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DIFFERENTIABLE STRUCTURES ASSOCIATED TO NON–STANDARD DIFFERENTIABLE MANIFOLDS

BY

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Dedicated to Professor Gh.Gheorghiev on the occasion of his 90-th birthday.

Starting from the field $\boldsymbol{R}$ of the real numbers as a model for the formal system of the theory of the real ordered closed fields, complete in the topological sense and Archimedian, A.Robinson [11] has found the extension ${}^*\boldsymbol{R}$ which is **ordered, nonarchimedian, not complete** and, moreover, the models of ${}^*\boldsymbol{R}$ are **not isomorphic**. We work on a model [6] based on an ultrafilter δ–incomlete, fixed and built as a consequence of a theorem due to Lowenheim–Skolem [4] which states that any infinite structure has an elementary extension. We use the three principles: of transfer, of idealisation and of standardisation and the rules stated in [10].

The first order properties are identically transformed, but the properties of higher order have restrictions on the quantifiers, only the internal sets are in the range of a transformed quantifier.

Let $M_0 = M$ be an infinite set and we define iteratively $M_{k+1} = \mathcal{P}\left(\bigcup_{i=0}^{k} M_i\right), k \in \boldsymbol{N}$ and $\Updownarrow = \bigcup_{k\in\boldsymbol{N}} M_k$. The elements of $\Updownarrow$ are called entities and the algebrical object $(\Updownarrow, \in)$ is called superstructure [11]. The individual constants of the supertructure are written ${}^*M$ and the new constructions will be based on the external sets that are closly related to the internal sets. We have obtaind a generalization of the geometrical space for the concept of **non–standard differentiable manifold**.

We associate the model space ${}^*\boldsymbol{R}^n$ to the **non–standard geometrical space** ${}^*M$ and we obtain the couple $({}^*M, {}^*\boldsymbol{R}^n)$. The fundamental objects is the **point**, that is supposed to be **non– standard** and **finite**. This point defines a standard frame and we consider that through this point is passing

a **non–standard** differentiable manifold.

1. The Definition of the Non–standard Differentiable Manifolds.

Definition 1. *A local chart of the couple* $({}^*M, {}^*\boldsymbol{R}^n)$ *is a pair* $({}^*U, \varphi) := {}^*h$, *where* ${}^*U \subset {}^*M, \varphi : {}^*U \to {}^*\boldsymbol{R}^n$ *is a standard injective function and* $\varphi({}^*U) \subset {}^*\boldsymbol{R}^n$ *is an open set in the* θ*–topology [10, p.99] of* ${}^*\boldsymbol{R}^n$.

For the map ${}^*h = ({}^*U, \varphi)$, ${}^*U$ is called geometrical zone, $\varphi({}^*U)$ is called arithmetical zone, and $\varphi : {}^*U \to \varphi({}^*U)$ is a standard one-one function of the chart.

Definition 2. *It is called atlas of the couple* $({}^*M, {}^*\boldsymbol{R}^n)$ *a set of charts* ${}^*A = \{{}^*h_a = ({}^*U_a, \varphi_a), a \in J\}$ *with the property:*

(A.1) $\bigcup\limits_{a\in J} {}^*U_a = {}^*M$ (i.e. the domain of charts constitutes a cover of ${}^*M$ or, equivalently, $\forall x \in {}^*M, \exists {}^*h_a = ({}^*U_a, \varphi_a) \in {}^*A$, s.t. $x \in {}^*U_a$).

An atlas ${}^*A$ of $({}^*M, {}^*\boldsymbol{R}^n)$ is called of class $C^k, k = \overline{1, \infty}$ if it satisfies the following properties:

(A.2) $\forall {}^*h_a, {}^*h_b \in {}^*A$ with ${}^*U_a \cap {}^*U_b \neq \emptyset$, then $\varphi_a({}^*U_a \cap {}^*U_b)$ and $\varphi_b({}^*U_a \cap {}^*U_b)$ are open sets in the θ–topology of ${}^*\boldsymbol{R}^n$.

(A.3) $\forall {}^*h_a, {}^*h_b \in {}^*A$ with ${}^*U_a \cap {}^*U_b \neq \emptyset$, then the standard application:

$$\varphi_b \circ \varphi_a^{-1} : \varphi_a({}^*U_a \cap {}^*U_b) \subset {}^*\boldsymbol{R}^n \to \varphi_b({}^*U_a \cap {}^*U_b) \subset \boldsymbol{R}^n. \tag{1}$$

is regular of class C^k.

The standard application (1) is called a change of charts or transformation of coordinates. Since the transformation of coordinates $\varphi_b \circ \varphi_a^{-1}$ is standard one–one, it is a diffeomorfism.

Definition 3. *A chart* ${}^*h = ({}^*U, \varphi)$ *of the couple* $({}^*M, {}^*\boldsymbol{R}^n)$ *is called compatible with the atlas* ${}^*A^k$, *if* $\forall {}^*h_a = ({}^*U_a, \varphi_a) \in {}^*A^k$ *so that* ${}^*U \cap {}^*U_a \neq \emptyset$, *we have:*

(i) $\varphi_a({}^*U \cap {}^*U_a), \varphi({}^*U \cap {}^*U_a)$ are open sets from ${}^*\boldsymbol{R}^n$,

(ii) the standard application

$$\varphi_b \circ \varphi_a^{-1} : \varphi_a({}^*U \cap {}^*U_a) \subset {}^*\boldsymbol{R}^n \to \varphi({}^*U \cap {}^*U_a) \subset {}^*\boldsymbol{R}^n$$

is regular of class C^k.

We put ${}^*h \sim {}^*A^k$.

Definition 4. *Two* C^k*–atlases* ${}^*A^k, {}^*A'^k$ *of the couple* $({}^*M, {}^*\boldsymbol{R}^n)$ *are called equivalent (we write* ${}^*A^k \sim {}^*A'^k$*) if* $\forall h_a \in {}^*A^k$ *we have* ${}^*h \sim {}^*A'^k$.

Proposition 1. *The relation of equivalence between atlases is reflexive, symetrical and transitive.*

Proof. Since the functions are standard and the induced topology is the θ–topology (the finest topology, see [1], p.99) the proof is identical with the standard one.

Definition 5. *It is called a differential structure of class C^k on the couple $({}^*M, {}^*\boldsymbol{R}^n)$ a coset of C^k atlases (written ${}^*\hat{A}^k$).*

Definition 6. *It is an n–dimensional* **non–standard differentiable manifold** *of class $C^k, k \geq 1$, the triple $({}^*M, {}^*\boldsymbol{R}^n, {}^*A^k)$ constituted from the couple $({}^*M, {}^*\boldsymbol{R}^n)$ and a C^k–differentiable structure on $({}^*M, {}^*\boldsymbol{R}^n)$ defined by the atlas ${}^*A^k$.*

Definition 7. *Let $({}^*M, {}^*\boldsymbol{R}^n, {}^*A^k)$, $({}^*N, {}^*\boldsymbol{R}^m, {}^*B^l)$ be two manifolds. A standard application $f : {}^*M \to {}^*N$ is called differentiable of class C^p, $o \leq p \leq$ min (l, q) if:*

(i) it is continous in the θ–topology,

(ii)$\forall x \in {}^*M, \exists {}^*h_a = ({}^*U_a, \varphi_a) \in {}^*A^k$ in x and ${}^*h_a = ({}^*V_a, \psi_a) \in {}^*B^l$ in $f(x)$ s.t. the standard application

(2) $\psi_a \circ f \circ \varphi_a^{-1} \circ : \varphi_a({}^*U_a \cap f^{-1}({}^*V_a)) \subset {}^*\boldsymbol{R}^n \to \psi_a(f^{-1}({}^*U_a) \cap V_a) \subset {}^*\boldsymbol{R}^n$

is differentiable of class C^p.

2. The Tangent Liniar Space to a Non–standard Differentiable Manifold.

a. Algebras of Real Function.

Definition 8. *Let $({}^*M, {}^*\boldsymbol{R}^n, {}^*A^k)$ be a C^p non–standard manifold and ${}^*D \subset {}^*M$ an open set in the θ–topology. A real standard function $f : {}^*D \subset {}^*M \to {}^*\boldsymbol{R}$ is called differentiable of class $C^p, p \leq k$, if $\forall a \in {}^*D$, if there exists a chart ${}^*h_a = ({}^*U_a, \varphi_a) \in {}^*A^k$ in x s.t. the standard function*

(3) $f \circ \varphi_a^{-1} : \varphi_a({}^*D \cap {}^*U_a) \subset {}^*\boldsymbol{R}^n \to {}^*\boldsymbol{R}$ *is differentiable of class C^p.*

The function $f \circ \varphi_a^{-1} := F$ is called the local representation of f in the chart ${}^*h_a$ and it is written (3′) $y = F(x', \ldots x^n)$, called the equation of the local representation of the function f in the chart ${}^*h_a$.

We shall set:

• $C_x^p({}^*M)$–for the set of the real functioons C^p–differentiable that are defined on the open sets from ${}^*M$ that contain the point x.

• $C_{{}^*D}^p({}^*M)$–for the set of the real functions C^p–differentiable that are defined on the set ${}^*D \subset {}^*M$.

• $C^p({}^*M)$–for the set of the real functions C^p–differentiable that are defined on the set ${}^*M$.

Proposition 2. *(i) If $f, g \in C^p_x(^*M)$, then $f+g, \lambda \cdot f, f \cdot g \in C^p_x(^*M)$; analogusly for $C^p_{^*D}(^*M), C^p(^*M)$.*

(ii) The sets of functions $C^p_x(^*M)$, $C^p_{^*D}(^*M)$, $C^p(^*M)$ are commutative real algebras (∞ dimensional, satisfying $C^p_x(^*M) \supset C^p_{^*D}(^*M) \supset C^p(^*M)$)

(iii) If the local representation in a chart $^*h_a \in {}^*A^k$ of the functions $f, g \in C^p_x(^*M)$ are F, G, then the local representation of the functions $f+g, \lambda \cdot f, f \cdot g$ are respectively $F+G, \lambda \cdot F, F \cdot G$. Analogies follows for $C^p_{^*D}(^*M), C^p(^*M)$.

Proof. By direct verification. Since the functions are standard the operations with them are defined like in the standard case and the induced topology being the θ–topology, the verification follows exactly like in the standard case.

b. The Tangent Linear Space $T_x{}^*M$.

Let be the C^k manifold $(^*M, {}^*\boldsymbol{R}^n, {}^*A^k), k \geq 1$ and the set of the triples:

$$^*T = \{(x \in {}^*M^k, h_a = (^*U_a, \varphi_a), (X^i) \in {}^*\boldsymbol{R}^n) :$$

$$: x \in {}^*V_a\} \subset {}^*M \times {}^*A^k \times {}^*\boldsymbol{R}^n.$$

In the set $^*T$ we define the next binary relation:

$$(4) \qquad ((x, {}^*h_a, (X^i)) \sim (y, {}^*h_b, (Y^i)) \Leftrightarrow x = y, \quad Y^i = \frac{\partial y^i}{\partial x^j}(x) \cdot X^j$$

where $y^i = y^i(x', \dots x^n), i = \overline{1,n}$ represent the transformation of coordinates $\varphi_b \circ \varphi_a^{-1}$ from the chart $^*h_a$ to $^*h_b$ that was anteriourly defined.

Evidently "$\sim$" is a relation of equivalence on $^*T$.

Definition 9. *It is called a tangent vector denoted by $^*X$, the coset modulo the relation "$\sim$", written $[x, {}^*h_a, (X^i)]$–that is defined by the triplet $[x, {}^*h_a, (X^i)) \in {}^*T$ (that is $^*X_x = [x, {}^*h_a, (X^i)]$).*

The hyperreal numbers $(X^i) \in^* \boldsymbol{R}^n$ are called the coordinates of the tangent vector X_x in the chart $^*h_a$.

The relations $(*)$ $Y^i = \frac{\partial Y^i}{\partial X^j}(x) \cdot X^j$, $i, j = \overline{1,n}$ constitute the formulas of transformation of the coordinates of the vector $^*X_x$ at a change of charts $^*h_a \to {}^*h_b$, the transformation of coordinates being $\varphi_b \circ \varphi_a^{-1}$.

A remarkable result is the fact that the tangent vector is surely standard, in the conditions that Y^i, X^j are not infinitely great.

Proposition 3. *The set of the tangent vectors at the C^k–non–standard manifold $(^*M, {}^*\boldsymbol{R}^n, {}^*A^k)$ in the point $x \in {}^*M$ forms a non–standard liniar space which is tangent to $^*M$ in x.*

Proof. The laws are defined conveniently in a chart

$$^*h_a[x, {}^*h_a, (X^i)] + [^*h_a[x, {}^*h_a, (Y^i)] \stackrel{def}{=} [x, {}^*h_a, (X^i + Y^i)] \text{ and}$$

$$\lambda[x, {}^*h_a, (X^i)] \stackrel{def}{=} [x, {}^*h_a, (\lambda X^i)].$$

The verifications follow the standard reasoning.

Definition 10. *The non–standard liniar space that is tangent in x at $^*M$ is called the non-standard liniar space of the non–standard manifold in x.*

Remark. The linear space is non-standard because it is defined in points that are not necessarely standard, but for all the points from the same monad the tangent vector is exactely one, that is the standard part of all the tangent vectors in the points of the monad.

It is obvious that the vector bundle T^*M of a manifold $^*M$ is standard and analogusly the fields of tangent vectors to a manifold and the square brackets of fields are also standard. The consequence of this result, the study of the important geometrical objects and of the geometrical structures on a non-standard differentiable manifold, is not difficult, being reduced to the standard case.

REFERENCES

1. A l b u, A.C. – *Primii paşi în analiza nestandard.* Tip.Univ.Timişoara 1990. *Varietăţi diferenţiabile* – curs anul III, Univ.Oradea 1994.
2. B a r w i s e, J.A. – *An introducţion to first–order logic Hand of Mathematical Logic. Model Theory.* Editor Joan Barwise 1978.
3. B u c u r, I. – *Capitole speciale de algebră.* E.A. R.S.R.Bucureşti 1980.
4. C o h e n, P.J. – *Set Theory and the Continuum Hypothesis.* W.A.Benjamin New–York 1966.
5. D a v i s, M. – *Applied Nonstandard Analysis.* John Wiley & Sons, New–York–London 1977.
6. D r ag o ş, P. – *Elemente de geometrie diferenţială definite şi calculate pe un model de analiză non– standard.* Analele Univ.Oradea 1991, 1992.
7. G o z e, M. – *Étude d'un point infiniment petit, dans Le labyrinthe du continu*, Colloq.Cerisy, Ed.Sinaceur, Springer V., 1993.
8. K a k u d a, Y. – *Theory of infinitesimals without non– standard models.* Kobe J.Math. 2. 1985.
9. L u x e m b u r g, W.A.J. – *What is non standard analysis?* Papers in the Found of Math.Part.III. Amer. Monthly 80 (6). 1973.

10. N e l s o n, Edw. – *Internal Set Theory: A new approach to Nonstandard Analysis*. Bul. A.M.S. 83 (6), 1988.
11. R o b i n s o n, A. – *Non–Standard Analysis*. North Holland Publ.Comp.1966.
12. S t r o y a n, K.D. – *Infinitesimal Analysis of Curves and Surface Hand of Mathematical Logic*. Editor Joan Barwise.

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