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COMPARISON BETWEEN STRONG SHAPE GROUPS AND SHAPE GROUPS FOR A POINTED TOPOLOGICAL SPACE

BY

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Abstract. **Strong homotopy groups** for inverse systems of pointed topological spaces are defined. Using resolutions of topological spaces we define **strong shape groups** and r **-stage shape groups** for arbitrary topological spaces. The main result of the paper asserts that for every pointed topological space exists an exact sequence involving strong shape groups and limits of r -stage shape groups.

1. Introduction. First, the theory of strong shape for arbitrary topological spaces based on an explicity constructed coherent prohomotopy category $CPHTop$ which involves higher homotopies of all orders, was introduced by S. M a r d e š i ć and Ju.T. L i s i c a in [5], 1984.

We denote $inv-Top_*$ the category of inverse systems of pointed topological spaces $(\underline{X}, *) = ((X_\lambda, *), p_\lambda^{\lambda'}, \Lambda)$ and morphismsof inverse systems

$$\underline{f} = (\varphi, f_\mu) : (\underline{X}, *) \longrightarrow (\underline{Y}, *) = ((Y_\mu, *), q_\mu^{\mu'}, M)$$

where $\varphi : M \longrightarrow \Lambda$ is an increasing function and $f_\mu : X_{\varphi(\mu)} \longrightarrow Y_\mu$ are morphisms in Top_* for all $\mu \in M$.

We define $pro-Top_*$ the category of inverse systems over cofinite directed index sets and of equivalence classes of morphisms of $inv-Top_*$, see [10, p.7].

The category $CPHTop_*$, defined in [5], has inverse systems as objects and coherent homotopy classes of coherent maps as morphisms.

Let $\Delta^n = \{t \in R^{n+1};\ t = (t_0, t_1, \ldots, t_n),\ t_i \geq 0,\ i = 0, \ldots, n,\ \sum_0^n t_i = 1\}$ denote the standard n-simplex.

By $\partial_j^n : \Delta^{n-1} \longrightarrow \Delta^n$ and $\sigma_j^n : \Delta^{n+1} \longrightarrow \Delta^n,\ 0 \leq j \leq n$, we denote the usual face and degeneracy operators.

They are given by

$$\partial_i^n(t_0, t_1, \dots t_{n-1}) = (t_0, t_1, \dots t_{j-1}, 0, t_j, \dots t_{n-1}),\ n \geq 1$$

$$\sigma_j^n(t_0, t_1, \dots t_{n+1}) = (t_0, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots t_{n+1}),\ n \geq 0.$$

Let M be a directed cofinite set and let $M^n = \{\bar{\mu} = (\mu_0, \dots \mu_n);\ \mu_i \in M,\ i = 0, \dots n,\ \mu_0 \leq \mu_1 \leq \dots \leq \mu_n\}$, for all $n \geq 0$ be the set of all finite increasing sequences in M. We denote $M^* = \cup M^n$. On M^* we also have face and degeneracy operators.

The j-th face operator for $0 \leq j \leq n$ and $n \geq 1$, takes $\bar{\mu} \in M^n$ to $\bar{\mu}_j = (\mu_0, \dots \mu_{j-1}, \mu_{j-1}, \mu_{j+1}, \dots \mu_n) \in M^{n-1}$.

The j-th degeneracy operator for $0 \leq j \leq n$ and $n \geq 0$, takes $\bar{\mu} \in M^n$ to $\bar{\mu}^j = (\mu_0, \mu_1, \dots, \mu_j, \mu_j, \dots, \mu_n) \in M^{n+1}$. These maps satisfy the usual commutative conditions.

Let $\underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$, $\underline{Y} = (Y_\mu, q_\mu^{\mu'}, M)$ denote inverse systems in $inv-Top$.

A coherent map of systems $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ (or just a coherent map) consists of an increasing function $\varphi : M \longrightarrow \Lambda$ and of continuous maps $f_{\bar{\mu}} : \Delta^n \times X_{\varphi(\mu_n)} \longrightarrow Y_{\mu_0}$, one for each increasing sequence $\bar{\mu} = (\mu_0, \dots, \mu_n) \in M^n$, $n \geq 0$, such that the following conditions hold:

$$(1) \qquad f_{\bar{\mu}}(\partial_j^n t, X) = \begin{cases} q_{\mu_0}^{\mu_1} f_{\bar{\mu_0}}(t, x) & j = 0 \\ f_{\bar{\mu_j}}(t, x) & j = 1, \dots, n-1 \\ f_{\bar{\mu_n}}\left(t, p_{\varphi(\mu_{n-1})}^{\varphi(\mu_n)}\right) & j = n \end{cases}$$

and

$$(2) \qquad f_{\bar{\mu}}(\sigma_j^n t, x) = f_{\bar{\mu}^j}(t, x),\ j = 0, \dots, n.$$

For $n = 0$, we identify $\Delta^0 \times X_{\varphi(\mu_0)}$ with $X_{\varphi(\mu_0)}$.

Note that inverse systems and coherent maps do not form a category, see [5].

A coherent homotopy, which connects coherent maps $\underline{f} = (\varphi, f_{\bar{\mu}})$, $\underline{f}' = (\varphi', f'_{\bar{\mu}}) : \underline{X} \longrightarrow \underline{Y}$ is a coherent map $\underline{F} : I \times \underline{X} \longrightarrow \underline{Y}$, given by $\Phi : M \longrightarrow \Lambda$, and $F_{\bar{\mu}} : \Delta^i \times I \times X_{\Phi(\mu_n)} \longrightarrow Y_{\mu_0}$ for $\bar{\mu} = (\mu_0, \dots, \mu_n) \in M^n$ such that

(a) $\Phi \geq \varphi,\ \varphi'$ and

(b) $F_{\bar{\mu}}(t, 0, x) = f_{\bar{\mu}}\left(t, p_{\varphi(\mu_n)}^{\Phi(\mu_n)}(x)\right)$

$F_{\bar{\mu}}(t, 1, x) = f'_{\bar{\mu}}(t, p_{\varphi'(\mu_n)}^{\Phi(\mu_n)}(x))$ for $t \in \Delta^i$ and $x \in X_{\Phi(\mu_n)}$.

We note $\underline{F} : \underline{f} \simeq \underline{f}'$

The relation $\simeq$ of coherent homotopy is an equivalence relation.

We define $CPHTop^r_*$ the category whose objects are inverse systems and morphisms are coherent homotopy classes of r-coherent morphisms, that is in the definition of a coherent map the index $\bar{\mu} \in M^n$ is considered only for $n = 0, \dots, r$.

In order to define the composition $\underline{h} = \underline{gf}$ of $\underline{f} = (\varphi, f_{\bar{\mu}}) : \underline{X} \longrightarrow \underline{Y}$ and $\underline{g} = (\psi, g_{\bar{\nu}}) : \underline{Y} \longrightarrow \underline{Z} = (Z_\nu, r^{\nu'}_\nu, N)$, one decomposes Δ^n into subpolyedra $P^n_i \cong \Delta^i \times \Delta^{n-i}$, $i = \overline{0,n}$ where $t = (t_0, \dots, t_n) \in P^n_i$, provided $\sum\limits_{j=0,i-1} t_j \leq \frac{1}{2} \leq \sum\limits_{j=0,i} t_j$.

One also considers maps $\alpha^n_i : P^n_i \longrightarrow \Delta^{n-i}$, $\beta^n_i : P^n_i \longrightarrow \Delta^i$ given by $\alpha^n_i(t) = (\#, 2t_{i+1}, \dots, 2t_n)$, $\beta^n_i(t) = (2t_0, \dots, 2t_{i-1}, \#)$ where $\#$ stands for 1 minus the sum of the remaining terms. We now put $\chi = \varphi\psi : N \longrightarrow \Lambda$ and define $h_{\bar{\nu}} : \Delta^n \times X_{\chi(\nu_n)} \longrightarrow Z_{\nu_0}$, $\bar{\nu} = (\nu_0, \dots, \nu_n) \in N^n$ by $h_{\bar{\nu}}(t, x) = g_{\nu_0 \dots \nu_i}(\beta^n_i(t), f_{\psi(\nu_i)\dots\psi(\nu_n)}(\alpha^n_i(t), x))$, $t \in P^n_i$

The construction of $Ho(proTop)$ introduced by Edwards and Hastings in [2] determines a category isomorphic to $CPHTop$, see also [3].

2. The group structure on the set of coherent homotopy classes of strong paths. We define the set of p-strong paths for an inverse system $(\underline{X}, *)$: $\Omega^s_{\underline{p}}(\underline{X}, *) = \{\underline{\omega} = (\omega_{\bar{\lambda}})_{\bar{\lambda} \in \Lambda^s, s=0,1,\dots} \mid \omega_{\bar{\lambda}} : \Delta^s \times (I^p, \partial I^p) \longrightarrow (X_{\lambda_0}, *)\}$, where $\bar{\lambda} = (\lambda_0, \dots, \lambda_s) \in \Lambda^s$, and $\underline{\omega}$ is a coherent map, that is

$$(1) \qquad \omega_{\bar{\lambda}}(\partial^s_j t, x) = \begin{cases} p^{\lambda_1}_{\lambda_0} \omega_{\lambda_1 \dots \lambda_s}(t, x), & j = 0 \\ \omega_{\lambda_0 \dots \lambda_{j-1} \lambda_{j+1} \dots \lambda_s}(t, x) & j = \overline{1, s-1} \text{ for } s \geq 1 \\ \omega_{\lambda_0 \dots \lambda_{s-1}}(t, x), & j = s \end{cases}$$

and

$$(2) \qquad \omega_{\bar{\lambda}}(\sigma^s_j t, x) = \omega_{\lambda_0 \dots \lambda_j \lambda_j \dots \lambda_s}(t, x), \ j = \overline{0, s} \text{ and } s \geq 0.$$

Using the homeomorphism $\varphi : (I^p, \partial I^p) \approx (D^p, S^{p-1})$ and $\bar{\theta} : D^p / S^{p-1} \longrightarrow S^p$ induced by the map $\theta : (D^p, S^{p-1}) \longrightarrow (S^p, *)$ we establish a bijection $[(S^p, *), (\underline{X}, *)]_{CPHTop_*} \xrightarrow{\simeq} [(I^p, \partial I^p), (\underline{X}, *)]_{CPHTop_*}$.

If we denote the coherent homotopy classes $[(S^p, *), (\underline{X}, *)]_{CPHTop_*}$ and the r-coherent homotopy classes $[(S^p, *), (\underline{X}, *)]_{CPHTop^r_*}$ we can define a group structure on these sets in an obvious way.

Let $\underline{\omega} = (\omega_{\bar{\lambda}})_{\bar{\lambda} \in \Delta^n, n=0,1\dots}$, $\underline{\pi} = (\pi_{\bar{\lambda}})_{\bar{\lambda} \in \Delta^n, n=0,1\dots}$ be p-strong paths for the inverse system $(\underline{X}, *)$, $p \geq 1$.

For $\omega_{\bar{\lambda}} = \omega_{\lambda_0\lambda_1\ldots\lambda_n}$, $\pi_{\bar{\lambda}} = \pi_{\lambda_0\lambda_1\ldots\lambda_n} : \Delta^n \times (I^p, \partial I^p) \longrightarrow (X_{\lambda_0}, *)$ we define the path $\theta_{\bar{\lambda}} = \omega_{\bar{\lambda}} * \pi_{\bar{\lambda}}$, $\theta_{\bar{\lambda}} : \Delta^n \times (I^p, \partial I^p) \longrightarrow (X_{\lambda_0}, *)$ to be

$$\theta_{\bar{\lambda}}(s, t_1, t_2, \ldots, t_p) = \begin{cases} \omega_{\bar{\lambda}}(s, t_1, t_2, \ldots, 2t_p) & , t_p \in [0, \frac{1}{2}] \\ \pi_{\bar{\lambda}}(s, t_1, t_2, \ldots, 2t_p - 1 & , t_p \in [\frac{1}{2}, 1] \end{cases}$$

where $s \in \Delta^n$, $(t_1, \ldots, t_p) \in I^p$.

Then $\underline{\theta} = (\theta_{\bar{\lambda}})_{\bar{\lambda} \in \Delta^n, n=0,1\ldots}$ is a coherent map $\underline{\theta} : (I^p, \partial I^p) \longrightarrow (\underline{X}, *)$. If $\underline{\omega}_1, \underline{\omega}_2$ and $\underline{\pi}_1, \underline{\pi}_2$ are p-strong paths in $(\underline{X}, *)$, coherently homotopic $\underline{\omega}_1 \simeq \underline{\omega}_2$ and $\underline{\pi}_1 \simeq \underline{\pi}_2$ then are coherently homotopic $\underline{\omega}_1 * \underline{\pi}_1 \simeq \underline{\omega}_2 * \underline{\pi}_2$. So in the set $[(I^p, \partial I^p), (\underline{X}, *)]_{CPHTop_*}$ we define $[\underline{\omega}][\underline{\pi}] = [\underline{\omega} * \underline{\pi}]$ for $\underline{\omega}$ and $\underline{\pi}$ p-strong paths. With this definition $[(I^p, \partial I^p), (\underline{X}, *)]_{CPHTop_*}$ has a group structure for $p \geq 1$ and for $p \geq 2$ the sets $[(I^p, \partial I^p), (\underline{X}, *)]_{CPHTop_*}$ are abelian groups.

For $CPHTop^r$ we consider in the above definitions the coherent maps only for $n = 0, \ldots, r$.

Now we define the coherent homotopy groups for an inverse system $(\underline{X}, *)$, $\pi_p^{coh}(\underline{X}, *) = [(S^p, *), (\underline{X}, *)]_{CPHTop_*}$ and the r-coherent homotopy groups ${}^r\pi_p^{coh}(\underline{X}, *)[(S^p, *), (\underline{X}, *)]_{CPHTop_*^r}$. So we have $\pi_p^{coh}(\underline{X}, *) = \{[\underline{\omega}] \mid \underline{\omega} \in \Omega_p^s(\underline{X}, *)\}$

Let be the inverse systems $\underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$, $\underline{Y} = (Y_\mu, q_\mu^{\mu'}, M)$ and $\underline{f} = (\varphi, f_{\bar{\mu}}) : \underline{X} \longrightarrow \underline{Y}$ a coherent map of inverse system of pointed topological spaces, that is an increasing function $\varphi : M \longrightarrow \Lambda$ and the continuous maps $f_{\bar{\mu}} : \Delta^n \times X_{\varphi(\mu_n)} \longrightarrow Y_{\mu_0}$ one for each increasing sequence $\bar{\mu} = (\mu_0, \ldots, \mu_n) \in M^n$, $n \geq 0$. Then the induced map $\underline{f}_\# : \Omega_p^s(\underline{X}, *) \longrightarrow \Omega_p^s(\underline{Y}, *)$ is given by the composition of coherent maps,

$\underline{f}_\#(\underline{\omega}) = ((\underline{f}_\#\underline{\omega})_{\bar{\mu}})_{\bar{\mu} \in M^s, s=0,1,\ldots}$ that is for $(t, x) \in \Delta^s \times I^p$ we have $(\underline{f}_\#\underline{\omega})_{\bar{\mu}}(t, x) = f_{\mu_0\ldots\mu_i}(\beta_i^s(t), \omega_{\varphi(\mu_i)\ldots\varphi(\mu_s)}(\alpha_i^s(t), x))$.

If $\underline{f} = (\varphi, f_\mu)$ is a map in inv–Top_*, that is $f_\mu : (X_{\varphi(\mu)}, *) \longrightarrow (Y_\mu, *)$ with the condition $q_{\mu_0}^{\mu_1} f_{\mu_1} = f_{\mu_0} p_{\varphi(\mu_0}^{\varphi(\mu_1)}$ for $\mu_1 \geq \mu_0$ and $\varphi(\mu_1) \geq \varphi(\mu_0)$, then $(\underline{f}_\#(\underline{\omega}))_{\bar{\mu}} = f_{\mu_0}\omega_{\varphi(\mu_0)\ldots\varphi(\mu_s)}$ for all $\bar{\mu} = (\mu_0, \ldots \mu_s) \in M^s$.

For $[\underline{f}] \in [(\underline{X}, *), (\underline{Y}, *)]_{CPHTop_*}$ we define

$\pi_p^{coh}[\underline{f}] : \pi_p^{coh}(\underline{X}, *) \longrightarrow \pi_p^{coh}(\underline{Y}, *)$

$\pi_p^{coh}[\underline{f}][\underline{\omega}] = [\underline{f}\underline{\omega}]$. It's not difficult to proof that the definition is correct, and $\pi_p^{coh}[\underline{f}]$ is a homomorphism of groups and

$\pi_p^{coh}[\underline{f}][\underline{g}] = \pi_p^{coh}[\underline{f}]\pi_p^{coh}[\underline{g}]$

3. Coherent shape group for a pointed topological space. The important notion for defining Ssh category is the ANR-resolution of a space

X, introduced by S. M a r d e š i ć in [10], 1981. It consists of an inverse system $\underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$ of ANR's X_λ and maps $p_\lambda^{\lambda'}$ and of a morphism of $pro-Top$, $\underline{p} = (p_\lambda) : X \longrightarrow \underline{X}$ satisfying certain additional conditions, see [10]. These conditions insure that the X_λ's give sufficiently good approximation of X.

The most important property of ANR-resolutions, fundamental to our construction of strong shape groups is expressed by this factorization theorem, proved in [9]:

"if $\underline{p} : X \longrightarrow \underline{X}$ is a resolution, $\underline{Y}$ is an inverse system of ANR's and $\underline{f} : X \longrightarrow \underline{Y}$ is a morphism of $CPHTop$, then there is a unique morphism of $CPHTop$, $\underline{g} : \underline{X} \longrightarrow \underline{Y}$ such that $\underline{g}C(\underline{p}) = \underline{f}$".

Here $C(\underline{p})$ is the coherent morphism induced by $\underline{p}$.

This theorem has its immediate consequence the following fact: if $\underline{p} : X \longrightarrow \underline{X}$ and $\underline{p'} : X \longrightarrow \underline{X'}$ are two ANR-resolutions of the same space X, then there is a unique isomorphism of $CPHTop$, $\underline{i} : \underline{X} \longrightarrow \underline{X'}$ such that $\underline{i}C(\underline{p}) = C(\underline{p'})$. Clearly, $\underline{i}$ induce a canonical isomorphism: $\underline{i}_* : \pi_p^{coh}(\underline{X}, *) \cong (\underline{X'}, *)$.

Since every X admits an ANR-resolution $\underline{p} : X \longrightarrow \underline{X}$, we can define the coherent shape groups or the strong shape groups of $(X, *)$ to be $\breve{\pi}_p^s(X, *) = \pi_p^{coh}(\underline{X}, *)$.

For every map $f : (X, *) \longrightarrow (Y, *)$ we consider the ANR-resolutions $\underline{p} : (X, *) \longrightarrow (\underline{X}, *)$, $\underline{q} : (Y, *) \longrightarrow (\underline{Y}, *)$ and $\underline{g} : (\underline{X}, *) \longrightarrow (\underline{Y}, *)$ the only morphism in $CPHTop_*$ for which $\underline{g}C(\underline{p}) = C(\underline{q}f)$. So, we define $f_{*p} : \breve{\pi}_p^s(X, *) \longrightarrow \breve{\pi}_p^s(Y, *)$ the homomorphism given by $\underline{g}_* : \pi_p^{coh}(\underline{X}, *) \longrightarrow \pi_p^{coh}(\underline{Y}, *)$ such that $\underline{g}_*\underline{p}_* = \underline{q}_* f_{*p}$.

It's readly seen that one obtains, in this way a functor:

$$\breve{\pi}_p^s : Top_* \longrightarrow \begin{cases} Set, & p = 0 \\ Grp, & p = 1 \\ Ab, & p \geq 2 \end{cases}$$

Remark 1. This definition of coherent groups isn't canonical because it involves a choice of ANR-resolutions. But this is also an advantage because in every case one can choose a suitable resolution.

4. Strong homotopy groups for pointed topological space. Using [4] and [7] we have for $\underline{G} = (G(\lambda), p_\lambda^{\lambda'}, \Lambda)$ an inverse system of abelian groups $G(\lambda)$ and homomorphisms $p_\lambda^{\lambda'} : G(\lambda') \longrightarrow G(\lambda)$, $\lambda' \geq \lambda$, over a directed cofinite set $(\Lambda, \leq)$ (every element has finitely many predecesors) the s-derived limit $\lim^s \underline{G}$ is the cohomology group $H^s(R(\underline{G}))$ of the cochain complex $R = R(\underline{G})$ of abelian groups $R^s = R^s(\underline{G})$ defined as follows:

(1) if $s < 0$ then $R^s = 0$

(2) if $s \geq 0$ let Λ^s denote the set of all increasing sequences $\bar{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_s)$ of $\lambda_i \in \Lambda$ for $i = 0, 1, \ldots s$. We denote $G(\bar{\lambda}) = G(\lambda_0)$. Then $R^s = \prod_{\bar{\lambda} \in \Lambda^s} G(\bar{\lambda}) = \prod_{(\lambda_0, \ldots \lambda_s) \in \Lambda^s} G(\lambda_0)$. We will denote the elements $x \in R^s$ by

$$x = \prod_{(\lambda_0, \ldots \lambda_s) \in \Lambda^s} x_{\lambda_0 \ldots \lambda_s}, x_{\lambda_0 \ldots \lambda_s} \in G(\lambda_0).$$

The coboundary operator of R is the Roos operator $\delta : R^{s-1} \longrightarrow R^s$ $(\delta x)_{\lambda_0 \ldots \lambda_s} = (p_{\lambda_0}^{\lambda_1})(x_{\lambda_1 \ldots \lambda_s}) + \sum_{j=\overline{1,s}} (-1)^j x_{\lambda_0 \ldots \lambda_j \ldots \lambda_s}$ for $x \in R^{s-1}$.

Now we have: $\lim^s \underline{G} = 0$ for $s < 0$, $\lim^0 \underline{G} = \lim \underline{G}$, $\lim^s \underline{G} = H^s(R(\underline{G}))$ for all $s \geq 1$.

A homomorphism of pro-groups $\underline{f} : \underline{G} \longrightarrow \underline{H}$ induces a cochain mapping $\underline{f} : R(\underline{G}) \longrightarrow R(\underline{H})$, consisting of homomorphisms $\underline{f}^s : R^s(\underline{G}) \longrightarrow R^s(\underline{H})$, $\underline{f}^s \delta = \delta \underline{f}^{s-1}$ where $(\underline{f}^s(x))_{\mu_0 \ldots \mu_s} = f_{\mu_0}(x_{\varphi(\mu_0) \ldots \varphi(\mu_s)})$. Beeing a cochain mapping, $\underline{f} : R(\underline{G}) \longrightarrow R(\underline{H})$ induces a homomorphism of cohomology groups $\lim^s \underline{f} : \lim^s \underline{G} \longrightarrow \lim^s \underline{H}$ defined $(\lim^s \underline{f})[x] = [\underline{f}^s(x)]$ for $x \in \ker(\delta : R^s(\underline{G}) \longrightarrow R^{s+1}(\underline{G}))$.

For the inverse system of pointed topological spaces $(\underline{X}, *) = ((X_\lambda, *), p_\lambda^{\lambda'}, \Lambda)$ we may consider the pro-groups $\pi_p(\underline{X}, *) = (\pi_p(X_\lambda, *), (p_\lambda^{\lambda'})_\#, \Lambda)$ for $p \geq 1$ and the corresponding homomorphisms of groups.

We may define $\lim^s \pi_p(\underline{X}, *)$ for $p \geq 2$ and all $s \geq 0$. For $p = 1$ we may consider the definition of $\lim^s \pi_p(\underline{X}, *)$ using the cohomotopy groups, for $s = 0, 1$ (see also [1]).

So we have $R^s(\pi_p(\underline{X}, *)) = \prod_{\bar{\lambda} \in \Lambda^s} \pi_p(X_{\lambda_0}, *)$ and the map $\delta : R^{s-1}(\pi_p(\underline{X}, *)) \longrightarrow R^s(\pi_p(\underline{X}, *))$ given by

$$(\delta \underline{\omega})_{\bar{\lambda} \in \Lambda^s, \bar{\lambda} = (\lambda_0, \ldots, \lambda_s)} = (p_{\lambda_0}^{\lambda_1})_\# [\omega_{\lambda_1 \ldots \lambda_s}] + \sum_{j=1, \ldots s} (-1)^j \left[\omega_{\lambda_0 \ldots \widehat{\lambda_j} \ldots \lambda_s}\right]$$

$$\text{for } \underline{\omega} \in \mathrm{R}^{\mathrm{s}-1}(\pi_{\mathrm{p}}(\underline{\mathrm{X}}, *)),$$

where we have denoted the operation in the abelian group $\pi_p(X_{\lambda_0},*)$ by "+". This map is a coboundary map because $\delta\circ\delta=0$.

Using [10, p.167] and [1, p.251] we define the morphisms between the pro-groups:

$\underline{f}_{*p}:\pi_p(\underline{X},*)\longrightarrow\pi_p(\underline{Y},*)$ where $\underline{f}_{*p}=(f_{\mu,\mu\in M})_{*p}$

$\lim\underline{f}_{*p}:\lim\pi_P(\underline{X},*)\longrightarrow\lim\pi_p(\underline{Y},*)$

$\lim^1\underline{f}_{*p}:\lim^1\pi_p(\underline{X},*)\longrightarrow\lim^1\pi_p(\underline{Y},*)$ for all $p\geq 1$ (for all groups abelian or not) and

$\lim^s\underline{f}_{*p}:\lim^s\pi_p(\underline{X},*)\longrightarrow\lim^s\pi_p(\underline{Y},*)$ for all $p\geq 2$ and all $s\geq 0$ (in the abelian case).

For the inverse system $(\underline{X},*)=((X_\lambda,*),p^{\lambda'}_\lambda,\Lambda)$ we define the set $K(\underline{X},*)=(K_n(\underline{X},*))_{n\geq 1}$ as follows

$$K_p(\underline{X},*)=\prod_{s=0}^{\infty}\prod_{\Lambda^s}\{\underline{\omega}\mid\underline{\omega}=(\omega_{\bar\lambda})_{\bar\lambda}\text{ , where}$$

$$\omega_{\bar\lambda}:\Delta^s\times(I^p,\partial I^p)\longrightarrow(X_{\lambda_0},*)\}\,.$$

As in first paragraph we may consider the composition " * " on $K_p(\underline{X},*)$.

If we denote $\underline{\omega}\in K_p(\underline{X},*)$ by $\underline{\omega}=\prod_{s=0}^{\infty}\prod_{\bar\lambda=(\lambda_0,\dots\lambda_s)\in\Lambda^s}\omega_{\lambda_0,\dots,\lambda_s}$,

we may consider the coboundary operator

$$d:\prod_{\Lambda^{s-1}}\{\underline{\omega}\mid\underline{\omega}=(\omega_{\bar\lambda}:\Delta^{s-1}\times(I^p,\partial I^p)\longrightarrow(X_{\lambda_0},*))_{\bar\lambda\in\Lambda^{s-1}}\}\longrightarrow$$

$$\longrightarrow\prod_{\Lambda^s}\{\underline{\omega}'\mid\underline{\omega}'=(\omega'_{\bar\lambda}:\Delta^s\times(I^p,\partial I^p)\longrightarrow(X_{\lambda_0},*))_{\bar\lambda\in\Lambda^s}\}$$

$d\underline{\omega}$ defined using the Roos operator, $d\underline{\omega}:\bar\lambda\in\Lambda^s\mapsto(d\underline{\omega})_{\bar\lambda}$

$$(d\underline{\omega})_{\bar\lambda\in\Lambda^s,\bar\lambda=(\lambda_0,\dots,\lambda_s)}=$$

$$=(p^{\lambda_1}_{\lambda_0}\omega_{\lambda_1\dots\lambda_s})*(\omega_{\lambda_0\widehat{\lambda_1}\dots\lambda_s})^{-1}*(\omega_{\lambda_0\lambda_1\widehat{\lambda_2}\dots\lambda_s})*\dots(\omega_{\lambda_1\dots\widehat{\lambda_j}\dots\lambda_s})^{(-1)^j}*\dots$$

where "*" is the composition of coherent paths in $\overline{\mathcal{J}Top}=Coh/_{\sim}$, the category defined in [11] and $()^{-1}$ is the inverse path.

For $p\geq 2$ we have the homotopic commutativity of the composition, so $d\circ d$ is coherent homotop with the constant path.

Now we define the strong homotopy groups (the Steenrod homotopy groups) of $(\underline{X},*)$ to be

$$\pi_p^s(\underline{X},*)=\{[\underline{\omega}] \mid [\underline{\omega}] \text{ is the homotopy class of}$$

$$\underline{\omega}=(\underline{\omega}_{\bar{\lambda}})_{\bar{\lambda}\in\Lambda^s,s=0,1,\ldots}\in K_p(\underline{X},*)\}\,.$$

These are isomorphic with the coherent groups of $(\underline{X},*)$.

In the same manner we define for the inverse system $(\underline{X},*)=((X_\lambda,*),p_\lambda^{\lambda'},\Lambda)$ and each integer $r\geq 0$ the complexes: $K^r(\underline{X},*)=(K_p^r(\underline{X},*))_{p\geq 1}$ as follows $K_p^r(\underline{X},*)=\prod_{s=0}^{r}\prod_{\Lambda^s}\{\underline{\omega} \mid \underline{\omega}=(\omega_{\bar{\lambda}})_{\bar{\lambda}}$, where $\omega_{\bar{\lambda}}$ is the map $\omega_{\bar{\lambda}}:\Delta^s\times(I^p,\partial P^p)\longrightarrow(X_{\lambda_0},*)$, for $s=0,\ldots,r\}$

For $r<0$ we put $K_p^r=0$. Then the r-strong homotopy groups of $(\underline{X},*)$ are ${}^r\pi_p^s(\underline{X},*)=\{[\underline{\omega}]|[\underline{\omega}]$ is the homotopy class of $\underline{\omega}=(\underline{\omega}_{\bar{\lambda}})_{\bar{\lambda}\in\Lambda^s,s=0,1,\ldots r}\in K_p^r(\underline{X},*)\}$.

Remark 2. For $[\underline{\omega}]\in{}^r\pi_p^s(\underline{X},*)$ we have that the component $\omega_{\bar{\lambda}}:\Delta^{r+1}\times(I^p,\partial I^p)\longrightarrow(X_{\lambda_0},*)$ is the constant path, where $\bar{\lambda}=(\lambda_0,\ldots,\lambda_{r+1})\in\Lambda^{r+1}$. So $\omega_{\lambda_0\ldots\lambda_r}(\sigma_j^r t,x)=\omega_{\lambda_0\ldots\lambda_j\lambda_j\ldots\lambda_r}(t,x)$, $j=\overline{0,r}$ and $r\geq 0$ is constant.

From the coherent conditions for $\omega_{\lambda_0\ldots\lambda_{r+1}}$ we have $p_{\lambda_0}^{\lambda_1}\omega_{\lambda_1\ldots\lambda_{r+1}}$, $\omega_{\lambda_0\widehat{\lambda_1}\ldots\lambda_{r+1}},\ldots\omega_{\lambda_0\ldots\widehat{\lambda_j}\ldots\lambda_{r+1}},\ldots,\omega_{\lambda_0\ldots\lambda_{r-1}}$ are constant.

For the construction of exact sequences in strong homotopy theory we consider the following groups and morphisms.

We may define the strong shape groups for a pointed topological space $(X,*)$ to be $\breve{\pi}_p^s(X,*)=\pi_p^s(\underline{X},*)$ where $(\underline{X},*)$ is a resolution of $(X,*)$.

For a map $\underline{f}=(\varphi,f_\mu)\in inv-Top_*((\underline{X},*),(\underline{Y},*))$ or in $pro-Top_*$, we have the induced maps:

$$\underline{f}_{\#}:K_p(\underline{X},*)\longrightarrow K_p(\underline{Y},*)$$

$$\underline{f}_{\#}:\underline{\omega}=\prod_{s=0}^{\infty}\prod_{\bar{\lambda}=(\lambda_0,\ldots\lambda_s)\in\Lambda^s}\omega_{\lambda_0,\ldots,\lambda_s}\mapsto\underline{f}_{\#}(\underline{\omega})=$$

$$=\prod_{s=0}^{\infty}\prod_{\bar{\lambda}=(\lambda_0,\ldots\lambda_s)\in\Lambda^s}f_{\mu_0}\omega_{\varphi(\mu_0),\ldots,\varphi(\mu_s)}$$

and in strong homotopy

$$\underline{f}_* : \pi_p^s(\underline{X}, *) \longrightarrow \pi_p^s(\underline{Y}, *) \text{ given by } \underline{f}_*[\underline{\omega}] = [\underline{f}\underline{\omega}].$$

We have also the coresponding maps

$$\underline{f}^r_{\#} : K_p^r(\underline{X}, *) \longrightarrow K_p^r(\underline{Y}, *) \text{ and } \underline{f}^r_* : {}^r\pi_p^s(\underline{X}, *) \longrightarrow {}^r\pi_p^s(\underline{Y}, *).$$

Now we consider the natural projections

$$\pi \overset{not}{=} \pi^r_{r-1} : K_p^r(\underline{X}, *) = \prod_{s=0}^{r} \prod_{\Lambda^s} \{\underline{\omega} \mid \underline{\omega} = (\omega_{\bar{\lambda}})_{\bar{\lambda} \in \Lambda^s, s=0,1,\dots r}\} \longrightarrow K_p^{r-1}(\underline{X})$$

$$\pi : \{\underline{\omega} \mid \underline{\omega} = (\omega_{\bar{\lambda}})_{\bar{\lambda}}\} \mapsto \pi(\underline{\omega}) = (\omega'_{\bar{\lambda}})_{\bar{\lambda} \in \Lambda^{r-1}},$$

$$\text{where } \omega'_{\bar{\lambda}} = \omega_{\bar{\lambda}} \text{ for } \bar{\lambda} \in \Lambda^s,\ s = 0, \dots, r-1\}.$$

Using the coherent conditions for coherent maps we have a mapping between the strong homotopy groups

$$\pi_{*p} : {}^r\pi_p^s(\underline{X}, *) \longrightarrow {}^{r-1}\pi_p^s(\underline{X}, *) \text{ with } \pi_{*p} = (\pi^r_{r-1})_*.$$

We define the r-stage shape groups for a pointed topological space $(X, *)$ or for its resolution $((\underline{X}, *),\ \underline{p} = (p_\lambda)_{\lambda \in \Lambda})$ by:
$\breve{\pi}_p^r(X, *) = (\pi_r^{r+1})_*({}^{r+1}\pi_p^s(\underline{X}, *))$ and
$\breve{\pi}_p^r(\underline{X}, *) = (\pi_p^{r+1})_*({}^{r+1}\pi_p^s(\underline{X}, *))$
Then we have (1) $\breve{\pi}_p^r(\underline{X}, *) = 0$ for $r < 0$ and
(2) $\breve{\pi}_p^0(\underline{X}, *) = \lim \pi_p(\underline{X}, *)$.

Remark 3. So is the elements of $\breve{\pi}_p^0(\underline{X}, *)$ are $(\pi_0^1)_*([\underline{\omega}]) \in {}^0\pi_p^s(\underline{X}, *)$ for $\underline{\omega} \in \Omega_p^1(\underline{X}, *)$ with $\underline{\omega} = (\omega_{\bar{\lambda}})_{\bar{\lambda} \in \Lambda^s, s=0,1}$ a 1-coherent map. Since $(\Delta^1, \partial_0^1(1), \partial_1^1(1)) \approx (I, 0, 1)$ and $\omega_{\lambda_0\lambda_1}(\partial_0^1(1), x) = p_{\lambda_0}^{\lambda_1}\omega_{\lambda_1}(x)$ and $\omega_{\lambda_0\lambda_1}(\partial_1^1(1), x) = \omega_{\lambda_0}(x),\ x \in I^p$ we have $[\omega_{\lambda_0}] = (p_{\lambda_0}^{\lambda_1})_{\#}[\omega_{\lambda_1}]$.

For $(X, *)$ a pointed topological space and $((\underline{X}, *), \underline{p})$ an ANR-resolution, for $(X, *)$ we denote shape groups of $(X, *)$ to be $\breve{\pi}_p(X, *) = \lim pro - \pi_p(X, *)$ where $pro{-}\pi_p(X, *) = \pi_p(\underline{X}, *)$. So we have $\breve{\pi}_p^0(\underline{X}, *) = \breve{\pi}_p(X, *)$. This explains why we call $\breve{\pi}_p^r(\underline{X}, *)$ the r-stage shape groups.

We can verify without difficulty that the following relations are satisfied: $\pi^r_{r-1}\pi^{r+1}_r = \pi^{r+1}_{r-1}$ and we see in the following diagram that $\pi_*(\breve{\pi}^r_p) \subset \breve{\pi}^{r-1}_p$

$$\begin{array}{ccccc} {}^{r+1}\pi^s_p(\underline{X},*) \to & {}^{r}\pi^s_p(\underline{X},*) & \stackrel{\pi_*}{\to} & {}^{r-1}\pi^s_p(\underline{X},*) \\ & \uparrow & & \uparrow \\ (\pi^{r+1}_r)_*({}^{r+1}\pi^s_p(\underline{X},*)) = & \breve{\pi}^r_p(\underline{X},*) & \to (\pi^r_{r-1})_*({}^{r}\pi^s_p(\underline{X},*)) = & \breve{\pi}^{r-1}_p(\underline{X},*) \end{array}$$

The restrictions of π_* to $\breve{\pi}^r_p(\underline{X},*)$ are homomorphisms and we denote these $\breve{\pi}_p = (\pi^r_{r-1})_p : \breve{\pi}^r_p(\underline{X},*) \longrightarrow \breve{\pi}^{r-1}_p(\underline{X},*)$.

So, for a morphism $\underline{f} = (\varphi, f_\mu) \in inv-Top_*((\underline{X},*),(\underline{Y},*))$ we have the induced homomorphism $\breve{f}^r_{*p} : \breve{\pi}^r_p(\underline{X},*) \longrightarrow \breve{\pi}^r_p(\underline{Y},*)$.

5. Exact sequences in strong homotopy theory. Now we establish some exact sequences for the strong homotopy groups of an inverse system of pointed topological spaces $(\underline{X},*)$. Similar results are established for strong homology by S. M a r d e š i ć and Z. M i m i n o s h v i l i in [7], 1990.

First we recall some facts on exact couple of bigraded abelian groups and then we use them in the theory of strong homotopy.

An **exact couple of bigraded abelian groups** consists of bigraded abelian groups $D = (D^r_n)$, $E = (E^r_n)$, $r,n \in \mathbb{Z}$ and homomorphisms $i : D \longrightarrow D$, $j : D \longrightarrow E$, $k : E \longrightarrow D$ of given bidegrees such that the diagram

$$\begin{array}{ccc} D & \longrightarrow & D \\ \searrow & & \swarrow \\ & E & \end{array} \quad \text{is exact at every vertex.}$$

A **homomorphism of exact couple** $(D,E,i,j,k) \longrightarrow (\tilde{D},\tilde{E},\tilde{i},\tilde{j},\tilde{k})$ consists of homomorphisms $f : D \longrightarrow \tilde{D}$ and $g : E \longrightarrow \tilde{E}$ of bidegree $(0,0)$ such that $fi = \tilde{i}f$, $gj = \tilde{j}f$, $fk = \tilde{k}g$.

With every exact couple (D,E,i,j,k) one associates its **derived exact couple** (D',E',i',j',k') defined as follows.

$D' = i(D)$. Since $(jk)^2 = 0$, (E,jk) is a bigraded differential group and E' is its homology group $E' = H(E,jk)$. Furthemore, $i' = i \mid D'$. If $d' = i(d)$, $d \in D$, then $j'(d') = [j(d)]$ is the homology class of $j(d)$. If $e' \in E'$ and e' is the homology class $[e]$ of a cycle $e \in \ker(jk) \subseteq E$, then $k'(e') = k(e)$.

Furthemore, a homomorphism of exact couple (f,g) induces a homomorphism between derived exact couple (f',g'). The homomorphisms f',

g' are defined as follows. $f' = f \mid D$. If $e' = [e] \in E'$, where $e \in \ker(jk) \subseteq E$, then $g'(e') = [g(e)]$ is the homology class of $g(e)$ in $\tilde{E}' = H(\tilde{E}, \tilde{j}\tilde{k})$. Then the following diagram commutes.

$$\begin{array}{ccccccc} E' & \xrightarrow{k'} & D' & \xrightarrow{i'} & D' & \xrightarrow{j'} & E' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{E}' & \xrightarrow{\tilde{k}'} & \tilde{D}' & \xrightarrow{\tilde{i}'} & \tilde{D}' & \xrightarrow{\tilde{j}'} & \tilde{E}' \end{array}$$

We apply this to our situation in strong homotopy theory.

For the inverse system $(\underline{X}, *) = ((X_\lambda, *), p_\lambda^{\lambda'}, \Lambda)$ we define the following sequence of spaces which is exact:

$$* \to \prod_{\bar{\lambda} \in \Lambda^r} X_{\lambda_0 \ldots \lambda_r} \to \prod_{s=0,\ldots,r} \prod_{\bar{\lambda} \in \Lambda^s} X_{\lambda_0 \ldots \lambda_s} \to \prod_{s=0,\ldots,r-1} \prod_{\bar{\lambda} \in \Lambda^s} X_{\lambda_0 \ldots \lambda_s} \to *$$

with $X_{\lambda_0 \ldots \lambda_s} = X_{\lambda_0}$. The first morphism is the inclusion and the second is the projction.

Now we consider the complex $K^r = (K_p^r(\underline{X}, *))_{p \geq 1}$, with $K_p^r = 0$ if $r < 0$ and the complex $L^r = L^r(\underline{X}, *) = (L_p^r(\underline{X}, *))_{p \geq 1}$, by putting $L_p^r = 0$ if $r < 0$ and $L_n^r = \prod_{\Lambda^r} \{\underline{\omega} \mid \underline{\omega} = (\omega_{\bar{\lambda}})_{\bar{\lambda} \in \Lambda^r}$, where $\omega_{\bar{\lambda}}$ is the map $\omega_{\bar{\lambda}} : \Delta^r \times (I^n, \partial I^n) \longrightarrow (X_{\lambda_0}, *)$, with $[\omega_{\bar{\lambda}}] \in \pi_{r+n}(X_{\lambda_0}, *)\}$.

Remark 4. If $[\underline{\omega}] \in \pi_n(L^r)$, it induces a path from $(I^{n+r}, \partial I^{n+r}) \approx (\Delta^r \times I^n, \Delta^r \times \partial I^n \cup (\bigcup_{j=0,\ldots,r} \partial_j^r \Delta^{r-1} \times I^n))$ and for $\overline{\lambda} \in \Lambda^r$ we have

$$\omega_{\bar{\lambda}}(\partial_j^r t, x) = \begin{cases} p_{\lambda_0}^{\lambda_1} \omega_{\lambda_1 \ldots \lambda_r}(t, x), & j = 0 \\ \omega_{\lambda_0 \ldots \lambda_{j-1} \lambda_{j+1} \ldots \lambda_r}(t, x), & j = \overline{1, r-1} \\ \omega_{\lambda_0 \ldots \lambda_{r-1}}, & j = r \end{cases}$$

constant. So $\pi_n(L^r) = \prod_{\bar{\lambda} \in \Lambda^r} \pi_{r+n}(X_{\lambda_0}, *)$.

For the construction of exact sequences in strong homotopy theory we define the following groups and morphisms using the complexes K^r and L^r.

$\pi_n(\prod_{\bar{\lambda} \in \Lambda^r} X_{\lambda_0 \ldots \lambda_r}, *) \overset{not}{=} \pi_n(L^r) = \{[\underline{\omega}] \mid \underline{\omega} = (\omega_{\bar{\lambda}})_{\bar{\lambda}}$, where $\omega_{\bar{\lambda}}$ is the map $\omega_{\bar{\lambda}} : \Delta^r \times (I^n, \partial I^n) \longrightarrow (X_{\lambda_0}, *),\ [\omega_{\bar{\lambda}}] \in \pi_{r+n}(X_{\lambda_0}, *)\}$ and

$\pi_n(\prod_{s=0,\dots,r}\prod_{\bar\lambda\in\Lambda^s} X_{\lambda_0\dots\lambda_s},*)\overset{not}{=}\pi_n(K^r)=\{[\underline{\omega}]\mid \underline{\omega}=(\omega_{\bar\lambda})_{\bar\lambda\in\Lambda^s},\ s=0,\dots,r$ and where $\omega_{\bar\lambda}$ is the map $\omega_{\bar\lambda}:\Delta^s\times(I^n,\partial I^n)\longrightarrow(X_{\lambda_0},*)$, for $s=0,\dots,r\}$.

Let $u:L^r\longrightarrow K^r,\ u=(u_n)$ the natural embedding. Then we have the short sequence of sets:

$$*\longrightarrow L_n^r\xrightarrow{u_n}K_n^r\xrightarrow{\pi=(\pi_{r-1}^r)_n}K_n^{r-1}\longrightarrow *$$

Consequently there exists a long exact sequence of homotopy groups:

$$\dots\longrightarrow\pi_{n+1}(K^{r-1})\xrightarrow{\Delta}\pi_n(L^r)\xrightarrow{u_*}\pi_n(K^r)\xrightarrow{\pi_*}\pi_n(K^{r-1})\longrightarrow\pi_{n-1}(L^r)\longrightarrow\dots. \tag{1}$$

Here we have defined Δ to be the map induced by the coboundary morphism:

$d:\prod_{\Lambda^{r-1}}\{\underline{\omega}\mid\underline{\omega}=(\omega_{\bar\lambda})_{\bar\lambda\in\Lambda^{r-1}}\}\longrightarrow\prod_{\Lambda^r}\{\underline{\omega}'\mid\underline{\omega}'=(\underline{\omega}'_{\bar\lambda})_{\bar\lambda\in\Lambda^r}\}$.

For $[\underline{\omega}]\in\pi_{n+1}(K^{r-1}),\ \underline{\omega}=(\omega_{\bar\lambda})_{\bar\lambda\in\Lambda^s,s=0,\dots,r-1}$,

$\omega_{\bar\lambda}:\Delta^s\times(I^{n+1},\partial I^{n+1})\longrightarrow(X_{\lambda_0},*)$, and $s=0,\dots,r-1$ we may consider the maps $\omega'_{\bar\lambda'}:\Delta^{s+1}\times(I^n,\partial I^n)\longrightarrow(X_{\lambda_{0,*})}$ induced by $\omega_{\bar\lambda}$ using the following facts:

$\Delta^{s+1}\times(I^n,\partial I^n)\approx\Delta^s\times(\Delta^1\times(I^n,\partial I^n))\approx$

$\approx\Delta^s\times(I^{n+1},\Delta^1\times\partial I^n)\subseteq\Delta^s\times(I^{n+1},\partial I^{n+1})$

Now if we put:

$D_n^r=\pi_n(K^r),\ E_n^r=\pi_n(L^r),\ i=\pi_*,\ j=\Delta,\ k=u_*$

then (1) shows that (D,E,i,j,k) is an exact couple of bigraded abelian groups with homomorphism i,j,k of bedegree $(-1,0),(1,-1),(0,0)$ respectively. We consider its derived exact couple defined as follows:

$(D')_n^r=i(D_n^{r+1})=\pi_*(\pi_n(K^{r+1}))=\breve\pi_n^r(\underline{X},*)$

$(E')_n^r=\lim^r\pi_{n+r}(X_{\lambda_0},*)$ (for we have $\pi_n(L^r)=\prod_{\Lambda^r}\pi_{n+r}(X_{\lambda_0},*)$) and $jk=\Delta u_*:E_n^r\longrightarrow E_{n-1}^{r+1}$ coincides with $\delta:\prod_{\Lambda^r}\pi_{n+r}(X_{\lambda_0},*)\longrightarrow\prod_{\Lambda^{r+1}}\pi_{n+r}(X_{\lambda_0},*)$. Then $i',\ j',\ k'$ have bedegrees $(-1,0),\ (2,-1),\ (0,0)$.

From the long exact sequences of groups:

$$\dots\longrightarrow(D')_{n+1}^{r-2}\xrightarrow{j'}(E')_n^r\xrightarrow{k'}(D')_n^r\xrightarrow{i'}(D')_n^{r-1}\xrightarrow{j'}(E')_{n-1}^{r+1}\longrightarrow\dots \tag{2}$$

we have the following results:

Proposition 1. *For every inverse system* $(\underline{X},*)$ *and* $p \geq 2$ *there exists an exact sequence:*

$$(S1) \qquad \ldots \longrightarrow \breve{\pi}^{r-2}_{p+1}(\underline{X},*) \longrightarrow \lim{}^{r}\pi_{p+r}(\underline{X},*) \longrightarrow \breve{\pi}^{r}_{p}(\underline{X},*)$$

$$\xrightarrow{\breve{\pi}} \breve{\pi}^{r-1}_{p}(\underline{X},*) \longrightarrow \lim{}^{r+1}\pi_{p+r}(\underline{X},*) \longrightarrow \ldots$$

which is natural with respect to morphisms of inverse systems. Since $\breve{\pi}^{r-2}_{p+1}(\underline{X},*) = 0$ *for* $r \leq 1$, $(S1)$ *begins with*

$$(3) \qquad 0 \longrightarrow \lim{}^{1}\pi_{p+1}(\underline{X},*) \longrightarrow \breve{\pi}^{1}_{p}(\underline{X},*) \xrightarrow{\breve{\pi}} \breve{\pi}^{0}_{p}(\underline{X},*) \longrightarrow \ldots .$$

Proof. Follows from the above considerations.

The naturality of this sequence results from the following commutative diagram induced by a morphism of inverse systems in $inv-Top_*$ or in $pro-Top_*$, $\underline{f} = (\varphi, f_\mu) : (\underline{X},*) \longrightarrow (\underline{Y},*)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^r(\underline{X},*) & \xrightarrow{u} & K^r(\underline{X},*) & \xrightarrow{\pi} & K^{r-1}(\underline{X},*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^r(\underline{Y},*) & \xrightarrow{u} & K^r(\underline{Y},*) & \xrightarrow{\pi} & K^r(\underline{Y},*) & \longrightarrow & 0 \end{array}$$

and from the corresponding strong homotopy sequences for $(\underline{X},*)$ and $(\underline{Y},*)$.

Now we discuss another exact sequence for an inverse system $(\underline{X},*)$. We already have ${}^{r}\pi^{s}_{p}(\underline{X},*)$, $\breve{\pi}^{r}_{p}(\underline{X},*)$ and the homomorphisms

$(\breve{\pi}^{r+1}_{r})_{*p} : \breve{\pi}^{r+1}_{p}(\underline{X},*) \longrightarrow \breve{\pi}^{r}_{p}(\underline{X},*)$ and

$(\pi^{r+1}_{r})_{*p} : {}^{r+1}\pi^{s}_{p}(\underline{X},*) \longrightarrow {}^{r}\pi^{s}_{p}(\underline{X},*)$.

Therefore we have towers (inverse sequences) of groups

${}^{*}\pi^{s}_{p}(\underline{X},*) = ({}^{r}\pi^{s}_{p}(\underline{X},*), (\pi^{r+1}_{r})_{*})_{r\geq 0}$

$\breve{\pi}^{*}_{p}(\underline{X},*) = (\breve{\pi}^{r}_{p}(\underline{X},*), (\breve{\pi}^{r+1}_{r})_{*})_{r\geq 0}$.

Moreover we have mappings $\pi^{r} = (\pi^{r}_{p})_{p\geq 1} : K_p(\underline{X},*) \longrightarrow K^{r}_{p}(\underline{X},*)$ given by the natural projections, which induce the homomorphism of strong homotopy groups: $\pi^{r}_{*} : \pi^{s}_{p}(\underline{X},*) \longrightarrow \breve{\pi}^{r}_{p}(\underline{X},*) \subseteq {}^{r}\pi^{s}_{p}(\underline{X},*)$.

Since $(\pi^{r}_{r-1})_{*}(\pi^{r})_{*} = (\pi^{r-1})_{*}$ these homomorphisms determine

$\pi^{*} : \pi^{s}_{p}(\underline{X},*) \longrightarrow \lim\limits_{r}{}^{n}\pi^{s}_{p}(\underline{X},*)$ and $\breve{\pi} : \pi^{s}_{p}(\underline{X},*) \longrightarrow \lim\limits_{r}\breve{\pi}^{r}_{p}(\underline{X},*)$.

Also a morphism of inverse systems in $inv-Top_*$ or in $pro-Top_*$ $\underline{f} = (\varphi, f_\mu) : (\underline{X},*) \longrightarrow (\underline{Y},*)$ induces the map

$\underline{f}^r_p : K^r_p(\underline{X}, *) \longrightarrow K^r_p(\underline{Y}, *)$, $r \geq 0$ which commute with π^r_{r-1} and therefore define the level preserving homomorphisms of towers:

$\underline{f}^*_p : {}^*\pi^s_p(\underline{X}, *) \longrightarrow {}^*\pi^s_p(\underline{Y}, *)$ and $\underline{\breve{f}}_p : \breve{\pi}^*_p(\underline{X}, *) \longrightarrow \breve{\pi}^*_p(\underline{Y}, *)$ which in turn induce the following homomorphisms of groups:

$\underline{\breve{f}}_p : \lim{}^s({}^*\pi^s_p(\underline{X}, *)) \longrightarrow \lim{}^s({}^*\pi^s_p(\underline{Y}, *))$ and

$\underline{\breve{f}}_p : \lim{}^s(\breve{\pi}^*_p(\underline{X}, *)) \longrightarrow \lim{}^s(\breve{\pi}^*_p(\underline{Y}, *))$ for $s = 0, 1$, $p \geq 1$.

Proposition 2. *For every inverse system $(\underline{X}, *)$ and $p \geq 2$ there exists an exact sequence*

$$(S2) \qquad 0 \longrightarrow \lim_r{}^1 \breve{\pi}^r_{p+1}(\underline{X}, *) \longrightarrow \pi^s_p(\underline{X}, *) \xrightarrow{\breve{\pi}} \lim_r \breve{\pi}^r_p(\underline{X}, *) \longrightarrow 0$$

which is natural with respect to morphisms of inverse systems.

The proof is based on a lemma of tower of spaces (see also [10, p.178]).

Remark 5. If $(\underline{X}, *)$ is such as $\breve{\pi}^*_{p+1}(\underline{X}, *) = (\breve{\pi}^r_{p+1}(\underline{X}, *), (\breve{\pi}^{r+1}_r)_*)_{r \geq 0}$ is Mittag–Leffler, then $\lim_r{}^1 \breve{\pi}^r_{p+1}(\underline{X}, *) = 0$ (see [10, p.173]) and the strong shape group is the limit of the r-stage shape groups, $\pi^s_p(\underline{X}, *) \approx \lim_r \breve{\pi}^r_p(\underline{X}, *)$.

Lemma 1. *Let $(\underline{X}, *) = ((X_n, *), p^{n+1}_n)$ be an inverse sequence in Top_* and let $\underline{p} = (p_n) : (X, *) \longrightarrow (\underline{X}, *)$ be an inverse limit. If all p^{n+1}_n are surjective (weak fibrations) then there exists an exact sequence:*

$$(5) \qquad 0 \longrightarrow \lim{}^1 \pi_{p+1}(\underline{X}, *) \longrightarrow \pi_p(\lim(\underline{X}, *)) \xrightarrow{\rho_{*n}} \lim \pi_p(\underline{X}, *) \longrightarrow 0$$

which is natural with respect to level-preserving morphisms of inverse systems indexed by N.

Remark 6. Here $\lim(\underline{X}, *)$ is the pointed topological space $(X, *)$. We consider the corresponding pro–set: $\Omega_p(\underline{X}, *) = (\Omega_p(X_n, *), \Omega_p(p^{n+1}_n), N)$ and the sets $\lim \Omega_p(\underline{X}, *)$ for each $p \geq 1$. The natural projection of the inverse limit are the mappings, $\rho_n : \lim \Omega_p(\underline{X}, *) \longrightarrow \Omega_p(X_n, *)$ which induce homomorphisms:

$$(\rho_n)_{*p} : \pi(\lim \Omega_p(\underline{X}, *)) \longrightarrow \pi(\Omega_p(X_n, *)),$$

$$\rho_{*p} : \pi_p(\lim(\underline{X}, *)) \longrightarrow \lim \pi_p((\underline{X}, *)).$$

Proof of lemma (a different proof is in [10, p.178]). Let $P = (P_n)$, $P_n = P_n(\underline{X}, *) = \prod_{k=0,1,\ldots} \Omega_n(X_k, *)$. We denote $\underline{\omega} = \prod_{k=0,1,\ldots} \omega_k$, $\omega_k \in$

$\Omega_n(X_k,*)$. We define a mapping $\widetilde{p} : P_n(\underline{X},*) \longrightarrow P_n(\underline{X},*)$, $\widetilde{p}(\underline{\omega}) = (\widetilde{p}(\underline{\omega})_k)_{k\geq 0}$, where $\widetilde{p}(\underline{\omega})_k = (p_k^{k+1}\omega_{k+1}) * (\omega_k)^{-1}$, $k \in N$. Then $\ker\widetilde{p} = \lim\limits_k \Omega_n(X_k,*) = \lim \Omega_n(\underline{X},*)$. Moreover, since each p_k^{k+1} is onto, so is $\widetilde{p}$. Therefore we have an exact sequence

$$* \longrightarrow \lim \Omega_n(\underline{X},*) \xrightarrow{i} P_n(\underline{X},*) \xrightarrow{\widetilde{p}} P_n(\underline{X},*) \longrightarrow *$$

where i is the inclusion and is therefore given by the restrictions of the projections $\rho_k : \lim \Omega_n(\underline{X},*) \longrightarrow \Omega_n(X_k,*)$. From the definition of P_n we have $\pi_n(P(\underline{X},*)) = \prod\limits_{k=0,1,\ldots} \pi_n(X_k,*)$. So the homotopy sequence for (4) has the form:

$$\ldots \longrightarrow \prod_N \pi_{n+1}(X_k,*) \xrightarrow{\widetilde{p}_*} \prod_N \pi_{n+1}(X_k,*) \longrightarrow \pi_n(\lim(\underline{X},*)) \longrightarrow$$

$$\xrightarrow{\rho_*} \prod_N \pi_n(X_k,*) \xrightarrow{\widetilde{p}_*} \prod_N \pi_n(X_k,*) \longrightarrow \ldots .$$

This implies:

$$0 \longrightarrow co\ker\widetilde{p}_* \longrightarrow \pi_n(\lim(\underline{X},*)) \xrightarrow{\rho_*} \ker\widetilde{p}_* \longrightarrow 0$$

with $co\ker\widetilde{p}_* \approx \lim^1 \pi_{n+1}(\underline{X},*)$ and $\ker\widetilde{p}_* \approx \lim \pi_n(\underline{X},*)$ (see also [7]).

Proof of the proposition 2. We will now apply Lemma 1 to the epimorphic tower

$K(\underline{X},*) = (K^r(\underline{X},*), \pi_r^{r+1})_{r\geq 0}$ with $K^r(\underline{X},*) = (K_n^r(\underline{X},*))_{n\geq 1}$

Clearly $\lim\limits_r K^r(\underline{X},*)$ can be identified with $K(\underline{X},*) = (K_n(\underline{X},*))$, ,

$K_n(\underline{X},*) = \prod\limits_{s=0}^{\infty} \prod\limits_{\Lambda^s} \{\underline{\omega} \mid \underline{\omega} = (\omega_{\overline{\lambda}})_{\overline{\lambda}\in\Lambda^s, s=0,1\ldots}$, where

$\omega_{\overline{\lambda}} : \Delta^s \times (I^n, \partial I^n) \longrightarrow (X_{\lambda_0},*)\}$ and therefore $\pi_n(\lim\limits_r K^r(\underline{X},*))$ coincides with the strong homotopy group $\pi_n^s(\underline{X},*)$. Moreover we have

$\pi_n(K^*(\underline{X},*)) = (\pi_n(K^r(\underline{X},*)), (\pi_{r-1}^r)_*) = {}^*\pi_n^s(\underline{X},*)$, with $\pi_n(K^r(\underline{X},*) = {}^r\pi_n^s(\underline{X},*)$.

Therefore, (5) from lemma 1 yields an exact sequence:

(6) $0 \longrightarrow \lim^1 {}^*\pi_{n+1}^s(\underline{X},*) \longrightarrow \pi_n^s(\underline{X},*) \xrightarrow{\rho_*} \lim {}^*\pi_n^s(\underline{X},*) \longrightarrow 0$

which is natural with respect to morphisms of inverse systems

$\underline{f} = (\varphi, f_\mu) : (\underline{X},*) \longrightarrow (\underline{Y},*)$. This is because $\underline{f}$ induces a level preserving homomorphism of towers: $\underline{f}_{-p}^r : K_p^r(\underline{X},*) \longrightarrow K_p^r(\underline{Y},*)$ for all

$r \geq 0$. In order to complete the proof note that $\breve{\pi}_p^r(\underline{X}, *) \subseteq \pi_p(K^r(\underline{X}, *)) = {}^r\pi_p^s(\underline{X}, *)$. Therefore the inclusions define homomorphisms: $\lim\limits_r \breve{\pi}_p^r(\underline{X}, *) \to \lim\limits_r({}^r\pi_p^s(\underline{X}, *))$ and $\lim\limits_r {}^1\breve{\pi}_p^r(\underline{X}, *) \longrightarrow \lim\limits_r {}^1({}^r\pi_p^s(\underline{X}, *)))$. Together with the identity $id : \pi_n^s(\underline{X}, *) \longrightarrow \pi_n^s(\underline{X}, *)$ they yield a homomorphism of sequences:

$$(7) \qquad \begin{array}{ccccccc} 0 \longrightarrow & \lim^1 \breve{\pi}_{p+1}^r(\underline{X}, *) & \longrightarrow \pi_p^s(\underline{X}, *) & \longrightarrow & \lim \breve{\pi}_p^r(\underline{X}, *) & \longrightarrow 0 \\ & \downarrow & \downarrow & & \downarrow & \\ 0 \longrightarrow & \lim^{1_*} \pi_{p+1}^s(\underline{X}, *) & \longrightarrow \pi_p^s(\underline{X}, *) & \longrightarrow & \lim^* \pi_p^s(\underline{X}, *) & \longrightarrow 0 \end{array}$$

Moreover it is easy to see that both of these homomorphisms are isomorphisms fot they satisfy M o r i t a condition [10, p.112].

$$(8) \qquad \begin{array}{ccc} \breve{\pi}_p^{r+1}(\underline{X}, *) & \overset{\breve{\pi}_r^{r+1}}{\longrightarrow} & \breve{\pi}_p^r(\underline{X}, *) \\ \downarrow & \nearrow & \downarrow \\ {}^{r+1}\pi_p^s(\underline{X}, *) & \overset{\pi_r^{r+1}}{\longrightarrow} & {}^r\pi_p^s(\underline{X}, *) \end{array}$$

All these facts implies that the exactness and naturality of the (6) determines the exactness and naturality of (S2).

Now we can state the main results:

Theorem 1. *Let* $(\underline{X}, *) = ((X_\lambda, *), p_\lambda^{\lambda'}, \Lambda)$, $(\underline{Y}, *) = ((Y_\mu, *), q_\mu^{\mu'}, M)$ *be inverse systems and* $\underline{f} = (\varphi, f_\mu)$ *a morphism in* $inv - Top_*$ *or in* $pro - Top_*$. *If* $\underline{f}$ *induces the isomorphisms :* $\lim^r \underline{f}_* : \lim^r \pi_p(\underline{X}, *) \longrightarrow \lim^r \pi_p(\underline{Y}, *)$ *for all* $r \geq 0$ *and all* $p \geq 2$ *(or for* $r = 0, 1$ *and* $p = 1$*) then* $\underline{f}$ *induces the corresponding isomorphisms of strong homotopy groups:* $\underline{f}_* : \pi_p^s(\underline{X}, *) \longrightarrow \pi_p^s(\underline{Y}, *)$.

Proof. Let $\underline{f} = (\varphi, f_\mu) : (\underline{X}, *) \longrightarrow (\underline{Y}, *)$ a morphism of inverse systems which induces isomorphisms $\lim^r \underline{f}_* : \lim^r \pi_p(\underline{X}, *) \longrightarrow \lim^r \pi_p(\underline{Y}, *)$ as in the hypothesis. We first show that $\underline{f}$ induces the isomorphisms $\breve{\underline{f}} : \breve{\pi}_p^r((\underline{X}, *)) \longrightarrow \breve{\pi}_p^r((\underline{Y}, *))$ for all $r \geq 0$. The proof is by induction on r. If $r < 0$, both groups vanish.

If $r = 0$, $\breve{\underline{f}}$ is an isomorphism, because it coincides with $\lim \underline{f}_* : \lim^0 \pi_p(\underline{X}, *) \longrightarrow \lim^0 \pi_p(\underline{Y}, *)$.

We now assume that the assertion holds for all $p \geq 2$ and $s < r$. Consider the commutative diagram whose rows are the homomorphisms induced by $\underline{f}$ as in Proposition 1. Then the assumption and the five lemma yield the desired conclusion that $\breve{\underline{f}} : \breve{\pi}_p^r(\underline{X}, *) \longrightarrow \pi_p^r(\underline{Y}, *)$ is an isomorphism.

Now we consider the commutative diagram whose rows are the sequences (S2) for $(\underline{X},*)$, $(\underline{Y},*)$ and the vertical arrows are the homomorphisms induced by f. Since $\underline{\breve{f}}_{*p} : \breve{\pi}_p^r(\underline{X},*) \longrightarrow \breve{\pi}_p^r(\underline{X},*)$ are isomorphisms, then $\lim \underline{\breve{f}} : \lim \breve{\pi}_p^r(\underline{X},*) \longrightarrow \lim \breve{\pi}_p^r((\underline{Y},*))$ and $\lim^1 \underline{\breve{f}} : \lim^1 \breve{\pi}_p^r((\underline{X},*)) \longrightarrow \lim^1 \breve{\pi}_p^r((\underline{Y},*))$ are also isomorphisms and the five lemma shows that $\underline{f}_* : \pi_p^s(\underline{X},*) \longrightarrow \pi_p^s(\underline{Y},*)$ is an isomorphism. This completes the proof of Theorem 1.

As consequences of these results we can prove without difficulty the following three corollaries.

Corolary 1. *If $\underline{f} = (\varphi, f_\mu)$ induces isomorphisms of homotopy pro-groups $\underline{f}_{*p} : \pi_p(\underline{X},*) \longrightarrow \pi_p(\underline{Y},*)$ for $p \geq 1$ then $\underline{f}$ induces isomorphisms of strong homotopy groups: $\underline{f}_* : \pi_p^s(\underline{X},*) \longrightarrow \pi_p^s(\underline{Y},*)$.*

Corollary 2. *For $f : (X,*) \longrightarrow (Y,*)$ a shape morphism such that $pro-\pi_p(X,*) \simeq pro-\pi_p(Y,*)$ for all $p \geq 1$, then the strong shape groups are isomorphic $\breve{\pi}_p^s(X,*) \approx \breve{\pi}_p^s(Y,*)$.*

Corollary 3. *Let $f : (X,*) \longrightarrow (Y,*)$ a shape morphism such that the shape groups and the r-stage shape groups of $(X,*)$ and $(Y,*)$ are isomorphic for $r \geq 1$. Then the strong shape groups are isomorphic.*

Proof. Let $\underline{p} : (X,*) \longrightarrow (\underline{X},*)$ and $\underline{q} : (Y,*) \longrightarrow (\underline{Y},*)$ resolutions of $(X,*)$ and $(Y,*)$, respectivelly, and the corresponding morphism in $pro-Top_*$, $\underline{f} = (\varphi, f_\mu) : (\underline{X},*) \longrightarrow (\underline{Y},*)$. From the hypotesis we have isomorphisms $\underline{\breve{f}}_{*p} : \breve{\pi}_p^r(\underline{X},*) \longrightarrow \breve{\pi}_p^r(\underline{Y},*)$ for all $r \geq 0$. Then we have, using Theorem 1, the isomorphisms of strong shape groups.

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REFERENCES

1. B o u s f i e l d, A.K. and Kan, D.M. – *Homotopy Limits. Completions and Localizations*, LNM 304, 1972.
2. E d w a r d s, D.A. and H a s t i n g s, H.M. – *Čech and Steenrod homotopy theory with applications to geometric topologies*, LNM 542, 1976.
3. G u e n t e r, B. – *Comparison of the coherent pro-homotopy theories of Edwards-Hastings, Lisica-Mardešić, Guenter*, Glasnik Mat. 26(46) (1991), 141–176.
4. J e n s e n, C.U. – *Les foncteur derivés de* $\varprojlim$ *et leurs applications en theorie des modules*, Springer Verlag, 1972.
5. L i s i c a, Ju.T. and M a r d e š i ć, S. – *Coherent prohomotopy and strong shape theory*, Glasnik Mat. 19(39), (1984), 335–399.
6. L i s i c a, Ju.T. and M a r d e š i ć, S. – *Strong homology of inverse systems of spaces*, Top. Appl. 19(1985) 29–43, 45–64.
7. M a r d e š i ć, S. and M i m i n o s h v i l i, Z. – *The relative homeomorphism and wedge axioms for strong homology*, Glasnik Mat. vol 25(45) 1990, 387–416.
8. M a r d e š i ć, S. – *Approximate polyhedra, resolutions of maps, shape fibrations*, Fund. Math. 114(1981) 53–78.
9. M a r d e š i ć, S. – *Coherent prohomotopy and strong shape for pairs*, Glasnik Mat. 20(1985), 419–434.
10. M a r d e š i ć, S. and S e g a l, J. – *Shape theory. The inverse system approach*, North Holland Publ 1983.
11. T h i e m a n n, H. – *Strong shape and fibrations*, Glasnik Mat. vol.30(50) (1995), 135–174.

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