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THE TANGENT BUNDLES and HARMONICITY[1]

BY

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Dedicated to Professor Gh.Gheorghiev on the occasion of his 90-th birthday.

1. Introduction. The problems studied in this paper are concerned with the harmonicity of the canonical projection $\pi : TM \to M$, where (M, g) is a Riemannian space and TM is its tangent bundle, and conversely, the harmonicity of the vector fields $\xi \in \chi(M)$ thought of as mappings from M to TM. These things have been studied when TM is endowed with the Riemannian Sasaki metric (see [4] and [8]) or when TM is endowed with a pseudo–Riemannian metric of complete lift type g^c (see [6] and [7]).

In this paper we compute the Levi–Civita connection ${}^G\widetilde{\nabla}$ and its curvature tensor field ${}^GK$ for the Cheeger–Gromoll metric G. Then we study the harmonicity of the natural projection $\pi : (TM, G) \to (M, g)$ and that of the vector fields $\xi : (M, g) \to (TM, G)$. We prove that if $\xi : (M, g) \to (TM, G)$ is an isometric immersion then ξ is harmonic.

Then we study the submanifold T_1M of the unit tangent vectors to M in (TM, S) and in (TM, G), where S is the Sasaki metric and the harmonicity of the projection $\widetilde{\pi} : (T_1M, G) \to (M, g)$, $\widetilde{\pi} : (T_1M, S) \to (M, g)$, where $\widetilde{\pi} = \pi_{/T_1M}$. For vector fields ξ with $||\xi|| = 1$ it has been established the relation between the harmonicity of $\xi : (M, g) \to (T_1M, G)$ and that of $\xi : (M, g) \to (T_1M, S)$.

In the last part of the paper, we study the harmonicity of the mapping $\xi : (M, g) \to (T_1M, \overline{\nabla})$, for $||\xi|| = 1$, where $\overline{\nabla}$ is the connection obtained

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from the Levi–Civita connection of g^c by projection on T_1M with respect to the Liouville vector field on TM, by using Gauss–Weingarten type formulas.

We study also the harmonicity of the metrics g^c, S and G, each with respect to the others.

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2. The tangent bundles and Riemannian metrics. Consider an n–dimensional, smooth, Riemannian manifold (M, g) and let $\pi : TM \to M$ be the tangent bundle of M. Then TM is a $2n$– dimensional manifold and on TM we may use some special local charts induced from local charts on M. If $(U, x^i);\ i = 1, \ldots, n$ is a local chart on M then the local chart $(\pi^{-1}(U), x^i, y^i)$ is induced on TM where $x^i = x^i \circ \pi$ and y^i are the vector space coordonates of $v \in \pi^{-1}(U) \subset TM$ with respect to the natural local frame $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ i.e. $v = y^i \left(\frac{\partial}{\partial x^i}\right)_{\pi(v)}$.

Denote by ∇ the Levi–Civita connection of g. The connection ∇ determines a horizontal distribution HTM on TM and the local vector fields:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^j \Gamma^h_{ij} \frac{\partial}{\partial y^h};\ i = 1, \ldots, n \tag{1}$$

define a local frame in HTM where Γ^k_{ij} are the Christoffel symbols. We have $TTM = HTM \oplus VTM$ where $VTM = Ker\ \pi_*$ is the vertical distribution on TM and the local vector fields $\frac{\partial}{\partial y^i};\ i = 1, \ldots, n$ define a local frame of VTM. Then we may define the horizontal lift X^H of a vector field $X = \xi^i \frac{\partial}{\partial x^i} \in \chi(U)$ by:

$$X^H = \xi^i \frac{\delta}{\delta x^i} \in \chi(\pi^{-1}(U))$$

and the vertical lift of X by:

$$X^V = \xi^i \frac{\partial}{\partial y^i} \in \chi(\pi^{-1}(U)).$$

It follows $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}, (\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$ (see [9]).

The system of local 1–forme $(dx^i, \delta y^i);\ i = 1, \ldots, n$ on TM defines the local dual frame of the local frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right);\ i = 1, \ldots, n$ on TM where:

$$\delta y^i = (dx^i)^H = dy^i + y^j \Gamma^i_{hj} dx^h. \tag{2}$$

We may consider several metrics on TM by using the above horizontal and vertical lifts. The most known is Sasaki metric, S, defined by:

$$S(X^H, Y^H) = g(X, Y), S(X^V, Y^V) = g(X, Y), S(X^H, Y^V) = 0. \tag{3}$$

S is given locally by:

$$S = g_{ij} dx^i dx^j + g_{ij} \delta y^i \delta y^j. \tag{4}$$

It follows that the matrix associated with S is: $\begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix}$.

The Sasaki metric is a Riemannian metric. The horizontal and vertical distributions are orthogonal with respect to S, and the fibres of horizontal and vertical distributions are canonically isometric with the tangent spaces on M in the corresponding points.

The Cheeger–Gromoll metric, G, is a Riemannian metric defined by:

$$\begin{cases} G(X^V, Y^V)_v = \frac{1}{1+||v||^2}\{g_p(X, Y) + g_p(X, v)g_p(Y, v)\} \\ G(X^H, Y^V) = 0,\ G(X^H, Y^H) = g(X, Y) \end{cases} \tag{5}$$

where $v \in TM, ||v||$ is the norm of $v \in T_pM$ with respect to $g_p, p = \pi(v)$ (see [5]). Locally, G is given by:

$$G = g_{ij} dx^i dx^j + \frac{1}{1 + g_{00}}\{g_{ij} + g_{i0}g_{j0}\}\delta y^i \delta y^j \tag{6}$$

where $g_{i0} = g_{ik}y^k$ and $g_{00} = g_{i0}y^i$. The matrix associated with G is

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{1}{1+g_{00}}(g_{ij} + g_{i0}g_{j0}) \end{pmatrix}$$

The horizontal and vertical distributions are orthogonal each other with respect to G and the fibres of the horizontal distribution are isometric with the tangent space on M in the corresponding points.

The complete lift type metric g^c is a pseudo–Riemannian metric on TM defined by:

$$g^c(X^H, Y^H) = 0,\ g^c(X^V, Y^V) = 0,\ g^c(X^V, Y^H) = g(X, Y) \tag{7}$$

g^c is written locally by:

$$g^c = 2g_{ij}\delta y^i dx^j; \tag{8}$$

its matrix being: $\begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$ and having the signature (n,n). We can see that the horizontal and vertical distributions are maximally isotropic and g^c defines a pairing between them.

A slight generalization of the complete lift consists in:

$$g^c(X^H, Y^H) = c(X, Y), g^c(X^V, Y^H) = g(X, Y), g^c(X^V, Y^V) = 0, \tag{9}$$

where c is a symmetric tensor field of type $(0, 2)$ on M. In this case, the horizontal distribution is no longer isotropic. Locally, if $c = c_{ij}dx^i dx^j$, then:

$$g^c = c_{ij}dx^i dx^j + 2g_{ij}\delta y^i dx^j. \tag{10}$$

The Levi–Civita connections ${}^S\widetilde{\nabla}$ and $\widetilde{\nabla}$ of S and g^c metrics respectively have been calculated so far. We also have calculated their curvature tensors ${}^sK$ and K respectively (see [8],[6]).

We may prove by a straightforward computation the following results:

Proposition 1. *The Levi–Civita connection* ${}^G\widetilde{\nabla}$ *of* G *is given locally by:*

$$\begin{cases} {}^G\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = \frac{1}{1+g_{00}}\{g_{ij}C - g_{i0}\frac{\partial}{\partial y^j} - g_{j0}\frac{\partial}{\partial y^i}\} + \frac{1}{(1+g_{00})^2}\{g_{ij} + g_{i0}g_{j0}\}C = \\ \qquad = A^l_{ij}\frac{\partial}{\partial y^l} \\ {}^G\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = -\frac{1}{2(1+g_{00})}R^h_{ji0}\frac{\delta}{\delta x^h} \\ {}^G\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = -\frac{1}{2(1+g_{00})}R^h_{ij0}\frac{\delta}{\delta x^h} + \Gamma^h_{ij}\frac{\partial}{\partial y^h}, \\ {}^G\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = -\frac{1}{2}R^h_{0ij}\frac{\partial}{\partial y^h} + \Gamma^h_{ij}\frac{\delta}{\delta x^h} \end{cases} \tag{11}$$

where $C = y^i\frac{\partial}{\partial y^i}$ *is Liouville vector field on* TM *and*

$$A^l_{ij} = \{\frac{1}{1+g_{00}}(2g_{ij}y^l - \delta^l_i g_{j0} - \delta^l_j g_{i0}) - \frac{1}{(1+g_{00})^2}(g_{ij}g_{00}y^l - g_{j0}g_{i0}y^l)\}$$

where R^h_{ijk} *are the local coordinate components of the curvature tensor field,* R *of* ∇.

From (10) it follows easily the global expression of ${}^G\widetilde{\nabla}$. By a straightforward computation we also get the following results:

Proposition 2. *The curvature tensor field ${}^{G}K$ of ${}^{G}\widetilde{\nabla}$ is locally given by:*

$$
\begin{aligned}
(12)\quad {}^{G}K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= \frac{1}{2}(\nabla_k R^l_{mij})y^m\frac{\partial}{\partial y^l}+\\
&+\left\{R^l_{kij}+\frac{1}{4(1+g_{00})}(R^h_{0jk}R^l_{ih0}-R^h_{0ik}R^l_{jh0}-2R^h_{0ij}R^l_{kh0})\right\}\frac{\delta}{\delta x^l}\\
{}^{G}K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= \frac{1}{2(1+g_{00})}(\nabla_j R^l_{ikm}-\nabla_i R^l_{jkm})y^m\frac{\delta}{\delta x^l}+\\
&+\left\{R^l_{kij}-\frac{1}{1+g_{00}}g_{k0}R^l_{0ij}+\frac{1}{4(1+g_{00})}(R^h_{jk0}R^l_{0ih}-R^h_{ik0}R^l_{0jh})\right\}\frac{\partial}{\partial y^l}+\\
&+\frac{1}{(1+g_{00})^2}R_{k0ij}(2+g_{00})C\\
{}^{G}K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} &= \frac{1}{1+g_{00}}\left\{R^l_{kij}+\frac{1}{1+g_{00}}\left[(g_{i0}R^l_{kj0}-g_{j0}R^l_{ki0})+\right.\right.\\
&\qquad\left.\left.+\frac{1}{4}(R^h_{kj0}R^l_{hi0}-R^h_{ki0}R^l_{hj0})\right]\right\}\frac{\delta}{\delta x^l}\\
{}^{G}K\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= -\frac{1}{2(1+g_{00})}R^h_{jki}\frac{\delta}{\delta x^h}+\\
&+\frac{1}{2(1+g_{00})^2}(g_{i0}R^h_{jk0}-g_{k0}R^h_{ji0})\frac{\delta}{\delta x^h}+\frac{1}{4(1+g_{00})^2}R^l_{jk0}R^h_{li0}\frac{\delta}{\delta x^h}\\
{}^{G}K\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= \frac{1}{2(1+g_{00})}(\nabla_j R^l_{kim})y^m\frac{\delta}{\delta x^l}+\\
&+\left\{-\frac{1}{2}R^l_{ijk}+\frac{1}{2(1+g_{00})}g_{i0}R^l_{0jk}-\frac{1}{4(1+g_{00})}R^h_{ki0}R^l_{0jh}\right\}\frac{\partial}{\partial y^l}-\\
&-\frac{1}{2(1+g_{00})^2}R_{i0jk}(2+g_{00})C\\
{}^{G}K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} &= \frac{3}{1+g_{00}}\left\{g_{jk}\frac{\partial}{\partial y^i}-g_{ik}\frac{\partial}{\partial y^j}\right\}+\\
&+\frac{1}{(1+g_{00})^2}\left\{5(g_{ik}g_{j0}-g_{jk}g_{i0})C+2\left(g_{ik}\frac{\partial}{\partial y^j}-g_{jk}\frac{\partial}{\partial y^i}\right)g_{00}\right\}+\\
&+\frac{1}{(1+g_{00})^3}\left\{2(g_{i0}g_{jk}-g_{j0}g_{ik})(1+2g_{00})C+\right.\\
&\left.+\left(g_{ik}\frac{\partial}{\partial y^j}-g_{jk}\frac{\partial}{\partial y^i}\right)g_{00}+g_{k0}g_{00}\left(g_{i0}\frac{\partial}{\partial y^j}-g_{j0}\frac{\partial}{\partial y^i}\right)\right\}
\end{aligned}
$$

From (12) we can easily get the global expression of ${}^{G}K$.

Remark. ${}^{G}\widetilde{\nabla}$ is flat if and only if ∇ is flat.

From (12) we have:

$$ {}^{G}K_v(X^H,Y^H)C=\frac{1}{1+||v||^2}\{R(X,Y)v\}^V, {}^{G}K_v(C,X^V)Y^H=$$

$$=\frac{1}{(1+||v||^2)^2}\{R(v,X)Y\}^H$$

$$ {}^{G}K_v(X^H,Y^H)C=-\frac{1}{2(1+||v||^2)^2}\{R(v,X)Y\}^H, {}^{G}K(C,X^H)C=0,$$

$$ {}^{G}K(C,S)C=0$$

$$ {}^{G}K(C,S)S=0, {}^{G}K_v(C,X^H)Y^H=-\frac{1}{2(1+||v||^2}\{R(X,Y)v\}^V$$

where $S=y^i\frac{\delta}{\delta x^i}$ is the geodesic spray defined by the connection ∇.

Let M and N be two Riemannian manifolds and let $f:M\to N$ be a smooth map. The local expression of the second fundamental form of f denoted by $\beta(f)$ is

$$\beta(f)(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})=\{f^\alpha_{ij}-{}^{M}\Gamma^k_{ij}f^\alpha_k+{}^{N}\Gamma^\alpha_{\beta\gamma}f^\beta_i f^\gamma_j\}\frac{\partial}{\partial u^\alpha} \tag{13}$$

and that of the tension field $\tau(f)$ of f is:

$$\tau(f)=g^{ij}\{f^\alpha_{ij}-{}^{M}\Gamma^k_{ij}f^\alpha_k+{}^{N}\Gamma^\alpha_{\beta\gamma}f^\beta_i f^\gamma_j\}\frac{\partial}{\partial u^\alpha}\quad \text{(see [2])}. \tag{14}$$

From (13) and (14) we remark that $\beta(f)$ and $\tau(f)$ may also be defined when N is only endowed with a linear connection without torsion or when M is only a pseudo–Riemannian manifold.

Thus is has sense to study the harmonicity of the vector fields regarded as maps $\xi:M\to TM$, $\xi:M\to T_1M$, when we no longer consider on TM,T_1M, the Levi–Civita connection of a Riemannian (pseudo–Riemannian) metric but another connection, without torsion, obtained from the Levi–Civita connection by various methods.

It also will have sense the study of the harmonicity of the canonical projection $\pi:TM\to M$, when we have a pseudo– Riemannian metric on TM.

Now, we remark that the canonical projection $\pi : (TM, G) \to (M, g)$ is a Riemannian submersion. The we get by a straightforward computation:

$$(15) \qquad \begin{cases} \beta(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \beta(\pi)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0 \\ \beta(\pi)_v\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \frac{1}{2(1+||v||^2)} y^l R^h_{jil}(\pi(v)) \frac{\partial}{\partial x^h} \end{cases}$$

From (15) and using results from the theory of the Riemannian submersions (see [3]) we obtain:

Theorem 1. *The Riemannian submersion $\pi : (TM, G) \to (M, g)$ has the following properties:*

a) π has its fibres totally geodesic, hence minimal

b) π is harmonic

c) the following relations are equivalent:

– π is totally geodesic

– the horizontal distribution is integrable

– ∇ is flat.

Let c be a Riemannian metric on M. We consider $\pi : (TM, G) \to (M, c)$. We have

$$(16) \qquad \begin{cases} \beta(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \beta(\pi)_v\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \frac{1}{2(1+||v||^2)} y^l R^h_{jil}(\pi(v)) \frac{\partial}{\partial x^h} \\ \beta(\pi)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \{{}^c\Gamma^h_{ij} - \Gamma^h_{ij}\} \frac{\partial}{\partial x^h} \end{cases}$$

where ${}^c\Gamma^h_{ij}$ are the Christoffel symbols of the metric c. So we obtain:

Proposition 3. *$\pi : (TM, G) \to (M, c)$ is totally geodesic if and only if (M, g) is flat and $1 : (M, g) \to (M, c)$ is totally geodesic.*

We remark that if $\pi : (TM, G) \to (M, c)$ is totally geodesic then (M, g) and (M, c) are flat.

Let g and c be two Riemannian metrics on M. We say that c is harmonic with respect to g if

$$(17) \qquad g^{ij}\{{}^c\Gamma^h_{ij} - \Gamma^h_{ij}\} = 0 \quad \text{(see [1])}.$$

We observe that this definition can also be extended for the pseudo–Riemannian case.

From (16) it follows that:

Proposition 4. *$\pi : (TM, G) \to (M, c)$ is harmonic if and only if c is harmonic with respect to g.*

Consider the tensor fields T and A on TM given by:

$$T(\widetilde{X},\widetilde{Y}) = H^G\widetilde{\nabla}_{V\widetilde{X}}V\widetilde{Y} + V^G\widetilde{\nabla}_{V\widetilde{X}}H\widetilde{Y},\ A(\widetilde{X},\widetilde{Y}) = H^G\widetilde{\nabla}_{H\widetilde{X}}V\widetilde{Y} + \tag{18}$$
$$+V^G\widetilde{\nabla}_{H\widetilde{X}}H\widetilde{Y}$$

familiar in the theory of Riemannian submersions. Remark that T is called O'Neill's tensor. Since $\pi : (TM, G) \to (M, g)$ is a Riemannian submersion with its fibres totally geodesic, or by straightforward computation, we obtain that $T = 0$. For A we obtain the following expression:

$$\begin{cases} A\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = A\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0 \\ A\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = -\frac{1}{2}R^h_{0ij}\frac{\partial}{\partial y^h},\ A\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = -\frac{1}{2(1+g_{00})}R^h_{ij0}\frac{\delta}{\delta x^h} \end{cases} \tag{19}$$

So we get:

Proposition 5. *The following relations are equivalent:*

a) $A = 0$

b) *the horizontal distribution is integrable*

c) $A(X^H, Y^H) = 0,\ \forall X, Y \in \chi(M)$

d) $A(X^H, Y^V) = 0,\ \forall X, Y \in \chi(M)$

Remark. From theorem 1 and proposition 5 it results that $\pi : (TM, G) \to (M, g)$ is totally geodesic if and only if $A = 0$.

The Schouten connection ${}^G\nabla$ associated with ${}^G\widetilde{\nabla}$ is defined by:

$${}^G\nabla_{\widetilde{X}}\widetilde{Y} = V^G\widetilde{\nabla}_{\widetilde{X}}V\widetilde{Y} + H^G\widetilde{\nabla}_{\widetilde{X}}H\widetilde{Y},\ \forall\widetilde{X},\widetilde{Y} \in \chi(TM) \tag{20}$$

Locally ${}^G\nabla$ is given by:

$$\begin{cases} {}^G\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = {}^G\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j},\ {}^G\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = {}^G\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} \\ {}^G\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \Gamma^h_{ij}\frac{\partial}{\partial y^h},\ {}^G\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \Gamma^h_{ij}\frac{\delta}{\delta x^h} \end{cases} \tag{21}$$

The mean connection of the connection ${}^G\nabla$ is ${}^G\overset{m}{\nabla} = {}^G\nabla - \frac{1}{2}T$ where T is the torsion tensor field of ${}^G\nabla$. In local coordinate ${}^G\overset{m}{\nabla}$ has the expression:

$$\begin{cases} {}^G\overset{m}{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = {}^G\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j},\ {}^G\overset{m}{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = \frac{1}{2}{}^G\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} \\ {}^G\overset{m}{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = {}^G\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} + \frac{1}{4(1+g_{00})}R^h_{ij0}\frac{\delta}{\delta x^h},\ {}^G\overset{m}{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = {}^G\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} \end{cases} \tag{22}$$

Obviously ${}^{G}\overset{m}{\nabla}$ is without torsion. We remark that ${}^{G}\overset{m}{\nabla} = {}^{G}\widetilde{\nabla}$ if and only if $R = 0$.

Let $1 : (TM, G) \to (TM, {}^{G}\overset{m}{\nabla})$ be the identity map. We obtain:

$$\beta(1)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \beta(1)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \beta(1)\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \tag{23}$$

$$= \frac{1}{4(1+g_{00})} R^h_{ji0} \frac{\delta}{\delta x^h}$$

hence $\tau(1) = \text{trace}\, \beta(1) = 0$. Obviously, if $R \neq 0$ then $\beta(1) \neq 0$.

Now, let ξ be a vector field. We consider ξ as a smooth map from M to TM, $\xi : (M, g) \to (TM, G)$.

Proposition 6. *$\xi : (M, g) \to (TM, G)$ is an isometric immersion if and only if ξ is parallel i.e. $\nabla\xi = 0$.*

Proof. We obtain immediatly:

$$\xi_{*,p} X = (X^H + (\nabla_X \xi)^V)_{\xi(p)}, \ \forall p \in M \tag{24}$$

If we consider

$$\overset{1}{g}_p(X, Y) = G_{\xi(p)}(\xi_{*,p}X, \xi_{*,p}Y) = g_p(X, Y) + \frac{1}{1+||\xi(p)||^2}\{g_p(\nabla_X\xi, \nabla_Y\xi) +$$

$$+ g_p(\nabla_X\xi, \xi) g_p(\nabla_Y\xi, \xi)\}$$

then ξ is isometric immersion if and only if $\overset{1}{g} = g$. We remark that $\overset{1}{g} = g$ if and only if $\nabla_X\xi = 0$ for all $X \in \chi(M)$ i.e. $\nabla\xi = 0$.

Considering the relations (24) and (11) we obtain by a straightforward computation:

Proposition 7. *The map $\xi : (M, g) \to (TM, G)$ has $\beta(\xi)$ and $\tau(\xi)$ given by:*

(25)

$$\beta(\xi)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \{-\frac{1}{2} R^h_{mij}\xi^m + \nabla_i\nabla_j\xi^h + (\nabla_i\xi^l)(\nabla_j\xi^k)A^h_{lk}\}\frac{\partial}{\partial y^h} -$$

$$- \frac{1}{2(1+||\xi||^2)}\{(\nabla_j\xi^k)R^h_{ikm}\xi^m + (\nabla_i\xi^l)R^h_{jlm}\xi^m\}\frac{\delta}{\delta x^h}$$

$$\tau(\xi) = \{g^{ij}(\nabla_i\nabla_j\xi^h) + g^{ij}(\nabla_i\xi^l)(\nabla_j\xi^k)A^h_{lk}\}\frac{\partial}{\partial y^h} - \tag{26}$$

$$- \frac{1}{1+||\xi||^2}\{g^{ij}(\nabla_j\xi^k)R^h_{ikm}\xi^m\}\frac{\delta}{\delta x^h}$$

In a similar way, from the relations (24) and (22) we obtain:

Proposition 8. *The map $\xi : (M, g) \to (TM, {}^{G}\overset{m}{\nabla})$ has the second fundamental form, $\overset{m}{\beta}(\xi)$, and the tension field $\overset{m}{\tau}(\xi)$ given by:*

(27)
$$\overset{m}{\beta}(\xi)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \{-\frac{1}{2}R^h_{mij}\xi^m + \nabla_i\nabla_j\xi^h + (\nabla_i\xi^l)(\nabla_j\xi^k)A^h_{lk}\}\frac{\partial}{\partial y^h} - \\ - \frac{1}{4(1+||\xi||^2)}\{(\nabla_j\xi^k)R^h_{ikm}\xi^m + (\nabla_i\xi^l)R^h_{jlm}\xi^m\}\frac{\delta}{\delta x^h}$$

(28)
$$\overset{m}{\tau}(\xi) = \{g^{ij}(\nabla_i\nabla_j\xi^h) + g^{ij}(\nabla_i\xi^l)(\nabla_j\xi^k)A^h_{lk}\}\frac{\partial}{\partial y^h} - \\ -\frac{1}{2(1+||\xi||^2)}\{g^{ij}(\nabla_j\xi^k)R^h_{ikm}\xi^m\}\frac{\delta}{\delta x^h}$$

Comparing the relations (25) and (26) to (27) and (28), we obtain:

Theorem 2. *$\beta(\xi) = 0$ if and only if $\overset{m}{\beta}(\xi) = 0$ and $\tau(\xi) = 0$ if and only if $\overset{m}{\tau}(\xi) = 0$.*

So, the use of the connection ${}^{G}\overset{m}{\nabla}$ has not led us to new results.

From (26) it follows that:

Theorem 3. *$\xi : (M, g) \to (TM, G)$ is harmonic if and only if it verifies the following system:*

(29)
$$g^{ij}(\nabla_i\nabla_j\xi^h) + g^{ij}(\nabla_i\xi^l)(\nabla_j\xi^k)A^h_{lk} = 0,\ g^{ij}(\nabla_j\xi^k)R^h_{ikm}\xi^m = 0$$

From the proposition 6 and (25) it follows that:

Theorem 4. *If $\xi : (M, g) \to (TM, G)$ is isometric immersion then ξ is totally geodesic, hence ξ is harmonic.*

We note that ξ is parallel if and only if ξ is an isometric minimal immersion.

Now we shall study the relations (25) and (26) become in some particular cases:

1) Suppose that $||\xi|| = 1$. Then $\beta(\xi)$ and $\tau(\xi)$ have the following expressions:

$$\beta(\xi)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \{-\frac{1}{2}R^h_{mij}\xi^m + \nabla_i\nabla j\xi^h + \frac{3}{4}(\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\xi^h\}\frac{\partial}{\partial y^h} - \\ - \frac{1}{4}\{(\nabla_j\xi^k)R^h_{ikm}\xi^m + (\nabla_i\xi^l)R^h_{jlm}\xi^m\}\frac{\delta}{\delta x^h}$$

$$\tau(\xi) = \{g^{ij}(\nabla_i\nabla_j\xi^h) + \frac{3}{4}g^{ij}(\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\xi^h\}\frac{\partial}{\partial y^h} -$$

$$-\frac{1}{2}\{g^{ij}(\nabla_j\xi^k)R^h_{ikm}\xi^m\}\frac{\delta}{\delta x^h}$$

2) Suppose that $||\xi|| = 1$ and (M, g) has its sectional curvature constant a. Then:

$$\beta(\xi)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \{-\frac{a}{2}(\xi_j\delta^h_i - \xi_i\delta^h_j) + \nabla_i\nabla_j\xi^h + \frac{3}{4}(\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\xi^h\}\frac{\partial}{\partial y^h} -$$
$$-\frac{a}{4}\{(\nabla_j\xi^h)\xi_i - \xi^h(\nabla_j\xi_i) + (\nabla_i\xi^h)\xi_j - \xi^h(\nabla_i\xi_j)\}\frac{\delta}{\delta x^h}$$

$$\tau(\xi) = \{g^{ij}(\nabla_i\nabla_j\xi^h) + \frac{3}{4}g^{ij}(\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\xi^h\}\frac{\partial}{\partial y^h} -$$

$$-\frac{a}{2}\{g^{ij}[(\nabla_j\xi^h)\xi_i - \xi^h(\nabla_j\xi_i)]\}\frac{\delta}{\delta x^h}$$

3) Suppose that $||\xi|| = 1$, ξ–Killing and (M, g) has sectional curvature constant a. Then:

$$\beta(\xi)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{a}{2}\{\xi_j\delta^h_i + \xi_i\delta^h_j - \frac{1}{2}g_{ij}\xi^h - \frac{3}{2}(\xi_i\xi_j)\xi^h\}\frac{\partial}{\partial y^h} -$$
$$-\frac{a}{4}\{(\nabla_j\xi^h)\xi_i + (\nabla_i\xi^h)\xi_j\}\frac{\delta}{\delta x^h}$$

$$\tau(\xi) = \frac{a}{4}(1-n)\xi^h\frac{\partial}{\partial y^h}$$

So if $a = 0$ then ξ is totally geodesic, and if $a \neq 0$ then ξ is not harmonic.
4) Suppose that $||\xi|| = 1$, $\Delta\xi = 0$ i.e. $g^{ij}(\nabla_i\nabla_j\xi^h) = R^h_l\xi^l$ where R_{jk} is the local coordinate expression of the Ricci tensor field $\rho(R)$, $R^i_k = R_{jk}g^{ij}$ and (M, g) has sectional curvature constant a. Then:

$$\tau(\xi) = -\frac{a}{4}(1-n)\xi^h\frac{\partial}{\partial y^h} - \frac{a}{2}\{g^{ij}[(\nabla_j\xi^h)\xi_i - \xi^h(\nabla_j\xi_i)]\}\frac{\delta}{\delta x^h}$$

So ξ is harmonic if and only if $a = 0$.
5) Suppose that $||\xi|| = 1$. Then $\tau(\xi)$ is colinear with $C = y^i\frac{\partial}{\partial y^i}$ if and only if:

$$g^{ij}(\nabla_i\nabla_j\xi^h) = f\xi^h,\ g^{ij}(\nabla_j\xi^k)\xi^m R^h_{ikm} = 0$$

where $f \in \mathcal{F}(M)$.
6) Suppose that $||\xi|| = 1$, (M, g) has its sectional curvature constant a, $\Delta\xi = 0$ and we also suppose that

$$g^{ij}[(\nabla_j \xi^h)\xi_i - \xi^h(\nabla_j \xi_i)] = 0.$$

Then $\tau(\xi)$ is colinear with C.

Now we return to Sasaki metric S. It is known that $\pi : (TM, S) \to (M, g)$ is a Riemannian harmonic submersion with totally geodesic fibres.

Let c be a Riemannian metric on M. One can prove easily:

Proposition 9. *The metric c is harmonic with respect to g if and only if $\pi : (TM, S) \to (M, c)$ is harmonic.*

The mean connection ${}^S\overset{m}{\nabla}$ of the Schouten connection associated with ${}^S\widetilde{\nabla}$ has the local coordinate expression:

$$(30) \quad \begin{cases} {}^S\overset{m}{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \ {}^S\overset{m}{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = -\frac{1}{4} R^h_{ji0} \frac{\delta}{\delta x^h} \\ {}^S\overset{m}{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij} \frac{\partial}{\partial y^h} - \frac{1}{4} R^h_{ij0} \frac{\delta}{\delta x^h}, \ {}^S\overset{m}{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} - \frac{1}{2} R^h_{0ij} \frac{\partial}{\partial y^h} \end{cases}$$

We remark that ${}^S\overset{m}{\nabla} = {}^S\widetilde{\nabla}$ if and only if $R = 0$.

Let $1 : (TM, S) \to (TM, {}^S\overset{m}{\nabla})$ be the identity map. We obtain by a straightforward computation $\tau(1) = \text{trace}\, \beta(1) = 0$; evidently, if $R \neq 0$ then $\beta(1) \neq 0$.

Now let ξ be a vector field. We consider ξ as a map from M to TM.

Proposition 10. *a) $\xi : (M, g) \to (TM, S)$ has its tension field $\tau(\xi)$ given locally by:*

$$(31) \qquad \tau(\xi) = \{g^{ij}(\nabla_i \nabla_j \xi^h)\} \frac{\partial}{\partial y^h} - \{(\nabla_j \xi^l) R^h_{ilm} \xi^m g^{ij}\} \frac{\delta}{\delta x^h}$$

b) $\xi : (M, g) \to (TM, {}^S\overset{m}{\nabla})$ has the tension field ${}^m\tau(\xi)$ given locally by:

$$(32) \qquad {}^m\tau(\xi) = \{g^{ij}(\nabla_i \nabla_j \xi^h)\} \frac{\partial}{\partial y^h} - \frac{1}{2}\{(\nabla_j \xi^l) R^h_{ilm} \xi^m g^{ij}\} \frac{\delta}{\delta x^h}.$$

From (31) and (32) it follows:

Theorem 5. *For $\xi \in \chi(M)$ we have $\tau(\xi) = 0$ if and only if $\overset{m}{\tau}(\xi) = 0$.*

Remarks.

1) From (30) it follows that $\xi : (M, g) \to (TM, S)$ is harmonic if and only if ξ verifies the equations system:

$$g^{ij}(\nabla_i \nabla_j \xi^h) = 0,\ g^{ij}(\nabla_j \xi^k)\xi^m R^h_{ikm} = 0$$

2) If ξ is a Killing vector field or a geodesic vector field or $\Delta\xi = 0$ and (M, g) is flat, then $\tau(\xi) = 0$, i.e. ξ is harmonic.

3) If ξ is a Killing vector field and (M, g) has the sectional curvature constant $a,\ a \neq 0$, then:

$$\tau(\xi) = a(1-n)\xi^h \frac{\partial}{\partial y^h} - a\{g^{ij}(\nabla_j \xi^h)\xi_i\}\frac{\delta}{\delta x^h}$$

Hence ξ is not harmonic.

4) If $\Delta\xi = 0$ and (M, g) has the sectional curvature constant $a,\ a \neq 0$, then:

$$\tau(\xi) = -a(1-n)\xi^h \frac{\partial}{\partial y^h} - a\{g^{ij}[(\nabla_j \xi^h)\xi_i - \xi^h(\nabla_j \xi_i)]\}\frac{\delta}{\delta x^h}.$$

Hence ξ is not harmonic.

5) If M is compact and orientable, then ξ is harmonic if and only if $\nabla\xi = 0$.

3. The submanifold T_1M of TM. Let $T_1M = \{v \in TM; ||v|| = 1\}$ be the $2n-1$ dimensional submanifold of TM. The local vector fields $\{Y_i, \frac{\delta}{\delta x^i}\}_{i=1}^n$ generate locally TT_1M, where

$$Y_i = \frac{\partial}{\partial y^i} - g_{i0}C. \tag{33}$$

We consider T_1M as a submanifold of (TM, G). We verify easily that $C = y^i \frac{\partial}{\partial y^i}$ is orthogonal on T_1M with respect to G. We get by a straightforward computation:

Proposition 11. *We have the following relations:*

$$\begin{cases} B\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = B\left(\frac{\delta}{\delta x^i}, Y_j\right) = 0,\ B(Y_i, Y_j) = \frac{1}{4}(g_{i0}g_{j0} - g_{ij})C \\ H_C\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = H_C\left(\frac{\delta}{\delta x^i}, Y_j\right) = 0,\ H_C(Y_i, Y_j) = \frac{1}{4}(g_{i0}g_{j0} - g_{ij}), \\ S_C\left(\frac{\delta}{\delta x^i}\right) = 0,\ S_C(Y_i) = -\frac{1}{2}Y_i \end{cases} \tag{34}$$

where B is the second fundamental form of submanifold T_1M of the manifold (TM, G) and:

$$H_C(\overline{X}, \overline{Y}) = G(B(\overline{X}, \overline{Y}), C),\ \forall \overline{X}, \overline{Y} \in \chi(T_1M)$$
$$S_C(\overline{X}) = -({}^G\widetilde{\nabla}_{\overline{X}} C)^T$$

We remark that T_1M is not totally geodesic in (TM, G), it is not minimal and it is not totally umbilical. Since T_1M is not minimal it follows that $i : (T_1M, G_{/T_1M}) \to (TM, G)$ is not harmonic.

Proposition 12. *The Levi–Civita connection ${}^G\overline{\nabla}$ of the Riemannian metric $G_{/T_1M}$ is given locally by:*

$$(35)\quad \begin{cases} {}^G\overline{\nabla}_{\frac{\delta}{\delta x^i}} Y_j = -\frac{1}{4} R^h_{ij0} \frac{\delta}{\delta x^h} + \Gamma^h_{ij} Y_h, \ {}^G\overline{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = -\frac{1}{2} R^h_{0ij} Y_h + \Gamma^h_{ij} \frac{\delta}{\delta x^h} \\ {}^G\overline{\nabla}_{Y_i} \frac{\delta}{\delta x^j} = -\frac{1}{4} R^h_{ji0} \frac{\delta}{\delta x^h}, \ {}^G\overline{\nabla}_{Y_i} Y_j = -g_{j0} Y_i \end{cases}$$

We consider now T_1M as a submanifold of (TM, S); in the same way we obtain that C is orthogonal on T_1M with respect to S and:

Proposition 13. *We have:*

$$(36)\quad \begin{cases} B\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = B(\frac{\delta}{\delta x^i}, Y_j) = 0, \ B(Y_i, Y_j) = (g_{i0} g_{j0} - g_{ij}) C, \\ H_C\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = H_C(\frac{\delta}{\delta x^i}, Y_j) = 0, \ H_C(Y_i, Y_j) = g_{i0} g_{j0} - g_{ij}, \\ S_C\left(\frac{\delta}{\delta x^i}\right) = 0, \ S_C(Y_i) = -Y_i \end{cases}$$

Hence we get that T_1M is not totally geodesic, is not minimal and is not totally umbilical in (TM, S); $i : (T_1M, S_{/T_1M}) \to (TM, S)$ is not harmonic.

Proposition 14. *The Levi–Civita connection ${}^S\overline{\nabla}$ of the Riemannian metric $S_{/T_1M}$ is given locally by:*

$$(37)\quad \begin{cases} {}^S\overline{\nabla}_{\frac{\delta}{\delta x^i}} Y_j = -\frac{1}{2} R^h_{ij0} \frac{\delta}{\delta x^h} + \Gamma^h_{ij} Y_h, \ {}^S\overline{\nabla}_{\frac{\delta}{\delta x_i}} \frac{\delta}{\delta x^j} = -\frac{1}{2} R^h_{0ij} Y_h + \Gamma^h_{ij} \frac{\delta}{\delta x^h} \\ {}^S\overline{\nabla}_{Y_i} \frac{\delta}{\delta x^j} = -\frac{1}{2} R^h_{ji0} \frac{\delta}{\delta x^h}, \ {}^S\overline{\nabla}_{Y_i} Y_j = -g_{j0} Y_i \end{cases}$$

Using the Gauss–Weingarten relations we project ${}^S\overset{m}{\nabla}$ on T_1M with respect to the vector field C and we get ${}^S\overset{m}{\overline{\nabla}} = {}^G\overline{\nabla}$.

Proposition 15. *a) T_1M is a hypersurface of the (TM, S) having the mean curvature constant:*

$$(38)\qquad H_S^{T_1M} = -\frac{n-1}{2n-1} C$$

b) T_1M is a hypersurface of the (TM, G) having the mean curvature constant:

$$(39)\qquad H_G^{T_1M} = -\frac{n-1}{2(2n-1)} C$$

Hence it follows that $i : (T_1M, G_{/T_1M}) \to (TM, G)$ is neither minimal nor pseudo-umbilical. Similarly for $i : (T_1M, S_{/T_1M}) \to (TM, S)$.

Now studying the harmonicity of the metrics G and S each with respect to other, from the expressions of ${}^G\widetilde{\nabla}$ and ${}^S\widetilde{\nabla}$, we obtain:

Proposition 16. *Let $1_{TM} : (TM, G) \to (TM, S)$ be the identity map. Then:*

a) 1_{TM} cannot be totally geodesic

b) 1_{TM} has the tension field $\tau_G(1_{TM})$ given by:

$$\tau_G(1_{TM}) = -\frac{(n-1)(g_{00}+2)}{g_{00}+1}C, \tag{40}$$

so $\tau_G(1_{TM})$ is orthogonal on T_1M and S cannot be harmonic with respect to G.

Proposition 17. *Let $1_{T_1M} : (T_1M, G) \to (T_1M, S)$ be the identity map. We have:*

a) 1_{T_1M} is totally geodesic if and only if $R = 0$

b) 1_{T_1M} is harmonic i.e. $S_{/T_1M}$ is harmonic with respect to $G_{/T_1M}$.

Proposition 18. *Let $1_{TM} : (TM, S) \to (TM, G)$ be the identity map. Then 1_{TM} has the tension field $\tau_G(1_{TM})$ given by:*

$$\tau_S(1_{TM}) = \frac{(n-1)(g_{00}+2)}{(g_{00}+1)^2}C \tag{41}$$

Proposition 19. *Let $1_{T_1M} : (T_1M, S) \to (T_1M, G)$ be the identity map. Then 1_{T_1M} is harmonic, so $G_{/T_1M}$ is harmonic with respect to $S_{/T_1M}$.*

Let $\widetilde{\pi}$ be defined by : $\widetilde{\pi} = \pi_{/T_1M}$. We observ that $\widetilde{\pi} : (T_1M, G) \to (M, g)$ is a Riemannian submersion. From the relations:

$$\widetilde{\pi}_*(Y_i) = 0,\ \widetilde{\pi}_*\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial x^i}, \tag{42}$$

and relations (35), we obtain:

Theorem 6. *a) $\widetilde{\pi} : (T_1M, G) \to (M, g)$ is totally geodesic if and only if $R = 0$*

b) $\widetilde{\pi} : (T_1M, G) \to (M, g)$ is harmonic.

In the same way, for $\widetilde{\pi} : (T_1M, S) \to (M, g)$ we get:

Theorem 7. *a) $\widetilde{\pi} : (T_1M, S) \to (M, g)$ is totally geodesic if and only if $R = 0$*

b) $\widetilde{\pi} : (T_1M, S) \to (M, g)$ is harmonic.

We remark that T_1M is orientable and if M is compact then T_1M is compact. So, if M is compact, $\widetilde{\pi}$ is harmonic with the meaning of its basic definition (see [2]).

Let ξ be a vector field with $||\xi|| = 1$. Denote by:

$$\xi_G : (M,g) \to (TM,G), \xi_G(p) = \xi(p)$$
$$\overline{\xi}_G : (M,g) \to (T_1M,G), \overline{\xi}_G(p) = \xi(p), \forall p \in M;$$
$$\xi_S : (M,g) \to (TM,S), \xi_S(p) = \xi(p)$$
$$\overline{\xi}_S : (M,g) \to (T_1M,S), \overline{\xi}_S(p) = \xi(p), \forall p \in M.$$

We have:

$$\overline{\xi}_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\delta}{\delta x^i} + (\nabla_i \xi^j) Y_j \tag{43}$$

From the relations (43), (35) and (37) we obtain:

$$\begin{cases} \tau(\overline{\xi}_G) = \{g^{ij}(\nabla_i \nabla_j \xi^h)\} Y_h - \frac{1}{2}\{(\nabla_j \xi^l) R^h_{ilm} \xi^m g^{ij}\} \frac{\delta}{\delta x^h}, \\ \tau(\overline{\xi}_G) = \tau(\xi_G) - g^{ij} B(\frac{\delta}{\delta x^i} + (\nabla_i \xi^l) Y_l, \frac{\delta}{\delta x^j} + (\nabla_j \xi^l) Y_l), \\ \tau(\overline{\xi}_S) = \{g^{ij}(\nabla_i \nabla_j \xi^h)\} Y_h - \{(\nabla_j \xi^l) R^h_{ilm} \xi^m g^{ij}\} \frac{\delta}{\delta x^h}. \end{cases} \tag{44}$$

Proposition 20. *$\tau(\overline{\xi}_G) = 0$ if and only if $\tau(\xi_G)$ is colinear with C.*

The conditions under which $\tau(\xi_G)$ is colinear with C have previously been studied.

Theorem 8. *Let ξ be a vector field with $||\xi|| = 1$. Then:*

a) $\tau(\overline{\xi}_G) = 0$ if and only if $\tau(\overline{\xi}_S) = 0$,

b) $\tau(\xi_G) = 0$ if and only if $\tau(\xi_S) = 0$.

Proof. a) It results from the relations (43).

b) It results from the relations (26), (31) and from the relations $\xi^h \xi_h = 1$, $(\nabla_i \nabla_j \xi^h)\xi_h = -(\nabla_j \xi^t) g_{tr} (\nabla_i \xi^r)$.

Remark. 1) If M is compact and orientable, $||\xi|| = 1$, then $\tau(\xi_G) = 0$ if and only if $\nabla \xi = 0$.

2) $\tau(\overline{\xi}_S) = 0$ if and only if $\tau(\xi_S) = 0$ is equivalent with $det(\delta^m_h - \xi_h \xi^m) \neq 0$. But $det(\delta^m_h - \xi_h \xi^m) = 0$ because $(\delta^m_h - \xi_h \xi^m)\xi_m = 0$ and $\xi \neq 0$. So it has sense to consider the case $||\xi|| = 1$.

4. The tangent bundles and the pseudo–Riemannian metric g^c. We consider the pseudo–Riemannian metric g^c on TM. Its Levi–Civita connection, $\widetilde{\nabla}$, has been calculated (see [5]). We consider the canonical projection $\pi : (TM, g^c) \to (M, g)$. We have:

$$\beta(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \beta(\pi)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \beta(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0 \tag{45}$$

Hence

Theorem 9. $\pi : (TM, g^c) \to (M, g)$ *has the properties:*
a) $\beta(\pi) = 0$
b) $\tau(\pi) = 0$ *i.e.* π *is harmonic.*

Let $\overset{1}{g}$ be a Riemannian metric on M. We consider $\pi : (TM, g^c) \to (M, \overset{1}{g})$. We have:

$$\beta(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \beta(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0, \tag{46}$$

$$\beta(\pi)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = ({}^1\Gamma^k_{ij} - \Gamma^k_{ij})\frac{\partial}{\partial x^k},$$

where ${}^1\Gamma^k_{ij}$ are the Christoffel symbols of $\overset{1}{g}$ metric.

Proposition 21. *a)* $\pi : (TM, g^c) \to (M, \overset{1}{g})$ *is totally geodesic if and only if* $1 : (M, g) \to (M, \overset{1}{g})$ *is totally geodesic;*
b) $\pi : (TM, g^c) \to (M, \overset{1}{g})$ *is harmonic.*

Let $\overset{1}{c}$ be a symmetric tensor field of type $(0, 2)$ on M, and let $\overset{1}{g^c}$ be the pseudo–Riemannian metric defined with respect to $\overset{1}{g}$ and $\overset{1}{c}$. We remark that:

$$\frac{\overset{1}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - ({}^1\Gamma^h_{0i})\frac{\partial}{\partial y^h} = \frac{\delta}{\delta x^i} - ({}^1\Gamma^h_{0i} - \Gamma^h_{0i})\frac{\partial}{\partial y^h} \tag{47}$$

Proposition 22. $\overset{1^{\overset{l}{c}}}{g}$ *is harmonic with respect to* g^c *if and only if* $\overset{1}{g}$ *is harmonic with respect to* g.

We remark that the harmonicity of $\overset{1^{\overset{1}{c}}}{g}$ with respect to g^c does not depend on c or $\overset{1}{c}$.
Studying the harmonicity of the metrics g^c, S, G each with respect to other, we obtain:

Proposition 23. *a) Let* $1 : (TM, g^c) \to (TM, S)$ *be the identity map. Then*

$$\tau(1) = R^h_0 \frac{\delta}{\delta x^h}$$

so that S is harmonic with respect to g^c if and only if $\rho(R)=0$.

b) We consider $1:(TM,g^c)\to(TM,G)$. We have:

$$\tau(1)=-\frac{(n-1)(g_{00}+2)}{(1+g_{00})^2}C+\frac{1}{1+g_{00}}R^h_\circ\frac{\delta}{\delta x^h}$$

hence G is not harmonic with respect to g^c.

c) We consider $1:(TM,S)\to(TM,g^c)$. We have: $\tau(1)=R^h_0\frac{\delta}{\delta x^h}$ so that $\tau(1)=0$ if and only if $\rho(R)=0$.

d) We consider $1:(TM,G)\to(TM,g^c)$. We obtain:

$$\tau(1)=R^h_0\frac{\partial}{\partial y^h}-\frac{(n-1)(g_{00}+2)}{1+g_{00}}C$$

hence g^c is not harmonic with respect to G.

The mean connection $\overset{m}{\nabla}$ of the Schouten connection associated with $\widetilde{\nabla}$ is locally given by

$$\begin{cases}\overset{m}{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=0,\ \overset{m}{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=0,\\ \overset{m}{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j}=\Gamma^k_{ij}\frac{\partial}{\partial y^k},\ \overset{m}{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\Gamma^h_{ij}\frac{\delta}{\delta x^h}-\frac{1}{2}R^h_{0ij}\frac{\partial}{\partial y^h}\end{cases}\tag{48}$$

We see that $\overset{m}{\nabla}$ does not depend on c.

Proposition 24. *We consider $1:(TM,g^c)\to(TM,\overset{m}{\nabla})$. We have:*
a) $\beta(1)=0$ if and only if $\nabla c=0$ and $R=0$.
b) $\tau(1)=0$.

Let ξ be a vector field. We consider $\xi:(M,g)\to(TM,\overset{m}{\nabla})$. By a straightforward computation we get:

$$\overset{m}{\beta}(\xi)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)=\{-\frac{1}{2}R^h_{mij}\xi^m+\nabla_i\nabla_j\xi^h\}\frac{\partial}{\partial y^h}\tag{49}$$

$$\overset{m}{\tau}(\xi)=\{g^{ij}(\nabla_i\nabla_j\xi^h)\}\frac{\partial}{\partial y^h}\tag{50}$$

For $\xi:(M,g)\to(TM,g^c)$ we have (see [6]):

$$\tau(\xi)=\{g^{ij}(\nabla_i\nabla_j\xi^h)+\xi^lR^h_l+g^{kh}(\nabla_ic^i_k-\frac{1}{2}\nabla_k(tr\ c))\}\frac{\partial}{\partial y^h}\tag{51}$$

Remark. 1) The use of the connection $\overset{m}{\nabla}$ has led us to new results.
2) If ξ is a Killing vector field, or a geodesic one, or $\Delta\xi = 0$ and (M, g) is flat or (M, g) is Ricci–flat, then $\overset{m}{\tau}(\xi) = 0$ i.e. ξ is harmonic.
3) Suppose that ξ is Killing and (M, g) has the sectional curvature constant $a \neq 0$. Then

$$\overset{m}{\tau}(\xi) = a(1-n)\xi^h \frac{\partial}{\partial y^h} = a(1-n)\xi^V$$

Hence ξ is not harmonic.
4) Assume that ξ is Killing or geodesic and $n = 2$. Then: $\overset{m}{\tau}(\xi) = -k\xi^V$, where $k \in \mathcal{F}(M)$; so $\overset{m}{\tau}(\xi) \neq 0$.
5) Assume that $\Delta\xi = 0$ and (M, g) has the sectional curvature constant $a \neq 0$. Then

$$\overset{m}{\tau}(\xi) = a(n-1)\xi^V$$

Hence ξ is not harmonic.
6) Suppose that $\Delta\xi = 0$ and $n = 2$. Then: $\overset{m}{\tau}(\xi) = k\xi^V,\ k \in \mathcal{F}(M)$;
7) If M is compact and orientable, then ξ is harmonic if and only if $\nabla\xi = 0$.

5. The submanifold T_1M of (TM, g^c). Using the Gauss– Weingarten relations we project $\overset{m}{\nabla}$ on T_1M with respect to the vector field C and we get $\overset{m}{\overline{\nabla}}$ given locally by:

$$\begin{cases} \overset{m}{\overline{\nabla}}_{Y_i} Y_j = -g_{j0}Y_i,\ \overset{m}{\overline{\nabla}}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} - \frac{1}{2} R^h_{0ij} Y_h \\ \overset{m}{\overline{\nabla}}_{Y_i} \frac{\delta}{\delta x^j} = 0,\ \overset{m}{\overline{\nabla}}_{\frac{\delta}{\delta x^i}} Y_j = \Gamma^h_{ij} Y_h \end{cases} \tag{52}$$

From the relations (43) and (52) we obtain:

Proposition 25. *Let ξ be a vector field with $||\xi|| = 1$. We consider $\xi : (M, g) \to (T_1M, \overset{m}{\overline{\nabla}})$. Then $\overset{m}{\overline{\beta}}(\xi)$ and $\overset{m}{\overline{\tau}}(\xi)$ are locally given by:*

$$\overset{m}{\overline{\beta}}(\xi) = \{-\frac{1}{2} R^h_{mij}\xi^m + \nabla_i\nabla_j\xi^h\} Y_h \tag{53}$$

$$\overset{m}{\overline{\tau}}(\xi) = g^{ij}(\nabla_i\nabla_j\xi^h) Y_h \tag{54}$$

Theorem 10. $\overset{m}{\tau}(\xi) = 0$ *if and only if* $\overset{m}{\tau}(\xi)$ *is colinear with* C.

Remark. 1) if $||\xi|| = 1$, ξ is Killing or $\Delta\xi = 0$ and (M, g) has the sectional curvature constant then $\overset{m}{\tau}(\xi) = 0$.

2) if $||\xi|| = 1$, ξ is Killing or $\Delta\xi = 0$ and $n = 2$ then $\overset{m}{\tau}(\xi) = 0$.

In what follows we shall consider $c = g$.

In this case the Levi–Civita connection becomes

$$\begin{cases} \widetilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \ \widetilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = 0, \\ \widetilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^k_{ij} \frac{\partial}{\partial y^k}, \ \widetilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} + R^h_{j0i} \frac{\partial}{\partial y^h} \end{cases} \tag{55}$$

Consider $P = S - C$, $S = y^i \frac{\delta}{\delta x^i}$ and observe that $P \notin$ spann $\{Y_i, \frac{\delta}{\delta x^i}\}$, $g^c(P, Y_i) = g^c\left(P, \frac{\delta}{\delta x^i}\right) = 0$, $g^c(P, P) = -1$.

Projecting $\widetilde{\nabla}$ on T_1M with respect to P and we obtain the torsion free connection $\overset{=}{\nabla}$, given locally by:

$$\begin{cases} \overset{=}{\nabla}_{Y_i} Y_j = (g_{i0}g_{j0} - g_{ij})S - g_{j0}Y_i, \ \overset{=}{\nabla}_{Y_i} \frac{\delta}{\delta x^j} = 0, \\ \overset{=}{\nabla}_{\frac{\delta}{\delta x^i}} Y_j = \Gamma^h_{ij} Y_h, \ \overset{=}{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \{\Gamma^h_{ij} + R_{0j0i}y^h\} \frac{\delta}{\delta x^h} + R^h_{j0i} Y_h \end{cases} \tag{56}$$

From the relations (43) and (56) we get:

Proposition 26. *Let* ξ *be a vector field with* $||\xi|| = 1$. *We consider* $\xi : (M, g) \to (T_1M, \overset{=}{\nabla})$. *Then* $\overset{=}{\beta}(\xi)$ *and* $\overset{=}{\tau}(\xi)$ *have the local coordinate expressions given by:*

$$\overset{=}{\beta}(\xi)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \{R_{mjli}\xi^m\xi^l - (\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\}\xi^h \frac{\delta}{\delta x^h} + \tag{57}$$

$$+\{R^h_{jmi}\xi^m + \nabla_i\nabla_j\xi^h\}Y_h$$

$$\overset{=}{\tau}(\xi) = \{R_{ml}\xi^m\xi^l - g^{ij}(\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\}\xi^h \frac{\delta}{\delta x^h} + \tag{58}$$

$$+\{R^h_m\xi^m + g^{ij}(\nabla_i\nabla_j\xi^h)\}Y_h$$

From (51), (58) we obtain:

Theorem 11. *The following relations are equivalent:*
a) $\overset{=}{\tau}(\xi)=0$
b) $\tau(\xi)=0$
c) $||\xi||=1$ *and* ξ *is a geodesic vector field.*
Remark. If $||\xi||=1$ and ξ Killing then $\overset{=}{\tau}(\xi)=0$.
We project now, $\widetilde{\nabla}$ on T_1M with respect to C and we get $\overline{\nabla}$, given by:

$$\begin{cases}\overline{\nabla}_{Y_i}Y_j=-g_{j0}Y_i,\ \overline{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\Gamma^h_{ij}\frac{\delta}{\delta x^h}+R^h_{j0i}Y_h,\\ \overline{\nabla}_{Y_i}\frac{\delta}{\delta x^j}=0,\ \overline{\nabla}_{\frac{\delta}{\delta x^i}}Y_j=\Gamma^h_{ij}Y_h\end{cases}\tag{59}$$

Proposition 27. *Let* ξ *be a vector field with* $||\xi||=1$. *We consider* $\xi:(M,g)\to(T_1M,\overline{\nabla})$. *The* $\overline{\beta}(\xi)$ *and* $\overline{\tau}(\xi)$ *have the local coordinate expression given by:*

$$\overline{\beta}(\xi)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)=\{R^h_{jmi}\xi^m+\nabla_i\nabla_j\xi^h\}Y_h\tag{60}$$

$$\overline{\tau}(\xi)=\{R^h_m\xi^m+g^{ij}(\nabla_i\nabla_j\xi^h)\}Y_h\tag{61}$$

Theorem 12. $\overline{\tau}(\xi)=0$ *if and only if* $\tau(\xi)$ *is colinear with* C.
Remark. 1) if $\overset{=}{\tau}(\xi)=0$ then $\overline{\tau}(\xi)=0$.
2) if $||\xi||=1,\ \Delta\xi=0$ and (M,g) has been the sectional curvature constant or $n=2$ then $\overline{\tau}(\xi)=0$.
We project $\overset{m}{\nabla}$ on T_1M with respect to P. We get:

$$\begin{cases}\overset{m}{\overline{\nabla}}_{Y_i}Y_j=(g_{i0}g_{j0}-g_{ij})S-g_{j0}Y_i,\ \overset{m}{\overline{\nabla}}_{Y_i}\frac{\delta}{\delta x^j}=0\\ \overset{m}{\overline{\nabla}}_{\frac{\delta}{\delta x^i}}Y_j=\Gamma^h_{ij}Y_h,\ \overset{m}{\overline{\nabla}}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\Gamma^h_{ij}\frac{\delta}{\delta x^h}-\frac{1}{2}R^h_{0ij}Y_h\end{cases}\tag{62}$$

Proposition 28. *For* ξ *with* $||\xi||=1$ *we have:*

$$\begin{aligned}\overset{m}{\overline{\beta}}(\xi)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)&=\{-\frac{1}{2}R^h_{mij}\xi^m+\nabla_i\nabla_j\xi^h\}Y_h-\\&-\{(\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\}\xi^h\frac{\delta}{\delta x^h}\end{aligned}\tag{63}$$

$$\overset{m}{\overline{\tau}}(\xi)=\{g^{ij}(\nabla_i\nabla_j\xi^h)\}Y_h-\{g^{ij}(\nabla_i\xi^l)g_{lk}(\nabla_j\xi^k)\}\xi^h\frac{\delta}{\delta x^h}\tag{64}$$

Theorem 13. $\overset{m}{\overline{\overline{\tau}}}(\xi) = 0$ *if and only if* $\overset{m}{\tau}(\xi) = 0$.

Remark. 1) if $\overset{m}{\overline{\overline{\tau}}}(\xi) = 0$, then $\overset{m}{\overline{\tau}}(\xi) = 0$.

2) if $\Delta\xi = 0$ or ξ is Killing vector field and $\rho(R) = 0$ or $R = 0$, then $\overset{m}{\overline{\overline{\tau}}}(\xi) = 0$.

3) if M is compact and orientable then $\overset{m}{\overline{\overline{\tau}}}(\xi) = 0$ if and only if $\nabla\xi = 0$.

It is more convenient to consider the projection on T_1M using the Liouville vector field C, because the use of P imposes very restrictive conditions for the harmonicity of the vector fields.

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