v \in x} f(v)$.

**Remark 4.1.** From the fact that we show the following assertion (expressed in terms of directional derrivatives):

1') If $\limsup_n f_n(x_n) \leq f(x)$ and there exists $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $\limsup_n (f_n')_{\varepsilon_n}(x_n; u) \leq f_+'(x; u), \forall u \in X$, then for any $v \in X$ there exists $v_n \in X$ such that $\limsup_n f_n(v_n) \leq f(v)$.

2') If, in addition $f(x) \leq \liminf_n f_n(x_n)$ and $f_+'(x; u) \leq \liminf_n (f_n')_{\varepsilon_n}(x_n; u), \forall u \in X$, then $f(x) = \min_{v \in X} f(v)$.

**Proof.** 1) The upper scalar convergence of the subdifferentials, for $x \in \operatorname{dom} f$, amounts to $\limsup_n (f_n')_\varepsilon(x_n; u) \leq (f_+'(x; u)) \leq f_\varepsilon'(x; u), \forall \varepsilon \geq 0, \forall u \in X$.

Let $v \in X$; we put $u = v - x$; then $\forall \delta > 0, \exists n_\delta$ such that $\forall n \geq n_\delta \Rightarrow (f_n')_\varepsilon(x_n; v - x) < f_\varepsilon'(x; u) + \delta < f(v) - f(x) + \varepsilon + \delta$ and so

$$(f_n')_+(x_n; v - x) < f(v) - f(x) + \varepsilon + \delta.$$

By the definition of directional derivative there exists $t_n > 0$ such that $\frac{f_n(x_n + t_n(v-x)) - f_n(x_n)}{t_n} < f(v) - f(x) + \varepsilon + \delta$.

We observe that we can choose $t_n < 1$ (even $t_n \searrow 0$) because the proportion is increasing in $t$; we denote $v_n = x_n + t_n(v - x)$. Then

$$f_n(v_n) \leq f_n(x_n) + f(v) - f(x) + \varepsilon + \delta$$

and taking limit superior we obtain $\limsup_n f_n(v_n) \leq f(v)$.

**Remark 4.2.** The conclusion holds also in the hypothesis

$$\begin{cases} \limsup_n f_n(x_n) \leq f(x) \\ \partial f_n(x_n) \overset{s+}{\to} \partial_\varepsilon f(x). \end{cases}$$

**Remark 4.3.** Since $(t_n)$ can be choseen such that $t_n \to 0$, it results that $v_n - x_n \to 0$ and the inequlaity $\limsup_n f_n(v_n) \leq f(v)$ gives us the behaviour of $f$ with respect to $x$ which is a privileged point.

2) Now we assume that $(f_+'(x; u) \leq) f_\varepsilon'(x; u) \leq \liminf_n (f_n')_{\varepsilon_n}(x_n; u)$. $\forall \delta > 0, \exists n_\delta$ such that for $\forall n \geq n_\delta$ we have

$$\inf_{t>0} \frac{f(x + t(v - x)) - f(x)}{t} < \frac{f_n(v_n) - f_n(x_n) + \varepsilon_n}{t_n} + \delta.$$

Hence there exists $t'_n,\ 0 < t'_n < t_n$, such that

$$\frac{f(x + t'_n(x - v)) - f(x)}{t'_n} < \frac{f_n(v_n) - f_n(x_n) + \varepsilon_n}{t_n} + \delta.$$

So, this implies $f(x + t'_n(x - v)) < f(x) + f_n(v_n) - f_n(x_n) + \varepsilon_n + t'_n\delta$; by the lower semicontinuity of $f$ at $x$ and $t'_n \to 0$ we find

$$f(x) \leq \liminf_n f(x + t'_n(x - v)) \leq \liminf_n f_n(v_n)$$

which get us (together with 1)) $f(x) \leq f(v), \forall v \in X$.

The last result, which follows the Kuratowski convergence of the functions, is not comparable to that of Theorem 4.1, where it does not required $x_n \to x$.

**Definition 4.2.** *A sequence of functions $f_n \in \Gamma(X)$ is Kuratowski convergent at $f \in \Gamma(X)(f_n \xrightarrow{K} f)$ if*

*i)* $\forall x \in X, \exists x_n \to x$ *such that* $f_n(x_n) \to f(x)$

*ii)* $\forall x \in X, \forall x_n \to x,\ f(x) \leq \liminf_n f_n(x_n)$.

**Theorem 4.2.** *Let $X$ be a reflexive Banach space, $f_n, f \in \Gamma(X)$, $f_n \xrightarrow{K} f$. Then $\forall x \in \operatorname{dom} f, \exists x_n \to x$ such that $\partial_{\varepsilon_n} f_n(x_n) \xrightarrow{s+} \partial_\varepsilon f(x), \forall \varepsilon, \varepsilon_n > 0$, with $\varepsilon_n \to 0$. (In term of directional derivative, this means $f'_+(x;u) \geq \limsup_n (f'_n)_+(x_n;u),\ \forall u \in X$).*

**Proof.** We fix $u \in X$. By $f_n \xrightarrow{K} f$ it follows that $\forall t > 0, \forall x \in X, \exists y_n^t \to x + tu$ such that $f_n(y_n^t) \to f(x + tu)$. We denote $x_n^t = y_n^t - tu$ and so $f_n(x_n^t + tu) \to f(x)$ with $x_n^t \to x$. But $f_n \xrightarrow{K} f \Rightarrow f(x) \leq \liminf_n f_n(x_n^t)$.
$f'_\varepsilon(x;u) \geq f'_+(x;u) = \inf_{\varepsilon>0} \inf_{t>0} \frac{f(x+tu)-f(x)+\varepsilon}{t} = \lim_{t \searrow 0} \lim_{\varepsilon \searrow 0} \frac{f(x+tu)-f(x)+\varepsilon}{t} \geq$
$\geq \limsup_n \frac{f_n(x_n^{t_n}+t_n u)-f_n(x_n^{t_n})+\varepsilon_n}{t_n}$ for $\forall \varepsilon_n \searrow 0, \forall t_n \searrow 0$.
We denote $x_n^{t_n} = x_n$. If $x_n \in \operatorname{dom} f_n$, this ensures that $f'_\varepsilon(x;u) \geq$
$\geq \limsup_n \inf_{t>0} \frac{f_n(x_n+tu)-f_n(x_n)+\varepsilon_n}{t} = \limsup_n (f'_n)_{\varepsilon_n}(x_n;u) \geq$
$\geq \limsup_n (f'_n)_+(x_n;u)$.

For $x_n \notin \operatorname{dom} f_n$ the inequality is obvious. ∎

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