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ON THE HYPERGROUPS WITH FOUR PROPER PAIRS AND THREE OF FOUR NON–SCALAR ELEMENTS[1]

BY

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Summary: In this paper the authors study the hypergroups, such that are exactly four pairs of elements, which define hyperproducts (i.e. with $P(W) = 4$), and such that there are three or four non–scaler elements.

They solve the combinatorial problem of finding, up to isomorphism, all the tables of the aforesaid hypergroups and therefore, counting also on the papers [3], [4], they bring to a close the problem of finding all the finite hypergroups such that $|P(H)| = 4$.

1. Introduction. We recall that a **hypergroup** H is a non–empty set equipped with a hyperoperation such that the following two conditions are satisfied:

$$\forall (x, y, z) \in H^3,\ (x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{(associativity)}; \tag{1.1}$$

$$\forall x \in H,\ H \cdot x = x \cdot H = H \quad \text{(reproducibility)}. \tag{1.2}$$

Moreover we denote:

$P(H) = \{(x, y) \in H^2 / |x \cdot y| \geq 2\}$ (the set of proper pairs);
$S_l(H) = \{x \in H / |x \cdot y| = 1,\ \forall y \in H\}$ (the set of left scalars);
$S_r(H) = \{x \in H / |y \cdot x| = 1,\ \forall y \in H\}$ (the set of right scalars);
$S(H) = S_l(H) \cap S_r(H)$ (the set of scalars);
$M(H) = H / S(H)$

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It is known that if $S(H)$ is non empty then it is **a group** (see [2]).

In [5], [6], [7], one had studied the hypergroups with $|P(H)| \leq 3$.

In [3], the authors have begun to study the hypergroups with $|P(H)| = 4$, determining their structure when $S(H) = \emptyset$, while in [4] they have studied the hypergroups such that $P(H) = 4$ and $S(H) \neq \emptyset$, solving completely the case $|M(H)| = 2$ and partially the case $|M(H)| = 3$. Note that $|M(H)| \in \{2, 3, 4\}$.

In the present paper, they bring to a successful conclusion this study, completing the case $|M(H)| = 3$ and handling the case $|M(H)| = 4$.

We advice that, in order to simplify the writing, when a hyperproduct $\prod x_i$ is a singleton $\{\alpha\}$, we can omit the brackets and write $\prod x_i = \alpha$ instead of $\prod x_i = \{\alpha\}$. While if $|\prod x_i| > 1$, we say that it is a **proper hyperproduct**.

Moreover, whenever a proper pair gives a hyperproduct equal to $S(H)$, it is undestood that $|(S(H)| > 1$.

2. Some preliminary lemmas. We premise some results which will be useful in the following.

Lemma 2.1. (D.Freni, M.Gutan, Y.Sureau [5]) *If H is a hypergroup such that $S(H) \neq 0$, then H has a unique scalar identity which is the identity of the group $S(H)$.*

Lemma 2.2. *If H is a hypergroup such that $S(H) \neq \emptyset$, and $\forall z \in M(H)$, $z \cdot S(H) = S(H) \cdot z = \{z\}$, then:*

(i) $\forall (a, b) \in M(H) \times M(H)$, $(a \cdot b) \cap S(H) \neq \emptyset \Leftrightarrow S(H) \subseteq a \cdot b$;

(ii) $\forall (x, y) \in M^2(H) \backslash P(H)$, $(x \cdot y) \in M(H)$.

Proof.

(i) – At once as consequence of the Prop.(5.1) of [5].

(ii) – If $|S(H)| > 1$ and, for absurd, there exists $(x, y) \in M^2(H) \backslash P(H)$ such that $(x \cdot y) \cap S(H) \neq \emptyset$, then for (i), $S(H) \subseteq x \cdot y$ and this is impossible because $|x \cdot y| = 1$. Now suppose that $S(H) = \{g\}$. Let, for absurd, $(x, y) \in M^2(H) \backslash P(H)$ such that $x \cdot y = g$. For (2.1), we obtain that $(t \cdot x) \cdot y = t \cdot (x \cdot y) = t \cdot g = t$, $\forall t \in H$. Observe that $t_1 \neq t_2 \Rightarrow (t_1 \cdot x) \cap (t_2 \cdot x) = \emptyset$; in fact if, for absurd $r \in (t_1 \cdot x) \cap (t_2 \cdot x)$ then $r \cdot y \subseteq (t_1 \cdot x) \cdot y = t_1 \cdot (x \cdot y) = t_1$ and $r \cdot y \subseteq (t_2 \cdot x) \cdot y = t_2 \cdot (x \cdot y) = t_2$, that is a contradiction. Thus, $|t \cdot x| = 1 \ \ \forall t \in H$ and, in consequence $x \in S_r(H) = S(H)$. This is an absurd because $x \in M(H) = H \backslash S(H)$. ∎

3. The case $|M(H)| = 3$. In this section we consider hypergroups H such that $|P(H)| = 4, S(H) \neq \emptyset$ and $|M(H)| = 3$. We pose $M(H) = \{a, b, c\}$.

In [4], one has proved that under the aforesaid conditions, there are four possibilities:

(I) $S(H)\cdot a = a\cdot S(H) = \{a,b\}$ and $S(H)\cdot c = c\cdot S(H) = \{c\}$;
(II) $S(H)\cdot a = \{a,b\},\ S(H)\cdot c = \{c\}$ and $z\cdot S(H) = \{z\},\ \forall z\in M(H)$;
(III) $a\cdot S(H) = \{a,b\},\ c\cdot S(H) = \{c\}$ and $S(H)\cdot z = \{z\},\ \forall z\in M(H)$;
(IV) $S(H)\cdot z = z\cdot S(H) = \{z\},\ \forall z\in M(H)$.

In [4], the authors have solved completely the cases (I), (II) and (III). In the present section, they take in consideration the case (IV). It is clear that, in this ambit, the hypergroup H can be defined by a table in which the group $S(H)$ **is considered as the scalar identity**.

Up to isomorphisms, it suffices to study the following eight cases:

$$(A)\colon P(H) = \big\{(a,a),(a,b),(b,b),(c,c)\big\};$$

$$(B)\colon P(H) = \big\{(a,a),(a,b),(b,a),(c,c)\big\};$$

$$(C)\colon P(H) = \big\{(a,a),(a,b),(b,c),(c,c)\big\};$$

$$(D)\colon P(H) = \big\{(a,b),(a,c),(b,b),(c,a)\big\};$$

$$(E)\colon P(H) = \big\{(a,c),(b,a),(b,b),(c,a)\big\};$$

$$(F)\colon P(H) = \big\{(a,b),(a,c),(b,a),(c,a)\big\};$$

$$(G)\colon P(H) = \big\{(a,b),(a,c),(b,a),(c,b)\big\};$$

$$(H)\colon P(H) = \big\{(a,a),(a,b),(b,c),(c,a)\big\}.$$

In order to front the cases (A) and (B), we premise an useful lemma:

Lemma 3.1. *Let H be a hypergroup which satisfies the condition (IV). If there exists an element $c\in M(H)$ such that $|z\cdot c| = |c\cdot z| = 1,\ \forall z\in M(H)\setminus\{c\}$, then:*

(i) $S(H)\subseteq c\cdot c$;
(ii) $c\cdot c\in\big\{S(H), S(H)\cup\{c\}, S(H)\cup\{a,b\}, H\big\}$;
(iii) $c\cdot c\in\big\{S(H), S(H)\cup\{c\}\big\}\Longrightarrow [a\cdot c = c\cdot a = a \text{ and } b\cdot c = c\cdot b = b]$;
(iv) $\{a,b\}\subseteq c\cdot c\Longrightarrow a\cdot c = c\cdot a = b\cdot c = c\cdot b = \{c\}$;
(v) $\forall (z,w)\in\{a,b\}^2, |z\cdot w| = 1\Longrightarrow z\cdot w\in\big\{\{a\},\{b\}\big\}$.

Proof.

(i). Immediately as consequence of (2.2), taking in account the reproducibility.

(ii). Let $c\cdot c\in\big\{S(H)\cup\{a\}, S(H)\cup\{a,c\}\big\}$. For (1.2), we have $a\cdot c = b$ or $b\cdot c = b$. If $a\cdot c = b$, then we obtain:

$\{a\}\cup(a\cdot a) = a\cdot S(H)\cup(a\cdot a) = a\cdot\big\{S(H)\cup\{a\}\big\} \subseteq a\cdot(c\cdot c) = (a\cdot c)\cdot c = b\cdot c$ whence, since $|b\cdot c| = 1$, we deduce $b\cdot c = a\cdot a = a$.
Besides, from $\{b\}\cup(b\cdot a) = b\cdot S(H)\cup(b\cdot a) \subseteq b\cdot(c\cdot c) = (b\cdot c)\cdot c = a\cdot c = b$, we obtain $b\cdot a = b$ and consequently $|a\cdot a| = |b\cdot a| = 1$, which is impossible since $|a\cdot a| > 1$ or $|b\cdot a| > 1$.
Therefore $a\cdot c \neq b$ and consequently $b\cdot c = b$.
Analogously one can prove that $c\cdot a \neq b$ and $c\cdot b = b$. But in this case $\{b\}\cup(b\cdot a) = b\cdot S(H)\cup(b\cdot a) = b\cdot\big\{S(H)\cup\{a\}\big\} \subseteq b\cdot(c\cdot c) = (b\cdot c)\cdot c = b\cdot c = b$ and $\{b\}\cup(a\cdot b) = S(H)\cdot b\cup(a\cdot b) = \big\{S(H)\cup\{a\}\big\}\cdot b \subseteq (c\cdot c)\cdot b = c\cdot(c\cdot b) = c\cdot b = b$, hence $b\cdot a = a\cdot b = b$ and this is impossible since $|b\cdot a| > 1$ or $|a\cdot b| > 1$.
Therefore $c\cdot c \notin \big\{S(H)\cup\{a\}, S(H)\cup\{a,c\}\big\}$.
In the same way, one shows that $c\cdot c \notin \big\{S(H)\cup\{b\}, S(H)\cup\{b,c\}\big\}$ and finally, for (i), one obtains the assertion.

(iii). If, for absurd, $a\cdot c = c$, then, by (1.2), $\{a,b\} \subseteq b\cdot c$, but this contradicts the hypothesis $|b\cdot c| = 1$. Analogously, one can prove that $b\cdot c \neq c, c\cdot a \neq c$ and $c\cdot b \neq c$.
Besides, by lemma (2.2), $(a\cdot c)\cap S(H) = (b\cdot c)\cap S(H) = (c\cdot a)\cap S(H) = \emptyset$ and thus $\forall z\in\{a,b\}, (z\cdot c)\cup(c\cdot z)\subseteq\{a,b\}$. Now if $a\cdot c = a$ and $c\cdot a = b$, using (1.2), we obtain $b\cdot c = b$ and $c\cdot b = a$, whence $a\cdot a = a\cdot(c\cdot b) = (a\cdot c)\cdot b = a\cdot b$ and $b\cdot a = b\cdot(c\cdot b) = (b\cdot c)\cdot b = b\cdot b$. Therefore $|a\cdot a| = |a\cdot b|$ and $|b\cdot a| = |b\cdot b|$, but this is a contradiction because $|P(H)| = 4$.
Reasoning in an analogous way, we also can prove that supposing $a\cdot c = b$ and $c\cdot a = a$ or $a\cdot c = c\cdot a = b$, we obtain a contradiction. Thus $a\cdot c = c\cdot a = a$ and for (1.2), $b\cdot c = c\cdot b = b$.

(iv). Obviously, $(a\cdot a)\cup(a\cdot b) = a\cdot\{b,b\} \subseteq a\cdot(c\cdot c) = (a\cdot c)\cdot c$.
If $a\cdot c\in\{a,b\}$, then $|(a\cdot c)\cdot c| = 1$ and so $|a\cdot a| = |a\cdot b| = 1$, which contradicts the hypothesis $|P(H)| = 4$. Therefore by lemma (2.2), we have $a\cdot c = c$. An analogous reasoning proves that $c\cdot a = c\cdot b = b\cdot c = c$.

(v). In virtue of (ii), there are two possibilities:

$$(I)\quad (c\cdot c)\in\big\{S(H), S(H)\cup\{c\}\big\},$$

$$(II)\quad (c\cdot c)\in\big\{S(H)\cup\{a,b\}, H\big\}.$$

Consider (I) and suppose $(z,w) = (a,a)$. In an analogous way, one can reason if $(z,w)\in\big\{(a,b),(b,a),(b,b)\big\}$. By lemma (2.2), we have that $(a\cdot a)\cap S(H) = \emptyset$. Moreover, if $a\cdot a = c$, by (1.2), we obtain $S(H)\subseteq a\cdot b$ and so, by (iii):
$a = a\cdot S(H)\subseteq a\cdot(a\cdot b) = (a\cdot a)\cdot b = c\cdot b = b$, i.e. an absurd. Therefore $(a\cdot a)\in\{a,b\}$. Consider now (II) and let $(z,w) = (b,a)$. In the same manner

one can handle the case in which $(z,w) \in \big\{(a,b),(a,a),(b,b)\big\}$. By lemma (2.1), $(b\cdot a)\cap S(H)=\emptyset$. Besides, if $b\cdot a=c$, we obtain, by (iv) and (1.2), that $b\in a\cdot a$ and thus $b\cdot b\subseteq b\cdot(a\cdot a)=(b\cdot a)\cdot a=c\cdot a=c$, which is impossible because $|b\cdot b|>1$.

Therefore $(b\cdot a)\in\{a,b\}$ and so we obtain the assertion. ∎

CASE (A). $P(H)=\big\{(a,a),(a,b),(b,b),(c,c)\big\}$**.**

For (3.1), we can start from following two tables:

	a	b	c	S(H)
a	M	N	a	a
b	α	Q	b	b
c	a	b	R	c
S(H)	a	b	c	S(H)

	a	b	c	S(H)
a	M	N	c	a
b	α	Q	c	b
c	c	c	S	c
S(H)	a	b	c	S(H)

where $\alpha\in\{a,b\}, R\in\big\{S(H),S(H)\cup\{c\}\big\}, S\in\big\{H,H\setminus\{c\}\big\}$ and M,N,Q are the three proper hyperproducts which remain to be determined.

We continue to complete the tables, by means of succesive steps.

Step 1: If $R=(c\cdot c)\in\big\{S(H),S(H)\cup\{c\}\big\}$ and $\alpha=a$, then $M=N=H$ and $Q=H\setminus\{a\}$.

By (1.2), we have that $H\setminus\{a\}\subseteq M$. Now, if $M=H\setminus\{a\}$, then we obtain $N=a\cdot b\subseteq a\cdot M=a\cdot(a\cdot a)=(a\cdot a)\cdot a=M\cdot a=\{a\}$, which is impossible. Therefore $M=H$. Besides, if $a\in Q$, then we have $H=M=a\cdot a\subseteq Q\cdot a=(b\cdot b)\cdot a=b\cdot(b\cdot a)=b\cdot a=\{a\}$. Thus $a\notin Q$ and, for (1.2), $Q\in\big\{S(H)\cup\{c\},S(H)\cup\{b,c\}\big\}$ and $a\in N$. But, if $Q=S(H)\cup\{c\}$, then one obtains that $N=a\cdot b\subseteq N\cdot b=(a\cdot b)\cdot b=a\cdot(b\cdot b)=a\cdot Q=\{a\}$, a contradiction.

Therefore $Q=S(H)\cup\{b,c\}$.

Finally, if $|N|>1$, we have $N\cap(H\setminus\{a\})\neq\emptyset$, and since $b\cdot S(H)=\{b\}, b\in(b\cdot b), b\cdot c=\{b\}$, the equality $N=a\cdot b=(b\cdot a)\cdot b=b\cdot(a\cdot b)=b\cdot N$ implies that $b\in a\cdot b=N$. In consequence $Q=b\cdot b\subseteq b\cdot N=N$, whence $N=H$.

Step 2: If $R = (c \cdot c) \in \big\{S(H), S(H) \cup \{c\}\big\}$ and $\alpha = b$, then $M = H \setminus \{b\}$ and $N = Q = H$.

If $b \in M$, one has $Q = b \cdot b \subseteq b \cdot M = (b \cdot a) \cdot a = b \cdot a = b$, which is impossible, and so $b \notin M$. Moreover, by (1.2), $S(H) \cup \{c\} \subseteq M$; but $M \neq S(H) \cup \{c\}$, because otherwise we obtain $b \in N$ and $N = a \cdot b \subseteq a \cdot N = (a \cdot a) \cdot b = M \cdot b = \{b\}$, and this is impossible.

Therefore $M = H \setminus \{b\}$. On the other hand, for (1.2), $M = H \setminus \{b\} \subseteq Q$ and supposing $M = Q$, always by (1.2), we have $b \in N$ with $N \cap (H \setminus \{b\}) \neq \emptyset$, whence $(a \cdot b) \cdot b \neq a \cdot (b \cdot b)$ in any way we set $N = a \cdot b$.
Consequently $Q = H$. Finally, it is easy to prove that $N = H$, since in all the other cases, we obtain that $(a \cdot a) \cdot b \neq a \cdot (a \cdot b)$.

Step 3: If $S = (c \cdot c) \in \big\{H \setminus \{c\}, H\big\}$ and $\alpha = a$, then $M = N = H \setminus \{c\}$ and $Q = S(H) \cup \{b\}$.

Because $\alpha = b \cdot a = a$, we have $a = b \cdot a = b \cdot (b \cdot a) = Q \cdot a$ and so $Q \cap \{a, c\} = \emptyset$, whence, by (1.2), $Q \in \big\{S(H), S(H) \cup \{b\}\big\}$ and $a \in N$.

If $Q = S(H)$, then $N = a \cdot b \subseteq (a \cdot b) \cdot b = a \cdot Q = \{a\}$, which is impossible. Thus $Q = S(H) \cup \{b\}$. Besides $|M \cdot c| = |a \cdot (a \cdot c)| = |a \cdot c| = |\{c\}| = 1$ and so $c \notin M$ and, by (1.2), one has that $M \in \big\{S(H) \cup \{b\}, S(H) \cup \{a, b\}\big\}$. But if $M = S(H) \cup \{b\}$ then $N = a \cdot b \subseteq \{a\} \cup (a \cdot b) = (a \cdot S(H)) \cup (a \cdot b) \subseteq a \cdot (S(H) \cup \{b\}) = a \cdot (a \cdot a) = (a \cdot a) \cdot a = \{a\}$, and this is impossible.

Therefore $M = S(H) \cup \{a, b\}$.

Step 4: If $S = (c \cdot c) \in \big\{H \setminus \{c\}, H\big\}$ and $\alpha = b$, then $N = Q = H \setminus \{c\}$ and $M = S(H) \cup \{a\}$.

The proof is quite similar to the preceding one. Ultimately we obtain the following eight hypergroups (five in the case $|S(H)| = 1$):

	a	b	c	S(H)
a	H	H	a	a
b	a	H\{a}	b	b
c	a	b	R	c
S(H)	a	b	c	S(H)

where $R \in \big\{S(H), S(H) \cup \{c\}\big\}$ and $R \neq S(H)$ if $|S(H)| = 1$.

	a	b	c	S(H)
a	H\{b}	H	a	a
b	b	H	b	b
c	a	b	R	c
S(H)	a	b	c	S(H)

where $R \in \big\{S(H), S(H) \cup \{c\}\big\}$ and $R \neq S(H)$ if $|S(H)| = 1$.

	a	b	c	S(H)
a	H\{c}	H\{c}	c	a
b	a	S(H)∪{b}	c	b
c	c	c	S	c
S(H)	a	b	c	S(H)

where $S \in \big\{H, H \setminus \{c\}\big\}$.

	a	b	c	S(H)
a	S(H)∪{a}	H\{c}	c	a
b	b	H\{c}	c	b
c	c	c	S	c
S(H)	a	b	c	S(H)

where $S \in \big\{H, H \setminus \{c\}\big\}$.

CASE (B). $P(H) = \big\{(a,a), (a,b), (b,a), (c,c)\big\}$.
By (3.1), we have the following initial tables:

	a	b	c	S(H)
a	M	N	a	a
b	Q	α	b	b
c	a	b	R	c
S(H)	a	b	c	S(H)

	a	b	c	S(H)
a	M	N	c	a
b	Q	α	c	b
c	c	c	S	c
S(H)	a	b	c	S(H)

where $\alpha \in \{a, b\}, R \in \big\{S(H), S(H) \cup \{c\}\big\}, S \in \big\{H, H \setminus \{c\}\big\}$ and M, N, Q are to be determined.

Step 1: If $R = (c \cdot c) \in \big\{S(H), S(H) \cup \{c\}\big\}$ and $\alpha = a$, then $N = Q \in \big\{S(H) \cup \{a, c\}, S(H) \cup \{b, c\}, H\big\}$. Besides if $N = Q \in \big\{S(H) \cup \{a, c\}, H\big\}$ then $M = H$, while if $N = Q = S(H) \cup \{b, c\}$, then $M = \{a, b\}$.

From $b \cdot b = a$, we obtain $N = a \cdot b = (b \cdot b) \cdot b = b \cdot a = Q$ whence, for (1.2), $S(H) \cup \{c\} \subseteq N = Q$. Now, if $N = S(H) \cup \{c\}$, then $M = a \cdot a = a \cdot (b \cdot b) = N \cdot b = \{b\}$, which is impossible, and so $N = Q \in \big\{S(H) \cup \{a, c\}, S(H) \cup \{b, c\}, H\big\}$.

Moreover, from $M = a \cdot a = a \cdot (b \cdot b) = N \cdot b$, we deduce that:

$M = H$ if $N \in \big\{S(H) \cup \{a, c\}, H\big\}$ and $M = \{a, b\}$, if $N = S(H) \cup \{b, c\}$.

Step 2: If $R = (c \cdot c) \in \big\{S(H), S(H) \cup \{c\}\big\}$ and $\alpha = b$, then $N = Q = H$ and $M \in \big\{\{a, b\}, S(H) \cup \{a, c\}, H\big\}$.

Since $b \cdot b = b$, using (1.2), we obtain $H \setminus \{b\} \subseteq N \cap Q$. Now, if $a \cdot b = N = H \setminus \{b\}$, then $(a \cdot b) \cdot b \neq a \cdot (b \cdot b)$ and so $N = H$. In the same manner one proves that $Q = H$. Finally $M \notin \big\{\{a, c\}, \{b, c\}, S(H), S(H) \cup \{a\}, S(H) \cup \{b\}, S(H) \cup \{a, b\}\big\}$ and $M \notin \big\{S(H) \cup \{c\}, S(H) \cup \{b, c\}\big\}$, because otherwise one obtains respectively $a \cdot (a \cdot c) \neq (a \cdot a) \cdot c$ and $a \cdot (a \cdot b) \neq (a \cdot a) \cdot b$. In consequence, we have that $M \in \big\{\{a, b\}, S(H) \cup \{a, c\}, H\big\}$.

Step 3: If $S = (c \cdot c) \in \big\{H \setminus \{c\}, H\big\}$ and $\alpha = a$, then $N = Q \in \big\{S(H) \cup \{a\}, S(H) \cup \{a, b\}\big\}$ and $M = S(H) \cup \{a, b\}$.

From $b \cdot b = a$, it follows that $N = Q$ (as in the step 1) and, by (1.2), $S(H) \subseteq N = Q$. Moreover $N \neq S(H), c \notin N$ and $N \neq S(H) \cup \{b\}$:

in fact, if $a \cdot b = N = S(H)$ then $M = a \cdot a = a \cdot (b \cdot b) = (a \cdot b) \cdot b = N \cdot b = S(H) \cdot b = \{b\}$; if $c \in N$ then $c \cdot c \subseteq N \cdot c = a \cdot (b \cdot c) = a \cdot c = c$; if $N = S(H) \cup \{b\}$ then $(a \cdot b) \cdot b \neq a \cdot (b \cdot b)$ and in every case we obtain a contradiction.

Therefore $N = Q \in \big\{S(H) \cup \{a\}, S(H) \cup \{a, b\}\big\}$ and, using the equality $M = a \cdot (b \cdot b) = (a \cdot b) \cdot b = N \cdot b$, one obtains that $M = S(H) \cup \{a, b\}$.

Step 4: If $S = (c \cdot c) \in \big\{H \setminus \{c\}, H\big\}$ and $\alpha = b$, then $N = Q = H \setminus \{c\}$ and $M \in \big\{\{a, b\}, S(H) \cup \{a\}, H \setminus \{c\}\big\}$.

By (1.2), as $b \cdot b = b$, one has that $S(H) \cup \{a\} \subseteq N \cap Q$. Now $c \notin N$, because otherwise $c \cdot c \subseteq N \cdot c = a \cdot (b \cdot c) = a \cdot c = c$ and this is impossible; besides $N \neq S(H) \cup \{a\}$, since differently $(a \cdot b) \cdot b \neq a \cdot (b \cdot b)$ and so $N = H \setminus \{c\}$. Analogously one can prove that $Q = H \setminus \{c\}$. Finally, if we suppose that $c \in M$ or $M \in \big\{S(H), S(H) \cup \{b\}\big\}$, we obtain respectively $c \cdot c = c$ or $(a \cdot a) \cdot b \neq a \cdot (a \cdot b)$. In consequence, we have $c \notin M$ and $M \notin \big\{S(H), S(H) \cup \{b\}\big\}$.

Therefore, it results that $M \in \big\{\{a, b\}, S(H) \cup \{a\}, H \setminus \{c\}\big\}$.

Ultimately, this case leads to the following 22 hypergroups (16, if $|S(H)| = 1$):

	a	b	c	S(H)
a	H	U	a	a
b	U	a	b	b
c	a	b	R	c
S(H)	a	b	c	S(H)

where $U \in \big\{H\backslash\{a\}, H\backslash\{b\}, H\big\}$ and $R \in \big\{S(H), S(H)\cup\{c\}\big\}$ with $R \neq S(H)$ if $|S(H)| = 1$.

	a	b	c	S(H)
a	U	H	a	a
b	H	b	b	b
c	a	b	R	c
S(H)	a	b	c	S(H)

where $U \in \big\{\{a, b\}, H \setminus \{b\}, H\big\}$ and $R \in \big\{S(H), S(H)\cup\{c\}\big\}$ with $R \neq S(H)$ if $|S(H)| = 1$.

	a	b	c	S(H)
a	H\{c}	U	c	a
b	U	a	c	b
c	a	c	R	c
S(H)	a	b	c	S(H)

where $U \in \big\{S(H) \cup \{a\}, H \setminus \{c\}\big\}$ and $S \in \big\{H \setminus \{c\}, H\big\}$.

	a	b	c	S(H)
a	U	H\{c}	c	a
b	H\{c}	b	c	b
c	c	c	S	c
S(H)	a	b	c	S(H)

where $U \in \big\{\{a, b\}, S(H) \cup \{a\}, H \setminus \{c\}\big\}$ and $S \in \big\{H \setminus \{c\}, H\big\}$.

CASE (C). $P(H) = \big\{(a, a), (a, b), (b, c), (c, c)\big\}$.

By (2.10) of [3], we have $b \cdot b = b$ and by (2.2), it also results $(b \cdot a) \cap S(H) = \emptyset$. Besides, $b \cdot a = c$ implies $b \cdot c = b \cdot (b \cdot a) = (b \cdot b) \cdot a = b \cdot a = c$, i.e. a contradiction and so $b \cdot a \in \big\{\{a\}, \{b\}\big\}$. Then we can consider the following two tables:

	a	b	c	S(H)
a	M	N	α	a
b	a	b	P	b
c	β	γ	Q	c
S(H)	a	b	c	S(H)

	a	b	c	S(H)
a	M	N	α	a
b	b	b	P	b
c	β	γ	Q	c
S(H)	a	b	c	S(H)

where we have to determine the hyperproducts $M, N, \alpha, \beta, \gamma, P, Q$.

Step 1: If $b \cdot a = a$, then $M = N = H, P = H \setminus \{a\}, Q = S(H) \cup \{c\}, \gamma = b$ and $\alpha = \beta = a$.

From the equality $(c \cdot b) \cdot a = c \cdot (b \cdot a) = c \cdot a$, it follows that $(c \cdot b) \in \big\{\{b\}, \{c\}\big\}$. Now, if $c \cdot b = c$ then, from $a \cdot (c \cdot b) = (a \cdot c) \cdot b$, we obtain $\alpha = \alpha \cdot b$ and so $\alpha \neq a$. Moreover, we have $b \cdot \alpha = b \cdot (a \cdot c) = (b \cdot a) \cdot c = \alpha$, whence $\alpha \neq c$, and thus $\alpha = b$. Consequently, we obtain $\beta \cdot c = (c \cdot a) \cdot c = c \cdot (a \cdot c) = c \cdot b = c$, which is impossible for every value of β.

Therefore $\gamma = c \cdot b = b$ and in consequence $\beta = c \cdot a = c \cdot (b \cdot a) = (c \cdot b) \cdot a = b \cdot a = a$ and, by (1.2), $M \in \big\{H, H \setminus \{a\}\big\}$ and $N \in \big\{H, H \setminus \{b\}\big\}$. Besides $Q \cdot b = (c \cdot c) \cdot b = c \cdot (c \cdot b) = c \cdot b = \{b\}$ and $P \cdot b = (b \cdot c) \cdot b = b \cdot (c \cdot b) = b \cdot b = \{b\}$ imply $a \notin Q \cup P$, whence, by (1.2), $\alpha = a, S(H) \cup \{c\} \subseteq P$ and $S(H) \subseteq Q$. On the other hand $a \cdot Q = a \cdot (c \cdot c) = (a \cdot c) \cdot c = \{a\}$ implies $\{a, b\} \cap Q = \emptyset$ and $Q \neq S(H)$, because otherwise one obtains $Q = c \cdot c \subseteq (b \cdot c) \cdot c = b \cdot (c \cdot c) = b \cdot S(H) = \{b\}$, which is clearly impossible.

Therefore $Q = S(H) \cup \{c\}$ and, by (1.2), $b \in P$, from which $P = H \setminus \{a\}$. Finally $M = N = H$, since if $M = H \setminus \{a\}$ or $N = H \setminus \{b\}$, one has respectively $(a \cdot a) \cdot a \neq a \cdot (a \cdot a)$ or $(a \cdot b) \cdot b \neq a \cdot (b \cdot b)$.

Step 2: If $b \cdot a = b$, then $M = S(H) \cup \{a\}, N = H \setminus \{c\}, P = Q = H$ and $\alpha = \beta = \gamma = c$. The proof is quite analogous to the preceding one.

Ultimately, we obtain the following two hypergroups:

	a	b	c	S(H)
a	H	H	a	a
b	a	b	H\{a}	b
c	a	b	S(H)∪{c}	c
S(H)	a	b	c	S(H)

	a	b	c	S(H)
a	S(H)∪{a}	H\{c}	c	a
b	b	b	H	b
c	c	c	H	c
S(H)	a	b	c	S(H)

Now we come to the:

CASE (D). $P(H) = \{(a,b),(a,c),(b,b),(c,a)\}$.

We denote the hyperproducts as indicated in the following table:

	a	b	c	S(H)
a	α	M	N	a
b	β	P	γ	b
c	Q	δ	θ	c
S(H)	a	b	c	S(H)

where, standing by the usual conventions, the capital letters represent the proper hyperproducts. By (1.2) and lemma (2.2), we have that $S(H) \subseteq N \cap P \cap Q$.

We begin with the

Step 1: $\theta = c$.

Suppose, for absurd, that $\theta \neq c$. Then there are two possibilities:

$$\text{(I)} \quad \theta = a; \qquad \text{(II)} \quad \theta = b.$$

(I) In the former case, because $c = S(H) \cdot c \subseteq N \cdot c = (a \cdot c) \cdot c = a \cdot (c \cdot c) = c \cdot c = \alpha$, one obtains $\alpha = c$ and so $N \cdot c = \{c\}$. But $N \cdot c = \{c\}$ implies $\{a, c\} \cap N = \emptyset$ and so $N \in \{S(H) \cup \{b\}, S(H)\}$. Besides, from $(a \cdot a) \cdot a = a \cdot (a \cdot a)$, one has $Q = N$.

If $N = S(H)$, (1.2) entails $\gamma = \delta = \beta = b$, whence $b = c \cdot b = (a \cdot a) \cdot b = a \cdot (a \cdot b) = a \cdot M$ and so $b \notin M$, but in this way $a \cdot H \neq H$.

If $N = Q = S(H) \cup \{b\}$, then $c = a \cdot a = (c \cdot c) \cdot a = c \cdot (c \cdot a) = c \cdot (S(H) \cup \{b\}) = c \cup (c \cdot b)$, hence $\delta = c \cdot b = c$. Now $c = (a \cdot a) \cdot b = a \cdot (a \cdot b) = a \cdot M$, hence $M \cap S(H) = \emptyset$ and so we have a contradiction, because $|M| \geq 2$. In conclusion $\theta \neq a$.

(II) In the case of $\theta = b$, it results $\beta = (c \cdot c) \cdot a = c \cdot (c \cdot a) = c \cdot Q \supseteq c \cdot S(H) = c$, whence $\beta = c$ and $c = c \cdot Q$. In consequence $Q \cap \{a, c\} = \emptyset$, i.e. $Q \in \big\{S(H) \cup \{b\}, S(H)\big\}$.

If $Q = S(H)$, then, for (1.2), $\alpha = b$ and $c = (a \cdot a) \cdot a = a \cdot (a \cdot a) = M$, which is impossible.

If $Q = S(H) \cup \{b\}$, then, because $c = c \cdot Q = \{c\} \cup (c \cdot b)$, one obtains $\delta = c \cdot b = c$, but so $a \notin c \cdot H$.

Therefore $\theta \neq b$ and this proves the assertion.

Step 2: $\{\gamma, \delta\} \subseteq \{b, c\}$.

From $\gamma = b \cdot c = b \cdot (c \cdot c) = (b \cdot c) \cdot c = \gamma \cdot c$, it follows that $\gamma \in \{b, c\}$. Analogously from $\delta = c \cdot b = (c \cdot c) \cdot b = c \cdot (c \cdot b) = c \cdot \delta$, we have $\delta \in \{b, c\}$.

We analyse the two cases:

$$\text{(i)} \quad \delta = c; \qquad \text{(ii)} \quad \delta = b.$$

(i). Suppose first $\delta = c$.

By (1.2), $\{a, b\} \cup S(H) \subseteq Q$. But $c = c \cdot b \in c \cdot Q = c \cdot (c \cdot a) = (c \cdot c) \cdot a = Q$ and thus $Q = H$. Moreover, as $\gamma \cdot a = (b \cdot c) \cdot a = b \cdot (c \cdot a) = H$, we obtain $\gamma = c$. In virtue of (1.2), one has $\{a, b\} \cup S(H) \subseteq N, c = (b \cdot c) \in N \cdot c = (a \cdot c) \cdot c = a \cdot (c \cdot c) = N$, and $N = H$.

Besides $\beta \cdot c = (b \cdot a) \cdot c = b \cdot (a \cdot c) = H$ implies $\beta = a$ and $c \cdot \alpha = c \cdot (a \cdot a) = (c \cdot a) \cdot a = H$ implies $\alpha = a$. From the equality $M \cdot a = (a \cdot b) \cdot a = a \cdot (b \cdot a) = a$, it follows that $c \notin M$. Moreover, if $S(H) \subseteq M$, then $b = b \cdot S(H) \subseteq b \cdot M = b \cdot (a \cdot b) = (b \cdot a) \cdot b = M$, and in consequence $M \in \big\{S(H) \cup \{b\}, S(H) \cup \{a, b\}, \{a, b\}\big\}$. In the case $M = S(H) \cup \{b\}$, we have $M = b \cdot M = \{b\} \cup P$, where $P \in \big\{S(H), S(H) \cup \{b\}\big\}$, but at all events $a \notin H \cdot b$. While, if $M = \{a, b\}$, then $M = b \cdot M = \{a\} \cup P$ and thus $P = \{a, b\}$. In this way, $S(H)$ is not contained in $H \cdot b$, and this contradicts (1.2). Consequently $M = S(H) \cup \{a, b\}$. From $M = b \cdot M = \{a, b\} \cup P$, we deduce that $c \notin P$.

Moreover, since $b = S(H) \cdot b \subseteq M \cdot b = (a \cdot b) \cdot b = a \cdot P$ and $c \notin P$, we obtain that $b \in P$. Therefore, taking also in account that $P \cup \{a, b\} = S(H) \cup \{a, b\}$, we have $P \in \big\{S(H) \cup \{b\}, S(H) \cup \{a, b\}\big\}$. But $P = S(H) \cup \{a, b\}$ entails that $H = (b \cdot b) \cdot c = b \cdot (b \cdot c) = \{c\}$, which is a contradiction. Then $P = S(H) \cup \{b\}$ and we obtain the following hypergroup:

	a	b	c	S(H)
a	a	H\{c}	H	a
b	a	S(H)∪{b}	c	b
c	H	c	c	c
S(H)	a	b	c	S(H)

We come to the second case:

(ii). $\delta = b$.

By (1.2), $S(H) \cup \{a\} \subseteq Q$. Moreover, as $S(H) \subseteq Q$, we have $c = c \cdot S(H) \subseteq c \cdot Q = c \cdot (c \cdot a) = (c \cdot c) \cdot a = Q$ and thus $Q \in \big\{S(H) \cup \{a, c\}, H\big\}$. If $Q = H$, reasoning as in the case (i), we obtain $Q = N = H$, $\gamma = c$, $\alpha = \beta = a$ and $M = S(H) \cup \{a, b\}$, whence, since $(b \cdot a) \cdot b = b \cdot (a \cdot b)$, we have $M = b \cdot M = \{a, b\} \cup P$ and so $c \notin H \cdot b$, a contradiction. Therefore, it must be $Q = S(H) \cup \{a, c\}$. Now $\gamma \cdot a = (b \cdot c) \cdot a = b \cdot (c \cdot a) = b \cdot Q = \big\{\beta, \gamma, b\big\}$ implies, according to step 2, $\gamma = \beta = b$ and $c \cdot \alpha = c \cdot (a \cdot a) = (c \cdot a) \cdot a = Q \cdot a = \big\{\alpha, c, a\big\} \cup S(H)$ implies $\alpha = a$. Besides, $b = (b \cdot a) \cdot c = b \cdot N$ implies that $b \notin N$ and thus, by (1.2), $N \in \big\{S(H) \cup \{a\}, S(H) \cup \{a, c\}\big\}$. But $N \neq S(H) \cup \{a\}$, since otherwise we obtain $a = a \cdot (a \cdot c) = (a \cdot a) \cdot c = N$ and so $N = S(H) \cup \{a, c\}$.

By (1.2), $b \in M$. Moreover, if $S(H) \subseteq M$, ane has $a = a \cdot S(H) \subseteq a \cdot M = a \cdot (a \cdot b) = M$ and $c = c \cdot S(H) \subseteq c \cdot M = c \cdot (a \cdot b) = M$, i.e. $S(H) \subseteq M \Longrightarrow M = H$.

Therefore, there are four possibilities:

(i) $M = \{a, b\}$;
(ii) $M = \{b, c\}$;
(iii) $M = \{a, b, c\}$;
(iv) $M = H$.

If $M = \{a, b\}$, then $c \cdot (a \cdot b) = (c \cdot a) \cdot b$ implies $H = \{a, b\}$, i.e. a contradiction. Analogously, if $M = \{b, c\}$, or $M = \big\{a, b, c\big\}$, the equality $a \cdot (a \cdot b) = (a \cdot a) \cdot b$ implies $H = \{b, c\}$ or, respectively $H = \big\{a, b, c\big\}$, which are impossible. So $M = H$.

Moreover, by (1.2), $P \in \big\{S(H) \cup \{a, c\}, H\big\}$. Finally, as $H = H \cdot b = (a \cdot b) \cdot b = a \cdot P$, we conclude that $P = H$, and the following table arises:

	a	b	c	S(H)
a	a	H	H\{b}	a
b	b	H	b	b
c	H\{b}	b	c	c
S(H)	a	b	c	S(H)

CASE (E). $P(H) = \{(a,c),(b,a),(b,b),(c,a)\}$.

In this case, we have a table which is symmetric to that one of the case (D). So we obtain the following two hypergroups, which are the symmetric hypergroups (see the section 2 of [3]) of those ones just found in the preceding case:

	a	b	c	S(H)
a	a	a	H	a
b	H\{c}	S(H)∪{ b}	c	b
c	H	c	c	c
S(H)	a	b	c	S(H)

	a	b	c	S(H)
a	a	b	H\{b}	a
b	H	H	b	b
c	H\{b}	b	c	c
S(H)	a	b	c	S(H)

CASE (F). $P(H) = \{(a,b),(a,c),(b,a),(c,a)\}$.

We call the hyperproducts as follows:

	a	b	c	S(H)
a	α	M	N	a
b	P	β	ρ	b
c	R	θ	γ	c
S(H)	a	b	c	S(H)

where, for (1.2), $S(H) \subseteq P \cap R \cap M \cap N$.

Step 1: $\alpha = a$.

Suppose, for absurd, $\alpha = b$. Then $\beta = b \cdot b = b \cdot \alpha = b \cdot (a \cdot a) = (b \cdot a) \cdot a = P \cdot a$ and so, as $S(H) \subseteq P, \beta = a$. From $a = P \cdot a$, one obtains that $P = S(H)$. Moreover $P = b \cdot a = (a \cdot a) \cdot a = a \cdot b = M$, and thus $M = S(H)$. But $c = c \cdot S(H) = c \cdot P = c \cdot (b \cdot a) = \theta \cdot a$ and this fact is clearly impossible. Now suppose $\alpha = c$. Taking in consideration the permutation $(a)(b\ c)\big(S(H)\big)$, we fall again in the case of $\alpha = b$.

Therefore, we have $\alpha = a$.

Step 2: $\beta \neq a$.

If, for absurd, $\beta = a$, then $a = a \cdot a = (b \cdot b) \cdot a = b \cdot (b \cdot a) = b \cdot P \supseteq b \cdot S(H) = b$, and this is a contradiction.

We can consider two subcases, according to $\beta = c$ or $\beta = b$.

Subcase 1: $\beta = c$.

Since $(b \cdot b) \cdot b = b \cdot (b \cdot b)$, we obtain $c \cdot b = b \cdot c$ and so, we can put $\theta = \rho = x$. Moreover $\gamma = c \cdot (b \cdot b) = (c \cdot b) \cdot b = x \cdot b$ implies $x \neq a$ and, for (1.2), $a \in P \cap M$. In consequence we have $P \subseteq R$; in fact $P = b \cdot a \subseteq b \cdot P = b \cdot (b \cdot a) = (b \cdot b) \cdot a = R$.

We distinguish more two possibilities:

$$\text{(i)} \quad x = b; \qquad \text{(ii)} \quad x = c.$$

(i). In this case we obtain $\theta = \rho = b$ and, as $\gamma = x \cdot b, \gamma = c$. Therefore, we have the following partial configuration:

	a	b	c	S(H)
a	a	M	N	a
b	P	c	b	b
c	R	b	c	c
S(H)	a	b	c	S(H)

with $S(H) \cup \{a\} \subseteq M \cap N \cap P \cap R$ and $P \subseteq R$. We shall prove that $M = N = P = R = H$.

We notice that $M \subseteq N$, because $M = a \cdot b \subseteq M \cdot b = (a \cdot b) \cdot b = a \cdot (b \cdot b) = N$, and so, by (1.2), we deduce that $N = H$.

On the other hand, $H = a \cdot c \subseteq a \cdot M = a \cdot (a \cdot b) = M$ and so, $M = H$.

By the symmetry of the case (F), we also obtain $P = R = H$, and consequently we find the following hypergroup:

	a	b	c	S(H)
a	a	H	H	a
b	H	c	b	b
c	H	b	c	c
S(H)	a	b	c	S(H)

(ii). In this case $\theta = \rho = c$ and, from $\gamma = x \cdot b$, one has $\gamma = c$.

We can consider the following table:

	a	b	c	S(H)
a	a	M	N	a
b	P	c	c	b
c	R	c	c	c
S(H)	a	b	c	S(H)

where, for (1.2), $S(H) \cup \{a, b\} \subseteq R \cap N$ and $S(H) \cup \{a\} \subseteq M \cap P$.
Now, it results $M \subseteq N$, because $M = a \cdot b \subseteq M \cdot b = (a \cdot b) \cdot b = a \cdot (b \cdot b) = N$. Using (1.2), one obtains $N = H$. By the symmetry of the case (F), we have $R = H$. Moreover $(a \cdot a) \cdot b = a \cdot (a \cdot b)$ implies $M = a \cdot M$, whence if $c \in M$, we obtain $M = H$; while if $c \notin M$, then $b \in M$ (since $S(H) \subseteq M$), and deduce that $M = S(H) \cup \{a, b\}$. In the same manner, starting from $b \cdot (a \cdot a) = (b \cdot a) \cdot a$, we deduce $P = H$, if $c \in P$ and $P = S(H) \cup \{a, b\}$, if $c \notin P$. In consequence we have the following hypergroups:

	a	b	c	S(H)
a	a	M	H	a
b	P	c	c	b
c	H	c	c	c
S(H)	a	b	c	S(H)

where $M, P \in \big\{H, H \setminus \{c\}\big\}$.

Subcase 2: $\beta = b$.
Notice that $\gamma \in \{b, c\}$. In fact, if $\gamma = a$ then we obtain $c = c \cdot S(H) \subseteq c \cdot R = c \cdot (c \cdot a) = (c \cdot c) \cdot a = a$, which is an evident contradiction. We can suppose $\gamma = c$, since if $\gamma = b$, exchanging b with c, we fall again in the subcase 1. From $b \cdot (c \cdot a) = (b \cdot c) \cdot a$ and $a \cdot (c \cdot b) = (a \cdot c) \cdot b$, it follows respectively $\rho \neq a$ and $\theta \neq a$, therefore there ore four possibilities:

(I) $\rho = \theta = c$;
(II) $\rho = b$; $\theta = c$;
(III) $\rho = c$; $\theta = b$;
(IV) $\rho = \theta = b$.

In the case (I), we have the following initial table:

	a	b	c	S(H)
a	a	M	N	a
b	P	b	c	b
c	R	c	c	c
S(H)	a	b	c	S(H)

where, for (1.2), $R \cap N \supseteq S(H) \cup \{a, b\}$ and $M \cap P \supseteq S(H) \cup \{a\}$.

Since $P = b \cdot a \subseteq R \cdot a = (c \cdot a) \cdot a = c \cdot (a \cdot a) = R$, considering (1.2), we obtain $R = H$ and, by the symmetry of the case (F), we also have $N = H$. In addition, the equality $(a \cdot a) \cdot b = a \cdot (a \cdot b)$ entails $M = a \cdot M$, whence if $c \in M$, then $M = H$, and if $c \notin M$, then $b \in M$ (since $S(H) \subseteq M$) and $M = S(H) \cup \{a, b\}$.

In the same manner, starting from the equality $b \cdot (a \cdot a) = (b \cdot a) \cdot a$, one can prove that in the case $c \in P, P = H$ and in the case $c \notin P, P = S(H) \cup \{a, b\}$.

In this way, we have found the following hypergroups:

	a	b	c	S(H)
a	a	M	H	a
b	P	b	c	b
c	H	c	c	c
S(H)	a	b	c	S(H)

where $M, P \in \big\{H, H \setminus \{c\}\big\}$.
In the case (II), we have the following table

	a	b	c	S(H)
a	a	M	N	a
b	P	b	b	b
c	R	c	c	c
S(H)	a	b	c	S(H)

with $P \supseteq S(H) \cup \{a, c\}$; $R \supseteq S(H) \cup \{a, b\}$; and $M \cap N \supseteq S(H) \cup \{a\}$.
At once $R = H$, since $b \in R$ and $P = b \cdot a \subseteq R \cdot a = (c \cdot a) \cdot a = c \cdot (a \cdot a) = R$. Analogously $P = H$ since $c \in P$ and $R = c \cdot a \subseteq P \cdot a = (b \cdot a) \cdot a = b \cdot (a \cdot a) = P$. Now $c \in N \cap M$; in fact $c = S(H) \cdot c \subseteq (M \cdot c) \cap (N \cdot c) = \big((a \cdot b) \cdot c\big) \cap \big((a \cdot c) \cdot c\big) = \big((a \cdot (b \cdot c)\big) \cap \big((a \cdot (c \cdot c)\big) = M \cap N$. Moreover $N \subseteq M$, since $N = a \cdot c \subseteq a \cdot M = a \cdot (a \cdot b) = (a \cdot a) \cdot b = M$. Finally $(a \cdot c) \cdot b = a \cdot (c \cdot b)$

implies $N \cdot b = N$, whence $b \in N$ and, as $a \in N, M \subseteq N$. It results that for the following hypergroup $M = N = H$.

	a	b	c	S(H)
a	a	H	H	a
b	H	b	b	b
c	H	c	c	c
S(H)	a	b	c	S(H)

Note that the case (III) can be brought again to the case (II), using the symmetry. So it gives rise to the following hypergroup:

	a	b	c	S(H)
a	a	H	H	a
b	H	b	c	b
c	H	b	c	c
S(H)	a	b	c	S(H)

In conclusion, the hypergroups of the case (IV) are isomorphic to those ones just found in the case (I), by means of the permutation $(a)(b\,c)\big((S(H)\big)$.

CASE (G). $P(H) = \big\{(a,b),(a,c),(b,a),(c,b)\big\}$.

By (2.10) of [3], we have $c \cdot c = c$ and so we can start from the following table:

	a	b	c	S(H)
a	α	M	N	a
b	P	β	γ	b
c	λ	Q	c	c
S(H)	a	b	c	S(H)

Notice that the equality $(a \cdot c) \cdot a = a \cdot (c \cdot a)$ implies $N \cdot a = a \cdot \lambda$.

Step 1: $\lambda \neq b$.

The assert follows from $\lambda = c \cdot a = (c \cdot c) \cdot a = c \cdot (c \cdot a) = c \cdot \lambda$.

Step 2: $\lambda \neq a$.

If, for absurd, $\lambda = a$, then $N \cdot a = \alpha$, whence $b \notin N$. But, since $S(H) \subseteq N = (c \cdot a) \cdot c = c \cdot (a \cdot c) = c \cdot N$, we obtain $b \in N$, a contradiction.

Step 3: $\lambda \neq c$.

If, for absurd, $\lambda = c$, then $N \cdot a = N$ and thus, as $S(H) \subseteq N$, $b \in N$. But, since $c = (c \cdot a) \cdot c = c \cdot (a \cdot c) = c \cdot N$, it results $b \notin N$, which is absurd.

Then we deduce that in this case such **hypergroups do not exist**.

CASE (H). $P(H) = \big\{(a,a),(a,b),(b,c),(c,a)\big\}$.

We can proceed as in the case (3_5) of [3] and conclude that in this case **there is no hypergroup satisfying this condition**.

4. The case $|M(H)| = 4$. The following result allows us to focus our attention on the case $|M(H)| = 4$.

Proposition 4.1. *If H is a hypergroup such that $S(H) \neq \emptyset$ and $|P(H)| = |M(H)|$, then:*

(a) $\forall z \in M(H),\ z \cdot S(H) = S(H) \cdot z = \{z\}$;

(b) $\forall (a,b) \in M(H)^2,\ (a \cdot b) \cap S(H) \neq \emptyset \Leftrightarrow S(H) \subseteq a \cdot b$;

(c) $\forall (x,y) \in M(H)^2 \setminus P(H),\ (x \cdot y) \cap S(H) = \emptyset$.

Proof. By (1.2) and lemma (2.2), it suffices to prove the statement (a). Assume that there exists $z \in M(H)$ such that $S(H) \cdot z \neq \{z\}$. Then we can choose $g \in S(H)$ and $y \in M(H) \setminus \{z\}$ so that $g \cdot z = y$. By hypothesis, there exists $x \in M(H)$ such that $|z \cdot x| > 1$. We then have $g \cdot (z \cdot x) = (g \cdot z) \cdot x = y \cdot x$, which is impossible because $|g \cdot (z \cdot x)| > 1$, while $|y \cdot x| = 1$ (for the fact that $|z \cdot x| > 1$ and $|P(H)| = |M(H)|$). This contradiction completes the proof of the equality $S(H) \cdot z = z$.

With regard to the equality $z \cdot S(H) = \{z\}$, we can proceed in a quite similar way. ∎

Henceforth H denotes a hypergroups such that $S(H) \neq \emptyset, |M(H)| = |P(H)| = 4$. In virtue of the preceding result, we can consider again the scalar group $S(H)$ **as the scalar identity of** H. Moreover we indicate $M(H) = \big\{a,b,c,d\big\}$.

It is easy to verify that there are, anless isomorphism, five possibilities:

$$(I)\quad P(H) = \big\{(a,b),(b,c),(c,d),(d,a)\big\};$$

$$(II)\quad P(H) = \big\{(a,b),(b,a),(c,d),(d,c)\big\};$$

$$(III)\quad P(H) = \big\{(a,a),(b,b),(c,d),(d,c)\big\};$$

$$(IV)\quad P(H) = \big\{(a,a),(b,b),(c,c),(d,d)\big\};$$

$$(V)\quad P(H) = \big\{(a,a),(b,c),(c,d),(d,b)\big\}.$$

We need now a lemma:

Lemma 4.2. *For each* $(x, y) \in P(H)$, *if* $x \neq y$ *then* $x \cdot y \notin \big\{\{x\} \cup S(H),\ \{y\} \cup S(H)\big\}$.

Proof. Suppose, by contradiction, that $x \cdot y = \{x\} \cup S(H)$. Therefore $(x \cdot x) \cdot y = x \cdot (x \cdot y) = x \cdot x \cup \{x\}$, whence $x \cdot x \in M(H) \setminus \{x\}$. But $\forall\ t \in M(H) \setminus \{x\}$, $x \cdot x = t$ implies $t \cdot y = \{t, x\}$, i.e. $|ty| = 2$. This is impossible, because there exists an unique proper hyperproduct in the column of the element y. Analogously, one proves that $x \cdot y \neq \{y\} \cup S(H)$. ∎

CASE (I). $P(H) = \big\{(a, b), (b, c), (c, d), (d, a)\big\}$.

Reasoning as in the case (3_5) of [3], we can prove that $a \cdot b \neq \{a, b\} \cup S(H)$ and $a \cdot b \cap \{c, d\} = \emptyset$. So, by lemma (4.2), $a \cdot b = S(H)$. In consequence, we obtain:

$$a = a \cdot S(H) \subseteq a \cdot (b \cdot c) = (a \cdot b) \cdot c = S(H) \cdot c = c,$$

an evident contradiction.

Therefore, **there is no hypergroup** corresponding to the case (I).

CASE (II). $P(H) = \big\{(a, b), (b, a), (c, d), (d, c)\big\}$.

	a	b	c	d	S(H)
a		M			a
b	N				b
c				P	c
d			Q		d
S(H)	a	b	c	d	S(H)

Reasoning as in the case (4_4) of [3] we can prove that $\exists (x, y) \in P(H)$ such that $x \cdot y \neq \{x, y\} \cup S(H)$. Therefore, by lemma (4.2), we can consider only the two following possibilities:

(1) $M \supseteq \{c\} \cup S(H)$; (2) $M = S(H)$, with $|S(H)| > 1$.

We begin with the case:

(1) $M \supseteq \{c\} \cup S(H)$.

Always as in the case (4_4) of [3], we get $b \cdot d = b$, $d \cdot a = a$, $P \cup Q \subseteq M$, $d \cdot d = d$. Besides $b \cdot c = b \cdot (d \cdot c) = b \cdot Q \supseteq b \cdot S(H) = b$, and so $b \cdot c = b$.

Analogously $c \cdot a = c \cdot (d \cdot a) = P \cdot a \supseteq S(H) \cdot a = a$ implies $c \cdot a = a$. Moreover $c \in N$, because otherwise, $b \cdot b \neq c$ (if $b \cdot b = c$, we obtain $P = c \cdot d = (b \cdot b) \cdot d = b \cdot d$, a contradiction) and by reproducibility, $b \cdot c = c$ and so $c = (b \cdot d) \cdot c = b \cdot Q \supseteq b \cdot S(H) = b$, that is a contradiction. As $c \in N$, we can use the symmetry of the table and write $d \cdot b = b, a \cdot d = a, c \cdot b = b$

and $a \cdot c = a$. For the equality $a = c \cdot a = c \cdot (d \cdot a) = P \cdot a$, one has that $b \notin P$;in the same manner, the equality $b = (b \cdot d) \cdot c = b \cdot Q$ implies $a \notin Q$. Using, as usual, the symmetry of the table, we can affirm that $b \notin Q$ and $a \notin P$. But from the relation $d \cdot (c \cdot c) = (d \cdot c) \cdot c = Q \cdot c \supseteq S(H) \cdot c = c$, we obtain $c \cdot c = c$, whence by (1.2), $P = Q = \{c, d\} \cup S(H)$. In addition the equality $(a \cdot a) \cdot b = a \cdot M$ entails $a \cdot a \in \{a, b\}$, because in the contrary one has $\{b\} = a \cdot M$ which is absurd.

We analyse now the two possibilities: **(i)** $a \cdot a = a$; **(ii)** $a \cdot a = b$;

(i) In the case $a \cdot a = a$, we get $a = a \cdot c \in a \cdot M = a \cdot (a \cdot b) = M$ and so, by (1.2), $M = H$. Obviously, by symmetry it also results $N = H$. Moreover, as $a \cdot (b \cdot b) = (a \cdot b) \cdot b = H$, we deduce $b \cdot b = b$ and this completes the following table:

	a	b	c	d	S(H)
a	a	H	a	a	a
b	H	b	b	b	b
c	a	b	c	T	c
d	a	b	T	d	d
S(H)	a	b	c	d	S(H)

with $T = S(H) \cup \{c, d\}$.

(ii) In the case $a \cdot a = b$, one has $b \cdot b = (a \cdot a) \cdot b = a \cdot M$, whence $b \cdot b = a$ and, by (1.2), $M = \{c, d\} \cup S(H)$. For the usual symmetry, one gets $N = \{c, d\} \cup S(H)$. It results the following final table:

	a	b	c	d	S(H)
a	b	T	a	a	a
b	T	a	b	b	b
c	a	b	c	T	c
d	a	b	T	d	d
S(H)	a	b	c	d	S(H)

with $T = S(H) \cup \{c, d\}$.

We come now to the second eventuality:

(2) $M = S(H)$ $(|S(H)| > 1)$.

We begin to prove that $N = S(H)$. In fact, as $H \cdot b = H$, $b = b \cdot S(H) = b \cdot (a \cdot b) = N \cdot b$ entails $\{a, b, c, d\} \cap N = \emptyset$. As consequence of (1.2), we can write:

$$(a \cdot a) \cup (a \cdot c) \cup (a \cdot d) = (a \cdot a) \cup (c \cdot a) \cup (d \cdot a) = \{b, c, d\};$$

$$(b \cdot b) \cup (b \cdot c) \cup (b \cdot d) = (b \cdot b) \cup (c \cdot b) \cup (d \cdot b) = \{a, c, d\}.$$

We have only two eventualities: **(j)** $a \cdot a = b$; **(jj)** $a \cdot a = c$; because if $a \cdot a = d$ then we fall again in the case (jj), by means of the permutation (a) (b) (cd) $(S(H))$.

(j) In this case, $a \cdot a = b$ and so, $b \cdot b = a$ (in fact $b \cdot b = (a \cdot a) \cdot b = a \cdot (a \cdot b) = a \cdot S(H) = a$). Supposing that $a \cdot c = d$ and $a \cdot d = c$, we obtain $Q = (a \cdot c) \cdot c = a \cdot (c \cdot c)$, whence $c \cdot c = b$ and $Q = S(H)$; but in this way, $d = S(H) \cdot d = (b \cdot a) \cdot d = b \cdot c$ and so $H \cdot c \neq H$, which is impossible. Therefore be $a \cdot c = c$ and $a \cdot d = d$. Besides $b \cdot c = b \cdot (a \cdot c) = S(H) \cdot c = c$ and, by (1.2), $b \cdot d = d$. As regards the hyperproduct $c \cdot c$, since $c \cdot c = (a \cdot c) \cdot c = a \cdot (c \cdot c)$, we deduce that $c \cdot c \in \{c, d\}$.

If assume $c \cdot c = d$, then we obtain $d \cdot d = (c \cdot c) \cdot d = c \cdot P \supseteq c \cdot S(H) = c$ and so $d \cdot d = c$. Moreover the equality $c = c \cdot P$ implies that $\{c, d\} \cap P = \emptyset$, whence, by (1.2), it results $P = \{a, b\} \cup S(H)$. By the symmetry of the table we obtain the following hypergroup:

	a	b	c	d	S(H)
a	b	S(H)	c	d	a
b	S(H)	a	c	d	b
c	c	c	d	U	c
d	d	d	U	c	d
S(H)	a	b	c	d	S(H)

with $U = S(H) \cup \{a, b\}$.

Alternatively, if $c \cdot c = c$, then from $d = d \cdot S(H) \subseteq d \cdot Q = d \cdot (d \cdot c) = (d \cdot d) \cdot c$ one has $d \cdot d = d$ and so, by symmetry and reproducibility, it results the following table:

	a	b	c	d	S(H)
a	b	S(H)	c	d	a
b	S(H)	a	c	d	b
c	c	c	c	H	c
d	d	d	H	d	d
S(H)	a	b	c	d	S(H)

Lastly, we have to analyse the case

(jj) $a \cdot a = c$.

At once one finds $c \cdot b = (a \cdot a) \cdot b = a$. Now we can write: $S(H) = a \cdot b = (c \cdot b) \cdot b = c \cdot (b \cdot b)$, whence $b \cdot b = d$ and $P = S(H)$. Now by (1.2), one deduces immediately $d \cdot b = c$. Besides $c \cdot a = (d \cdot b) \cdot a = d \cdot S(H) = d$, and

again by reproducibility, $d \cdot a = b$. Finally, by symmetry and reproducibility, we obtain the following table:

	a	b	c	d	S(H)
a	c	S(H)	d	b	a
b	S(H)	d	a	c	b
c	d	a	b	S(H)	c
d	b	c	S(H)	a	d
S(H)	a	b	c	d	S(H)

Before studying the remaining three cases, it is convenient to prove some lemmas, in every one of which one will denote $x \cdot x = M_x$ for each element $x \in M(H)$ such that $|x \cdot x| \geq 2$:

Lemma 4.3. *If $M_x = S(H)$ then $\forall y \in M(H) \setminus \{x\}$, it results: $[x \cdot t = y \Leftrightarrow x \cdot y = t]$ and $[t \cdot x = y \Leftrightarrow y \cdot x = t]$*

Proof. Assume $x \cdot y = t$; then $x \cdot t = x \cdot (x \cdot y) = (x \cdot x) \cdot y = S(H) \cdot y = y$. Analogously, one can prove the converse and the other implication. ∎

Lemma 4.4. *If $M_x = S(H) \cup \{x\}$, then $\forall y \in M(H) \setminus \{x\}, x \cdot y = y \cdot x = y$.*

Proof. From $(x \cdot x) \cdot y = x \cdot (x \cdot y)$, it follows that $\{y\} \cup (x \cdot y) = x \cdot (x \cdot y)$, whence, as obviously $x \cdot x \neq x$ and $|x \cdot y| = 1$, we deduce that $x \cdot y = y$. Analogously, starting from $y \cdot (x \cdot x) = (y \cdot x) \cdot x$, one obtains $y \cdot x = y$. ∎

Lemma 4.5. *Let $|x \cdot x| \geq 2$ and $|y \cdot y| \geq 2$. If $y \in M_x \cap M(H) \setminus \{x\}$, then $x \cdot y = y \cdot x = x$, $M_y \subseteq M_x$ and $x \notin M_y$.*

Proof. Since $M_y = y \cdot y \subseteq (x \cdot x) \cdot y = x \cdot (x \cdot y)$, we have $x \cdot y = x$. Analogously, one proves that $y \cdot x = x$. Moreover $M_y = y \cdot y \subseteq (x \cdot x) \cdot y = x \cdot (x \cdot y) = M_x$. Finally $x \notin M_y$, because otherwise $x = y \cdot x = y \cdot (y \cdot x) = M_y \cdot x \supseteq x \cdot x = M_x$, and this is impossible. ∎

Lemma 4.6. *Let $M_x = S(H) \cup \{x\}$ and $|y \cdot y| \geq 2$, with $x \neq y$. Then it results $M_y \setminus \{y\} \neq S(H)$ $(i.e M_y \notin \{S(H) \cup \{y\}, S(H)\})$.*

Proof. By lemma (4.4), $x \cdot y = y \cdot x = y$. So $M_y \neq S(H)$, because in the contrary case, applying lemma (4.3), we obtain $M_y = y \cdot y = x$, a contradiction. Finally supposing $M_y = S(H) \cup \{y\}$, we get, as consequence of lemma (4.4), $y \cdot x = x$, which is in opposition to what we have written before. This completes the proof. ∎

We are now ready to study the

CASE (III). $P(H) = \big\{(a, a), (b, b), (c, d), (d, c)\big\}$.

According to the preceding notations, we have the following initial table:

	a	b	c	d	S(H)
a	M_a				a
b		M_b			b
c				P	c
d			R		d
S(H)	a	b	c	d	S(H)

Considering the permutations $(ab)\ (c)\ (d)$ and $(a)\ (b)\ (cd)$, we can limit ourselves to study the following two possibilities:

$$(i) \qquad (M_a \cup M_b) \cap \{c, d\} = \emptyset; \qquad\qquad (ii) \qquad c \in M_a.$$

We beging with the former one: $(i)\ (M_a \cup M_b) \cap \{c, d\} = \emptyset$.

We prove the

Step 1: $M_a \in \{S(H) \cup \{b\}, S(H) \cup \{a, b\}\}$.

By lemma (4.6), up to isomorphisms, the following cases can befall:

$$M_a = M_b = S(H) \text{ or } M_a \in \{S(H) \cup \{b\}, S(H) \cup \{a, b\}\}.$$

But if $M_a = M_b = S(H)$, lemma (4.3) allows us to write $a \cdot b \in \{c, d\}$ and, assuming for instance $a \cdot b = c$, it follows, always by the same lemma, that $a \cdot c = b$ and $b \cdot c = a$. In this way, one obtains $c \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c) = M_a$, a contradiction.

We proceed with the

Step 2: $a \cdot b = b \cdot a = a;\ M_b \in \{S(H), S(H) \cup \{b\}\};\ b \cdot c = c \cdot b;\ b \cdot d = d \cdot b = d$.

Lemma (4.5) implies $a \cdot b = b \cdot a = a$, $a \notin M_b$ and $M_b \subseteq M_a$, whence $M_b \in \{S(H), S(H) \cup \{b\}\}$. By (1.2), $(b \cdot c) \cup (b \cdot d) = (c \cdot b) \cup (d \cdot b) = \{c, d\}$. Besides the equality $a \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c)$ and reproducibility imply $b \cdot c = c$ and $b \cdot d = d$. By the symmetry of the table, one obtains also $c \cdot b = c$ and $d \cdot b = d$.

Again by reproducibility, we have $(a \cdot d) \cup (a \cdot c) = (d \cdot a) \cup (c \cdot a) = \{c, d\}$ and this fact leads to two subcases:

$$(i_1)\ \ a \cdot d = d, \quad a \cdot c = c; \qquad\qquad (i_2)\ \ a \cdot d = c; \quad a \cdot c = d.$$

($\mathbf{i_1}$). From $(c \cdot a) \cdot d = c \cdot P$, it follows immediately $c \cdot a = c$ and $d \cdot a = d$. Concerning the hyperproduct $c \cdot c$, one has that $c \cdot c \in \{c, d\}$ (in fact, if

$c \cdot c = z$, with $z \in \{a, b\}$, then $M_z = z \cdot z = (c \cdot c) \cdot z = c \cdot (c \cdot z)c = c \cdot c$, a contradiction).

If $c \cdot c = d$ then $d \cdot d = (c \cdot c) \cdot d = c \cdot P$, whence $d \cdot d = c$ and so $P \cap \{c, d\} = \emptyset$. For symmetry, one also gets $R \cap \{c, d\} = \emptyset$. By reproducibility, it results $P = R = S(H) \cup \{a, b\}$ and so we obtain:

	a	b	c	d	S(H)
a	M_a	a	c	d	a
b	a	M_b	c	d	b
c	c	c	d	T	c
d	d	d	T	c	d
S(H)	a	b	c	d	S(H)

where $M_a \in \big\{S(H) \cup \{b\}, S(H) \cup \{a, b\}\big\}$; $M_b \in \big\{S(H), S(H) \cup \{b\}\big\}$ with $M_b \neq S(H)$ when $|S(H)| = 1$; $T = S(H) \cup \{a, b\}$.

Alternatively, if $c \cdot c = c$ then $P = (c \cdot c) \cdot d = c \cdot (c \cdot d) = c \cdot P$ and thus $c \in P$. Besides (1.2) implies that $P = H$ and, by symmetry, $R = H$. Finally, as $H = (c \cdot d) \cdot d = c \cdot (d \cdot d)$, we have $d \cdot d = d$ and this completes the following table:

	a	b	c	d	S(H)
a	M_a	a	c	d	a
b	a	M_b	c	d	b
c	c	c	c	H	c
d	d	d	H	d	d
S(H)	a	b	c	d	S(H)

where $M_a \in \big\{S(H) \cup \{b\}, S(H) \cup \{a, b\}\big\}$; $M_b \in \big\{S(H), S(H) \cup \{b\}\big\}$ with $M_b \neq S(H)$ if $|S(H)| = 1$.

We come now to the subcase

($\mathbf{i_2}$). $a \cdot c = d$; $a \cdot d = c$. We have $P = c \cdot (a \cdot c) = (c \cdot a) \cdot c$, whence $c \cdot a = d$ and $P = R$. By (1.2), $d \cdot a = c$. But $(a \cdot d) \cdot c = a \cdot (d \cdot c)$ implies $c \cdot c = a \cdot R \supseteq a \cdot S(H) = a$, whence $c \cdot c = a$ and $R \cap \{a, c, d\} = \emptyset$; therefore, by (1.2), $R = P = S(H) \cup \{b\}$. Lastly, again by (1.2), it results $d \cdot d = a$, and the following table arises:

	a	b	c	d	S(H)
a	M_a	a	d	c	a
b	a	M_b	c	d	b
c	d	c	a	U	c
d	c	d	U	a	d
S(H)	a	b	c	d	S(H)

where $M_a \in \big\{S(H) \cup \{b\}, S(H) \cup \{a,b\}\big\}$; $M_b \in \big\{S(H), S(H) \cup \{b\}\big\}$ with $M_b \neq S(H)$ if $|S(H)| = 1$; $U = S(H) \cup \{b\}$.

To complete the case (III), we have to study now the eventuality

(ii). $c \in M_a$. By lemma (4.5), we can write $a \cdot c = c \cdot a = a$. Furthermore, as $P = c \cdot d \subseteq M_a \cdot d = (a \cdot a) \cdot d = a \cdot (a \cdot d)$, we deduce $a \cdot d = a$ and $P \subseteq M_a$. By symmetry, it also results $d \cdot a = a$ and $R \subseteq M_a$.

We distinguish the case when $a \cdot b = a$ and that one when $a \cdot b \neq a$.

If $a \cdot b = a$, then $\forall (x,y) \in [M(H) \setminus \{a\}]^2$, one has $(x \cdot y) \cdot a = x \cdot (y \cdot a)$ whence $(x \cdot y) \cdot a = a$, and so $x \cdot y \cap \{a\} = \emptyset$. Then, is this case **we obtain hypergroups which are all and alone the a–enlargements** (see (6.1) of [3]) **of the hypergroups** K **such that** $S(K) \neq \emptyset$ **and** $|P(K)| = 3$, whose tables have been found in [5].

If $a \cdot b \neq a$, it results immediately $b \notin M_a$, because otherwise, by lemma (4.5), $a \cdot b = b \cdot a = a$. Therefore, the reproducibility implies that $a \cdot b = b \cdot a = b$. Besides, $d \cdot d \in \{c, d\}$, because $\forall x \in \{a, b\}, d \cdot d = x$ implies $(d \cdot d) \cdot a \neq d \cdot (d \cdot a)$, which is absurd.

Assume first $d \cdot d = c$. Consequently, $P = (d \cdot d) \cdot d = R$ and, since $d = d \cdot S(H) \subseteq d \cdot R = (d \cdot d) \cdot c = c \cdot c$, one has $c \cdot c = d$ and $R \subseteq S(H) \cup \{b\}$. As $R \subseteq M_a$ and $b \notin M_a$, we get $P = R = S(H)$. Ultimately, using (1.2), we obtain $d \cdot b = b \cdot d = b \cdot c = c \cdot b = b$. These hypergroups are isomorphic, by means of the permutation $(ab)\ (c)\ (d)\ (S(H))$, to those ones of the preceding case (when $a \cdot b = a$).

Suppose now $d \cdot d = d$. We have $c \cdot b = c \cdot (b \cdot a) = (c \cdot b) \cdot a$ and so $c \cdot b = b$. Analogously, from $d \cdot b = d \cdot (b \cdot a) = (d \cdot b) \cdot a$, it results $d \cdot b = b$. By symmetry, we also have $b \cdot c = b \cdot d = b$. Moreover, $c \cdot c \notin \{a, b\}$, because otherwise the equality $(c \cdot c) \cdot a = c \cdot (c \cdot a)$ is impossible. Then, exchanging a with b, we fall again in the preceding case (when $a \cdot b = a$).

We take in consideration now the

CASE (IV). $P(H) = \big\{(a,a), (b,b), (c,c), (d,d)\big\}$.

We put $x \cdot x = M_x, \forall x \in H \setminus S(H)$.

Let $\mu = MAX\big\{|M_x \setminus (S(H) \cup \{x\})| : x \in H \setminus S(H)\big\}$. Clearly, we have that $\mu \in \big\{0, 1, 2, 3\big\}$. We consider successively the four possibilities:

(i) $\mu = 0 (|S(H)| > 1)$.

In this case, applying lemma (4.6), we obtain $M_x = S(H), \forall x \in H \setminus S(H)$. Let $x \cdot y = z$, with $x \neq y, \{x, y\} \subseteq H \setminus S(H)$. By lemma (4.3), it follows that $y \cdot z = x, y \cdot x = z, z \cdot x = y, z \cdot y = x$, with $z \neq x$ and $z \neq y$.Thus, if we suppose $a \cdot b = c$, then it results $a \cdot b = b \cdot a = c,\ a \cdot c = c \cdot a = b,\ b \cdot c = c \cdot b = a$. By (1.2), one gets $a \cdot d = b \cdot d = c \cdot d = d$, from which $H \cdot d \neq H$, a contradiction. Therefore, **there are no hypergroups such that** $\mu = 0$.

(ii) $\mu = 1$.

Let $|M_a \setminus (S(H) \cup \{a\})| = 1$ and let $b \in M_a$ (in consequence $M_a \subseteq S(H) \cup \{a, b\}$). By lemma (4.5), it follows that $a \cdot b = b \cdot a = a,\ M_b \subseteq M_a$ and $a \notin M_b$, hence $M_b \in \{S(H), S(H) \cup \{b\}\}$ with $M_b \neq S(H)$ in case $|S(H)| = 1$. By (1.2), we can write: $(a \cdot c) \cup (a \cdot d) = (c \cdot a) \cup (d \cdot a) = \{c, d\}$.

We are now ready to prove the

Step 1: $a \cdot c = c \cdot a = d$ and $a \cdot d = d \cdot a = c$.

Suppose, by contradiction that $a \cdot c = c$. Thus we obtain $c = a \cdot c = a \cdot (a \cdot c) = M_a \cdot c \supseteq b \cdot c$, whence $b \cdot c = c$. Accordingly to (1.2), it results $|M_c \setminus (S(H) \cup \{c\})| \geq 2$, a contradiction. Therefore $a \cdot c = d$ and $a \cdot d = c$. By symmetry, it follows that $c \cdot a = d$ and $d \cdot a = c$.

Step 2: $b \cdot c = c \cdot b = c,\ b \cdot d = d \cdot b = d$ and $a \notin M_a$.

We have $b \cdot c \subseteq M_a \cdot c = (a \cdot a) \cdot c = a \cdot (a \cdot c) = c$ from which $b \cdot c = c$. Moreover, by (1.2), $b \cdot d = d$. By symmetry, one gets at once $c \cdot b = c$ and $d \cdot b = d$. Finally, as $a \cdot c = d, c = M_a \cdot c$ implies that $a \notin M_a$.

Step 3: $d \cdot c = c \cdot d = a$.

$a = a \cdot S(H) \subseteq a \cdot M_c = a \cdot (c \cdot c) = (a \cdot c) \cdot c = d \cdot c$ and so $d \cdot c = a$. The assertion follows from the symmetry.

Step 4: $M_a = M_c = M_d = S(H) \cup \{b\}$ and $M_b \in \{S(H), S(H) \cup \{b\}\}$.

By step 2, $M_a = S(H) \cup \{b\}$. Besides $a = (a \cdot c) \cdot c = a \cdot (c \cdot c) = a \cdot M_c$ implies that $M_c = S(H) \cup \{b\}$ and, as $(a \cdot c) \cdot d = a \cdot (c \cdot d)$, we deduce $M_d = M_a = S(H) \cup \{b\}$. The last assertion follows from the fact that $M_b \subseteq M_a$.

Hence, we have the following table:

	a	b	c	d	S(H)
a	M_a	a	d	c	a
b	a	M_b	c	d	b
c	d	c	M_c	a	c
d	c	d	a	M_b	d
S(H)	a	b	c	d	S(H)

where $M_a = M_c = M_d = S(H) \cup \{b\}$; $M_b \in \big\{S(H), S(H) \cup \{b\}\big\}$ with $M_b \neq S(H)$ if $|S(H)| = 1$.

(iii) $\mu = 2$.

Assume $|M_a \setminus (S(H) \cup \{a\})| = 2$ and $\{b, c\} \subseteq M_a$. By lemma (4.5), we have: $a \cdot b = b \cdot a = a \cdot c = c \cdot a = a$; $M_b \cup M_c \subseteq M_a$; $a \notin M_b \cup M_c$. By (1.2), $a \cdot d = d$ and by symmetry $d \cdot a = d$. Besides $c \cdot b \neq d$ (otherwise $d = d \cdot a = (c \cdot b) \cdot a = c \cdot (b \cdot a) = c \cdot a = a$). Since $M_c \subseteq M_a$, by (1.2), we have $c \cdot d = d$, whence, by symmetry $d \cdot c = d$.

Analogously, one can prove that $d \cdot b = b \cdot d = d$ and so $|M_d \setminus (S(H) \cup \{d\})| = 3$, **which contradicts the hypothesis** $\mu = 2$.

Finally, consider the case

(iv) $\mu = 3$.

Let $|M_a \setminus (S(H) \cup \{a\})| = 3$, that is $M_a \in \big\{H \setminus \{a\}, H\big\}$. By lemma (4.5), we can write: $\forall x \in \{b, c, d\}$, $a \cdot x = x \cdot a = a$; $M_x \subseteq M_a$; $a \notin M_x$. Moreover $\forall (x, y) \in \{b, c, d\}^2$, $x \neq y$, we have $(x \cdot y) \cdot a = x \cdot (y \cdot a) = a$ and so $x \cdot y \neq a$. For those facts, H can be obtained as an–elargement of hypergroups K with $S(K) \neq \emptyset$ and $|P(K)| = 3$ (see [3]).

Lastly, we study the case

CASE (V). $P(H) = \big\{(a, a), (b, c), (c, d), (d, b)\big\}$.

In conformity with the usual conventions, we can denote the proper hyperproducts as indicated in the following table:

	a	b	c	d	S(H)
a	M				a
b			N		b
c				R	c
d		P			d
S(H)	a	b	c	d	S(H)

where, by (1.2), $S(H) \subseteq M \cap N \cap P \cap R$.

Turning to account the permutations (a) (bcd) $(S(H))$ and (a) (bdc) $(S(H))$, we can suppose that one of the following hypothesis is valid:

(i) $M \supseteq S(H) \cup \{d\}$;

(ii) $M = S(H) \cup \{a\}$;

(iii) $M = S(H)$, whith $|S(H)| > 1$.

In the case (i), reasoning as in the case (4_3) of [3], we obtain a contradiction.

We come to the case (ii): $M = S(H) \cup \{a\}$. Let $x \in \{b, c, d\}$. By lemma (4.4), $a \cdot x = x \cdot a = x$. From $(x \cdot x) \cdot a = x \cdot (x \cdot a)$ and $(x \cdot x) \cdot x = x \cdot (x \cdot x)$, it follows that $x \cdot x = x$.Now we prove that $d \notin N$.

If we assume $d \in N$, then from $P = d \cdot b \subseteq N \cdot b = (b \cdot c) \cdot b = b \cdot (c \cdot b)$, we must have $c \cdot b = c$ and so $c = c \cdot c = (c \cdot b) \cdot c = c \cdot (b \cdot c) = c \cdot N \supseteq R$, which is absurd.

Therefore, by (1.2), it must be $b \cdot d = d$. From $R = c \cdot d = c \cdot (b \cdot d) = (c \cdot b) \cdot d$, it results $c \cdot b = c$ and from the equality $(b \cdot c) \cdot b = b \cdot (c \cdot b)$, one has $N \cdot b = N$ and so $N = \{b, c\}$. But this is a contradiction because it can not be $N \cap S(H) = \emptyset$.

We have now to consider the case (iii): $M = S(H)$, with $|S(H)| > 1$.

We begin remarking that from the equalities $(b \cdot c) \cdot d = b \cdot (c \cdot d)$, $(c \cdot d) \cdot b = c \cdot (d \cdot b)$ and $(d \cdot b) \cdot c = d \cdot (b \cdot c)$ it follows, respectively, $N \cdot d = b \cdot R$, $R \cdot b = c \cdot P$ and $P \cdot c = d \cdot N$ whence $N \neq S(H)$, $R \neq S(H)$ and $P \neq S(H)$.

Now, we prove that $\forall x \in \{b, c, d\}, a \cdot x = x \cdot a = x$. Suppose, by absurd, that there exists $x \in \{b, c, d\}$ such that $a \cdot x = y (x \neq y)$. From (1.2), we have that $y \in \{b, c, d\}$. Let, for instance, $x = b$ and $y = c$. The equality $(a \cdot b) \cdot d = a \cdot (b \cdot d)$ implies $R = a \cdot (b \cdot d)$ and so $R = S(H)$, which is impossible. In the same manner we can prove that all the other eventualities are not possible. Now, $\forall x \in \{b, c, d\}$, from $(x \cdot x) \cdot x = x \cdot (x \cdot x)$, it follows that $x \cdot x \in \{x, a\}$. If we suppose $x \cdot x = a$, taking y such that $|x \cdot y| > 1$, from $y = a \cdot y = (x \cdot x) \cdot y = x \cdot (x \cdot y)$, as $S(H) \subseteq x \cdot y$, one has $y = x$ and this fact is impossible, so $x \cdot x = x, \forall x \in \{b, c, d\}$.

Henceforth we can continue as in (ii) and conclude that **in the case (V) there are no such hypergroups**.

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