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SOME CLASSES OF PARAHERMITIAN STRUCTURES ON COTANGENT BUNDLES *

BY

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Dedicated to Professor Gh.Gheorghiev on the occasion of his 90-th birthday.

1. Introduction. P.M.Gadea and J.M.Masqué [3] (see also [1]) have classified the almost parahermitian manifolds obtaining characterizations of the 136 esentially different classes.

In this paper we present examples of almost parahermitian manifolds belonging to some classes of [3]. The first examples are constructed on the cotangent bundle T^*M of a smooth manifold M, considering the vertical distribution VT^*M and the horizontal distribution HT^*M defined by a symmetric nonlinear connection on T^*M as the eigendistributions of the fundamental almost product structure. Further, considering a nondegenerate skew-symmetric $M-$tensor field of type (0,2) on T^*M, and the "musical" isomorphism between VT^*M and HT^*M defined by this $M-$tensor field, we get other classes of almost parahermitian manifolds. In this case, the base manifold must have an even dimension.

2. The classes of almost parahermitian manifolds. Let (M, g, J) be an almost parahermitian manifold. Then M is a $2n-$ dimensional smooth manifold, g is a pseudoriemannian metric on M and J is an almost product structure such that:

$$g(JX, JY) = -g(X, Y) \ , \ \forall X, Y \in \mathcal{X}(M)$$

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The tangent space $T = T_pM$ at each point $p \in M$ splits as the direct sum of $n-$dimensional subspaces V, H where V (respectively H) is the eigenspace of T corresponding to the eigenvalue $+1$ (respectively -1) of J. It follows that both spaces V, H are maximally isotropic with respect to g. Denote by $\mathcal{V}$ (respectively $\mathcal{H}$) the set of vector fields on M with values in V (respectively in H).

Denote by Ω the $2-$form associated with the considered almost parahermitian structure, defined by

$$\Omega(X,Y) = g(JX,Y) \ , \ \forall X, Y \in \mathcal{X}(M)$$

Denote by ∇ the Levi Civita connection of g. The classification given in [3] is related to the decomposition of the space W of the tensors $\nabla\Omega$, of type (0,3) in invariant and irreducible subspaces under the action of the group $GL(n, \mathbf{R})$, thought of as a subgroup in $GL(2n, \mathbf{R})$ by

$$A \in GL(n, \mathbf{R}) \longrightarrow \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix}$$

For $n \geq 3$, the authors find eight terms $W_1, \ldots, W_8$ in this decomposition, thus there are 2^8 classes of almost parahermitian manifolds. However, since V and H are equivalent by the change $J \to -J$, there are only 136 essentially different classes. In the case $n = 2$, there are 10 essentially different classes.

A local adapted frame *(l.a.f.)* to the almost parahermitian structure (M, g, J) is defined by the set $\{A_i, U_i\}_{i=1,\ldots,n}$ of vector fields defined on an open set in M, where $\{A_i\}_{i=1,\ldots,n}$ is a local frame in $\mathcal{V}$ and $\{U_i\}_{i=1,\ldots,n}$ is a local frame in $\mathcal{H}$ such that:

$$g(A_i, U_j) = \delta_{ij} \ ; \ i, j = 1, \ldots, n$$

In order to get the characteristic conditions for each of the 136 essentially different classes, consider the following 8 properties:

$$(1) \qquad \underset{(A,B,C)}{S} (\nabla_A \Omega)(B,C) = 0 \ , \ \forall A, B, C \in \mathcal{V}$$

$$(2) \qquad \nabla_A A \in \mathcal{V} \ , \ \forall A \in \mathcal{V}$$

$$(3) \qquad (\nabla_A \Omega)(U,V) = \theta(V) g(A,U) - \theta(U) g(A,V) \ , \ \forall A \in \mathcal{V} \ ; \ U, V \in \mathcal{H}$$

$$\sum_{i=1}^{n}(\nabla_{A_i}\Omega)(U_i,U)=0\ ,\ \forall U\in\mathcal{H},\ \text{where } \{A_i,U_i\}_{i=1,\dots,n} \text{ is a } \textit{l.a.f.} \tag{4}$$

$$\underset{(U,V,W)}{S}(\nabla_U\Omega)(V,W)=0\ ,\ \forall U,V,W\in\mathcal{H} \tag{5}$$

$$\nabla_U U\in\mathcal{H}\ ,\ \forall U\in\mathcal{H} \tag{6}$$

$$(\nabla_U\Omega)(A,B)=\theta(A)g(U,B)-\theta(B)g(U,A)\ ,\ \forall A,B\in\mathcal{V}\ ,\ U\in\mathcal{H} \tag{7}$$

$$\sum_{i=1}^{n}(\nabla_{U_i}\Omega)(A_i,A)=0\ ,\ \forall A\in\mathcal{V},\ \text{where } \{A_i,U_i\}_{i=1,\dots,n} \text{ is a } \textit{l.a.f.} \tag{8}$$

In the above relations $\underset{(X,Y,Z)}{S}$ denotes the sum consisting of three terms obtained by cyclic permutations of X,Y,Z.

The class $\mathcal{PK}$ of the parakaehlerian manifolds is the class of almost parahermitian manifolds satisfying all the 8 conditions. The primitive class $\mathcal{W}_i\ ; i=1,\dots,8$ of the almost parahermitian manifolds for which $\nabla\Omega\in W_i$ in each point of M is characterized by the fullfilment of all but i) conditions. Further, the class $\mathcal{W}_i\oplus\mathcal{W}_j$ is characterized by the fullfilment of all but i) and j) conditions, etc. The fullfilment of the condition i) only, means that the almost manifold (M,J,g) is in the class $\underset{j\neq i}{\oplus}\mathcal{W}_j$.

3. The cotangent bundle. Let M be an $n-$dimensional smooth manifold and denote by $\pi:T^*M\longrightarrow M$ its cotangent bundle with fibres the cotangent spaces to M. Then T^*M is a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M, may be used. Let $(U,\mathbf{x})$ be a local chart on M with the domain $U\subset M$ and the coordinate map $\mathbf{x}=[x^i];i=1,\dots,n$. Then the local chart $(\pi^{-1}(U),(\mathbf{q},\mathbf{p}))$ is induced on T^*M where the coordinate map $(\mathbf{q},\mathbf{p})=[q^i,p_k]$ is defined as follows. Firstly $q^i=x^i\circ\pi;i=1,\dots,n$, i.e. the first n local coordinates of a cotangent vector from $\pi^{-1}(U)$ are the local coordinates of its base point, thought of as functions on $\pi^{-1}(U)\subset T^*M$. Then $p_i\ ;\ i=1,\dots,n$, are the vector space coordinates with respect to the natural local frame $(dx^1,\cdots,dx^n)$ in T^*M defined by the local chart $(U,\mathbf{x})$. The $M-$tensor

fields and the linear $M-$connections may be considered on T^*M and the usual tensor fields and linear connections on the base manifold M may be thought of naturally as $M-$tensor fields and linear $M-$connections on T^*M (see [5],[6]).

Let $VT^*M = Ker\ \pi_* \subset TT^*M$ be the vertical distribution over T^*M. Then VT^*M is involutive with fibre dimension n and the local vector fields $\partial^i = \dfrac{\partial}{\partial p_i}; i = 1, \ldots, n$, define a local frame in VT^*M. A (nonlinear) connection on T^*M is defined by a complementary distibution HT^*M (horizontal distribution) to VT^*M in TT^*M. A local frame in HT^*M, related to the induced local chart $(\pi^{-1}(U), (\mathbf{q}, \mathbf{p}))$ is defined by the vector fields $\delta_i = \dfrac{\delta}{\delta q^i}; i = 1, \ldots, n$, where:

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - N_{ij}\frac{\partial}{\partial p_j}$$

The functions $N_{ij} = N_{ij}(q, p)$; $i, j = 1, \ldots, n$, are the connection coefficients of the considered connection in the induced local chart $(\pi^{-1}(U), (\mathbf{q}, \mathbf{p}))$. We shall assume $N_{ij} = N_{ji}$, i.e. the considered connection is symmetric (there always exists a symmetric connection on T^*M). Then:

$$\left[\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right] = \Phi^i_{jk}\frac{\partial}{\partial p_k}\ ;\ \left[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right] = -R_{kij}\frac{\partial}{\partial p_k} \tag{9}$$

where:

$$\Phi^i_{jk} = -\frac{\partial N_{jk}}{\partial p_i}\ ,\ R_{kij} = \frac{\delta N_{kj}}{\delta q^i} - \frac{\delta N_{ki}}{\delta q^j} \tag{9'}$$

Remark that the components Φ^i_{jk} ; $i, j, k = 1, \ldots, n$, define a linear M-connection on T^*M and the components R_{kij} ; $i, j, k = 1, \ldots, n$ define an $M-$tensor field of type $(0, 3)$ on T^*M. Due to the symmetry of the considered connection the following identity

$$\underset{(i,j,k)}{S} R_{ijk} = 0 \tag{10}$$

is obvious, where, just like above, $\underset{(i,j,k)}{S}$ denotes the sum consisting of three terms, obtained by cyclic permutations of i, j, k.

The pseudo-Riemannian metric G with the signature (n,n) is defined on T^*M by :

(11)
$$G\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right)=G\left(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j}\right)=0\ ,\ \ G\left(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j}\right)=G\left(\frac{\delta}{\delta q^j},\frac{\partial}{\partial p_i}\right)=\delta^i_j$$

and has the following local coordinate expression (see [5]):

$$G=2(dp_i+N_{ij}dq^j)dq^i.$$

Its Levi Civita connection $\widetilde{\nabla}$ is given by:

$$\widetilde{\nabla}^i\frac{\partial}{\partial p_j}=0\ ,\ \widetilde{\nabla}^i\frac{\delta}{\delta q^j}=0\ ,\ \widetilde{\nabla}_i\frac{\partial}{\partial p_j}=-\Phi^j_{ik}\frac{\partial}{\partial p_k}\ , \tag{12}$$

$$\widetilde{\nabla}_i\frac{\delta}{\delta q^j}=\Phi^k_{ij}\frac{\delta}{\delta q^k}+R_{ijk}\frac{\partial}{\partial p_k}$$

where we have denoted:

$$\widetilde{\nabla}^i=\widetilde{\nabla}_{\ \frac{\partial}{\partial p_i}}\ ,\ \widetilde{\nabla}_i=\widetilde{\nabla}_{\ \frac{\delta}{\delta q^i}}\ .$$

From $\widetilde{\nabla}$ we get its Schouten connection $\overline{\nabla}$, given by:

$$\overline{\nabla}^i\frac{\partial}{\partial p_j}=0\ ,\ \overline{\nabla}^i\frac{\delta}{\delta q^j}=0\ \ ,\ \ \overline{\nabla}_i\frac{\partial}{\partial p_j}=-\Phi^j_{ik}\frac{\partial}{\partial p_k}\ ,\ \overline{\nabla}_i\frac{\delta}{\delta q^j}=\Phi^k_{ij}\frac{\delta}{\delta q^k}. \tag{13}$$

Next, we may consider the covariant derivatives of the $M-$tensor fields on T^*M with respect to $\overline{\nabla}$. E.g., if the components $h_{jk}(q,p)$; $j,k=1,\ldots,n$, define an $M-$tensor field of type $(0,2)$, then

$$\overline{\nabla}^i h_{jk}=\partial^i h_{jk}\ ,\ \overline{\nabla}_i h_{jk}=\frac{\delta h_{jk}}{\delta q^i}-\Phi^l_{ij}h_{lk}-\Phi^l_{ik}h_{jl}. \tag{14}$$

The components $\overline{\nabla}^i h_{jk}$ define an $M-$tensor field of type $(1,2)$ and the components $\overline{\nabla}_i h_{jk}$ define an $M-$tensor field of type $(0,3)$ on T^*M.

Consider the almost product structure P on T^*M, determined by the splitting $TT^*M=VT^*M\oplus HT^*M$ where $V=VT^*M, H=HT^*M$ i.e.:

$$P(\frac{\partial}{\partial p_i})=\frac{\partial}{\partial p_i}\ ;\ P(\frac{\delta}{\delta q^i})=-\frac{\delta}{\delta q^i} \tag{15}$$

One can check easily that (T^*M, G, P) is an almost parahermitian manifold (see [2], for the case of the Riemann extensions, i.e. the case where HT^*M is defined by a linear connection on M) and $\{\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^i}\}_{i=1,\dots n}$ is a local adapted frame. The corresponding $2-$form Ω is given by:

(16)
$$\Omega(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = 0 \ , \ \Omega(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = 0 \ , \ \Omega(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = -\Omega(\frac{\delta}{\delta q^j}, \frac{\partial}{\partial p_i}) = \delta^i_j$$

The local coordinate expression of Ω is:

$$\Omega = dp_i \wedge dq^i$$

The following expressions are obtained by a straightforward computation:

$$(17) \qquad (\widetilde{\nabla}^i\Omega)(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k}) = (\widetilde{\nabla}^i\Omega)(\frac{\partial}{\partial p_j}, \frac{\delta}{\delta q^k}) = (\widetilde{\nabla}^i\Omega)(\frac{\delta}{\delta q^j}, \frac{\delta}{\delta q^k}) =$$

$$= (\widetilde{\nabla}_i\Omega)(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k}) = 0 \ , \ (\widetilde{\nabla}_i\Omega)(\frac{\delta}{\delta q^j}, \frac{\delta}{\delta q^k}) = -2R_{ijk}.$$

Next, it can be checked that all but 6) conditions in section 2 are fullfilled. Thus:

Theorem 1. *The almost parahermitian manifold (T^*M, G, P) is in the class $\mathcal{W}_6$ (or (T^*M, G, P) is an almost parakaehlerian manifold, according to the terminology from* [3]).

Remark. If we consider the almost product structure $\overline{P} = -P$ on T^*M with $V = HT^*M$ and $H = HT^*M$, it follows easily that $(T^*M, G, \overline{P})$ is an almost parahermitian manifold in the class $\mathcal{W}_2$ (or a $(+)-$almost parakaehlerian manifold).

From the above considerations it follows that the almost parahermitian manifold (T^*M, G, P) is parakaehlerian if and only if $R_{ijk} = 0$, i.e. the horizontal distribution HT^*M is involutive. In the following we shall get some classes of manifolds whose cotangent bundles carry parakaehlerian structures.

Firstly, let (M, g) be a (pseudo-)Riemannian manifold with g_{ij} the local coordinate components of the metric tensor field g in the local chart $(U, \mathbf{x})$. Denote by ∇ the Levi Civita connection of (M, g) and let Γ^k_{ij} be its connections coefficients in the local chart $(U, \mathbf{x})$ (i.e. Γ^k_{ij} are the usual Christoffel

symbols). We may consider on T^*M the (nonlinear) connection with the connection coefficients

$$N_{ij} = -\Gamma^k_{ij} p_k + g_{ij} + a p_i p_j \tag{18}$$

where a is a constant. Then, according to a result from [4], it follows easily that, in this case, $R_{kij} = 0$ if and only if

$$R^h_{ijk} = a(\delta^h_j g_{ik} - \delta^h_k g_{ij}) \tag{19}$$

where R^h_{ijk} are the components of the curvature tensor field of ∇. But the condition (19) is just the condition for (M, g) to have constant sectional curvature a. Thus:

Theorem 2. *Let (M, g) be a (pseudo-)Riemannian space. The almost parahermitian manifold (T^*M, G, P) defined by the connection given in (18) is parakaehlerian if and only if (M, g) has constant sectional curvature a.*

Another parakaehlerian structures on the cotangent bundles can be obtained in the cases of the Kaehler and quaternion Kaehler manifolds.

Let (M, g, F) be a Kaehler manifold with the Riemannian metric g and the almost complex structure defined by the tensor field F of type (1,1) such that $F^2 = -I, g(FX, FY) = g(X, Y)$. If ∇ is the Levi Civita connection of (M, g) then we have $\nabla F = 0$. Consider the following (nonlinear) connection on the cotangent bundle T^*M of the Kaehler manifold (M, g, F) :

$$N_{ij} = -\Gamma^k_{ij} p_k + g_{ij} + a(p_i p_j - F^k_i F^l_j p_k p_l) \tag{20}$$

where F^h_i are the componets of F and a is a constant. Then it follows by a straightforward computation that the condition $R_{kij} = 0$ is equivalent to

$$R^h_{kij} = a\{\delta^h_i g_{jk} - \delta^h_j g_{ik} - F^h_i g_{jl} F^l_k + F^h_j g_{il} F^l_k + 2 g_{il} F^l_j F^h_k\}. \tag{21}$$

This is just the condition for the Kaehler manifold (M, g, F) to have constant holomorphic curvature $4a$. Thus:

Theorem 3. *Let (M, g, F) be a Kaehler manifold. The almost parahermitian manifold (T^*M, G, P) defined by the connection given in (20) is parakaehlerian if and only if (M, g, F) has constant holomorphic sectional curvature $4a$.*

Consider now (M, g, D) a quaternion Kaehler manifold. Then M is a $4m-$dimensional manifold, D is a subbundle with fibre dimension 3 in the

bundle of the tensors of type $(1,1)$ on M and, locally, D has a canonical base (F_1, F_2, F_3) such that:

$$F_\alpha^2 = -I \quad , \quad F_\alpha \circ F_\beta = -F_\beta \circ F_\alpha = F_\gamma$$

where $\alpha = 1,2,3$ and (α, β, γ) is any cyclic permutation of $(1,2,3)$. Moreover, we have

$$g(F_\alpha X, F_\alpha Y) = g(X,Y) \ ; \ \forall X, Y \in \mathcal{X}(M) \ , \ \alpha = 1,2,3$$

and the Levi Civita connection ∇ of g preserves D i.e., locally

$$\nabla F_\alpha = \sum_{\beta=1}^{3} \eta_{\alpha\beta} \otimes F_\beta.$$

Consider the following (nonlinear) connection on T^*M :

$$N_{ij} = -\Gamma_{ij}^k p_k + g_{ij} + a\{p_i p_j - \sum_{\alpha=1}^{3} (F_\alpha)_i^k (F_\alpha)_j^l p_k p_l\} \tag{22}$$

where a is a constant and $(F_\alpha)_i^k$ are the local coordinate components of F_α.

It follows by a straightforward computation that the condition $R_{kij} = 0$ is equivalent to:

$$R_{kij}^h = a\{\delta_i^h g_{jk} - \delta_j^h g_{ik} + \sum_{\alpha=1}^{3} [-(F_\alpha)_i^h (F_\alpha)_k^l g_{jl} + (F_\alpha)_j^h (F_\alpha)_k^l g_{il} +$$

$$+2(F_\alpha)_j^l (F_\alpha)_k^h g_{il}]\}$$

This is just the condition for M to have constant $Q-$ sectional curvature $4a$. Hence:

Theorem 4. *Let (M, g, D) be a quaternion Kaehler manifold. The almost parahermitian manifold (T^*M, G, P) defined by the connection given in (20) is parakaehlerian if and only if (M, g, D) has constant $Q-$sectional curvature $4a$.*

4. Other almost parahermitian structures on T^*M. Let $h_{ij}(q,p)$ be the components of a nondegenerate $M-$tensor field of type $(0,2)$ on T^*M. Denote by $h^{ij}(q,p)$ the components of the inverse of the matrix $(h_{ij})_{i,j=1,\cdots,n}$

i.e.$h_{ik}h^{jk} = \delta_i^j$. It follows that the components h^{ij} define an $M-$tensor field of type (2,0) on T^*M. Assume that on T^*M it is given a symmetric (nonlinear) connection defined by the connection coefficients N_{ij} and consider the automorphism J of TT^*M expressed in local adapted frames by:

$$J(\frac{\delta}{\delta q^i}) = h_{ik}\frac{\partial}{\partial p_k} \quad , \quad J(\frac{\partial}{\partial p_i}) = -h^{ik}\frac{\delta}{\delta q^k}. \tag{24}$$

The following result is obtained by a straightforward computation:

Proposition 5. *The automorphism J given by (24) defines an almost product structure on T^*M if and only if the $M-$tensor field h_{ij} is skew-symmetric i.e.:*

$$h_{ji} = -h_{ij}.$$

From now on we shall assume that M is an even-dimensional manifold whose cotangent bundle T^*M carries a nondegenerate skewsymmetric $M-$tensor field h_{ij} of type $(0, 2)$.

We may check easily that the pseudo-Riemannian metric G defined by (11) and the almost product structure J are related by

$$G(JX, JY) = -G(X, Y) \ ; \ \forall X, Y \in X(M).$$

Thus:

Proposition 6. *(T^*M, G, J) is an almost parahermitian manifold.*

We shall find the eigen distributions $\mathcal{V}, \mathcal{H}$ of J on T^*M by writing down the local vector fields on T^*M defining an adapted local frame. A local frame for $\mathcal{V}$ can be given by:

$$A^i = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial p_i} - h^{ik}\frac{\delta}{\delta q^k})$$

and the corresponding local frame of $\mathcal{H}$ can be given by:

$$U_i = \frac{1}{\sqrt{2}}(\frac{\delta}{\delta q^i} - h_{ik}\frac{\partial}{\partial p_k})$$

as it can be checked easily. We have also:

$$\partial^i = \frac{1}{\sqrt{2}}(A^i + h^{ik}U_k) \quad , \quad \delta_i = \frac{1}{\sqrt{2}}(U_i + h_{ik}A^k).$$

Next, it follwos by a straightforward computation the expression of the Levi Civita connection $\widetilde{\nabla}$ in the local adpted frame $\{A^i, U_k\}_{i=1,...,n}$:

$$\widetilde{\nabla}_{U_i}U_j = H_{ijk}A^k + (H_{ijl}h^{lk} + \frac{\sqrt{2}}{2}\Phi^k_{ij})U_k$$

$$\widetilde{\nabla}_{A^i}U_j = (-h^{il}H_{ljk} - \frac{\sqrt{2}}{2}\partial^i h_{jk})A^k +$$

$$+(-\frac{\sqrt{2}}{2}h^{il}\Phi^k_{lj} - \frac{\sqrt{2}}{2}\partial^i h_{jl}h^{lk} - h^{il}H_{ljh}h^{hk})U_k$$

$$\widetilde{\nabla}_{U_i}A^j = (h_{il}K^{ljk} + \frac{\sqrt{2}}{2}h_{il}\partial^l h^{jk})U_k + (-\frac{\sqrt{2}}{2}\Phi^j_{ik} - H_{ikl}h^{lj})A^k$$

$$\widetilde{\nabla}_{A^i}A^j = K^{ijk}U_k + (\frac{\sqrt{2}}{2}h^{il}\Phi^j_{lk} + \frac{\sqrt{2}}{2}h^{ia}h^{jb}R_{abk} + K^{ijl}h_{lk})A^k$$

where the expressions of the $M-$tensor fields H_{ijk}, K^{ijk} are given by:

$$H_{ijk} = \frac{1}{2\sqrt{2}}(h_{il}\partial^l h_{jk} - \overline{\nabla}_i h_{jk} + R_{ijk}),$$

$$K^{ijk} = \frac{1}{2\sqrt{2}}(-\partial^i h^{jk} + h^{il}\overline{\nabla}_l h^{jk} - h^{ia}h^{jb}h^{kc}R_{abc}).$$

These $M-$tensor fields are related by the following relations:

$$H_{ijk} + \frac{\sqrt{2}}{2}\overline{\nabla}_i h_{jk} = h_{ia}h_{jb}h_{kc}K^{abc},$$

$$K^{ijk} - \frac{\sqrt{2}}{2}h^{il}\overline{\nabla}_l h^{jk} = -H_{abc}h^{ia}h^{jb}h^{kc}.$$

Then we get by a straightforward computation:

$$(\widetilde{\nabla}_{A^i}\Omega)(A^j, A^k) = 2K^{ijk} \quad , \quad (\widetilde{\nabla}_{A^i}\Omega)(A^j, U_k) = 0,$$

$$(\widetilde{\nabla}_{A^i}\Omega)(U_j, U_k) = 2(h^{il}H_{ljk} + \frac{\sqrt{2}}{2}\partial^i h_{jk}),$$

$$(\widetilde{\nabla}_{U_i}\Omega)(A^j, A^k) = 2(h_{il}K^{ljk} + \frac{\sqrt{2}}{2}h_{il}\partial^l h^{jk}),$$

$$(\widetilde{\nabla}_{U_i}\Omega)(U_j, A^k) = 0 \quad , \quad (\widetilde{\nabla}_{U_i}\Omega)(U_j, U_k) = -2H_{ijk}.$$

Proposition 7. *The conditions(1)-(8) in the second section are expressed, in the case of* (T^*M, G, J) *by:*

i) $\underset{(i,j,k)}{S}(\overline{\nabla}_i h_{jk} + h_{il}\partial^l h_{jk}) = 0,$

ii) $3R_{ijk} = h_{kl}\partial^l h_{ij} - h_{jl}\partial^l h_{ik} - 2h_{il}\partial^l h_{jk} + \overline{\nabla}_k h_{ij} - \overline{\nabla}_j h_{ik} - 2\overline{\nabla}_i h_{jk},$

iii) $R_{ijk} = h_{il}\partial^l h_{jk} + \overline{\nabla}_i h_{jk} + \frac{1}{n-1}\{h_{ik}(\partial^l h_{lj} +$

$$+h^{lh}\overline{\nabla}_l h_{hj} - h^{lh}R_{lhj}) - h_{ij}\,(\partial^l h_{lk} + h^{lh}\overline{\nabla}_l h_{hk} - h^{lh}R_{lhk})\},$$

iv) $h^{ij}R_{ijk} = h^{ij}\overline{\nabla}_i h_{jk} + \partial^i h_{ik},$

v) $\underset{(i,j,k)}{S}(h_{il}\partial^l h_{jk} - \overline{\nabla}_i h_{jk}) = 0,$

vi) $3R_{ijk} = 2\overline{\nabla}_i h_{jk} + \overline{\nabla}_j h_{ik} - \overline{\nabla}_k h_{ij} - 2h_{il}\partial^l h_{jk} - h_{jl}\partial^l h_{ik} + h_{kl}\partial^l h_{ij},$

vii) $R_{ijk} = h_{il}\partial^l h_{jk} - \overline{\nabla}_i h_{jk} + \frac{1}{n-1}\{h_{ik}(\partial^l h_{lj} -$

$$-h^{lh}\overline{\nabla}_l h_{hj} - h^{lh}R_{lhj}) - h_{ij}(\partial^l h_{lk} - h^{lh}\overline{\nabla}_l h_{hk} - h^{lh}R_{lhk})\},$$

viii) $h^{ij}R_{ijk} = \partial^i h_{ik} - h^{ij}\overline{\nabla}_i h_{jk}.$

In the following we shall try to obtain another classes of parahermitian manifolds on T^*M by considering some special expressions for the (nonlinear) connection defined by N_{ij} and the $M-$tensor field h_{ij}. These expressions are given essentially by polynomials of degree 2 in the (co-)tangential coordinates p_i.

Let M be a $2n-$dimensional smooth manifold, η a $1-$form on M with the components η_i and $a_{ij}(q)$ a nondegenerate skew-symmetric tensor field of type $(0,2)$ on M. Denote by Γ^k_{ij} the connection coefficients of a torsion-free linear connection ∇ on M and consider on the cotangent bundle T^*M the components $h_{ij}(q,p)$ of a nondegenerate skew-symmetric $M-$tensor field of type $(0,2)$ given by:

$$h_{ij}(q,p) = a_{ij}(q) + p_i\eta_j - p_j\eta_i. \tag{25}$$

Consider also the following (nonlinear) connection on T^*M

$$N_{ij}(q,p) = -\Gamma^k_{ij}p_k + c_{ij}(q,p) \tag{26}$$

and assume that the $M-$tensor field $c_{ij}(q,p)$ is given by

$$c_{ij}(q,p) = b_{ij}(q) + \lambda p_i p_j$$

where $b_{ij}(q)$ are the components of a symmetric tensor field of type $(0,2)$ on M and λ is a real constant.

Then we get easily the components Φ^i_{jk} of the $M-$connection defined by N_{ij} and the covariant derivative of the $M-$tensor field h_{jk} with respect to ∇:

$$\Phi^i_{jk} = \Gamma^i_{jk} - \partial^i c_{jk} \ ; \ \nabla_i h_{jk} = \frac{\partial h_{jk}}{\partial q^i} + \Gamma^l_{ih} p_l \partial^h h_{jk} - \Gamma^l_{ij} h_{lk} - \Gamma^l_{ik} h_{jl}.$$

Next we obtain by a straightforward computation:

$$\overline{\nabla}_i h_{jk} = \nabla_i h_{jk} - c_{ih}\partial^h h_{jk} + h_{hk}\partial^h c_{ij} + h_{jh}\partial^h c_{ik} \tag{27}$$

$$R_{kij} = -p_h R^h_{kij} + \nabla_i c_{jk} - \nabla_j c_{ik} - c_{ih}\partial^h c_{jk} + c_{jh}\partial^h c_{ik} \tag{28}$$

where R^h_{kij} denotes the components of the curvature tensor of the linear connection ∇ on M. Finally, we have:

$$\partial^l h_{jk} = \delta^l_j \eta_k - \delta^l_k \eta_j \ ; \ \partial^l c_{jk} = \lambda(\delta^l_j p_k + \delta^l_k p_j) \tag{29}$$

$$\underset{(i,j,k)}{S} \nabla_i h_{jk} = \underset{(i,j,k)}{S} \delta^0_i h_{jk} \ ; \ \nabla_i h_{jk} = \nabla_i a_{jk} + ((\nabla_i \eta_k)\delta^l_j - (\nabla_i \eta_j)\delta^l_k)p_l \tag{30}$$

$$\nabla_i c_{jk} = \nabla_i b_{jk} \ ; \ h^{ij} = a^{ij} + \frac{1}{(\beta - 1)} a^{ik} a^{jh}(p_k \eta_h - p_h \eta_k) \tag{31}$$

where we have denoted:

$$\delta^0_i = \frac{\partial}{\partial q^i} + \Gamma^l_{ij} p_l \frac{\partial}{\partial p_j} \ ; \ \beta = a^{kh}\eta_k p_h.$$

By using (27)-(31) we obtain that on the almost parahermitian manifold (T^*M, G, J) where G and J are defined by (11) and respectively (24) and with h_{ij} and N_{ij} expressed by (25) and respectively (26), the conditions (i) and (v) from Proposition 7 become:

$$\underset{(i,j,k)}{S} (\partial_i a_{jk} + 2a_{ij}\eta_k) = 0 \ ; \ d\eta = 0 \tag{32}$$

and respectively

$$\underset{(i,j,k)}{S}(\partial_i a_{jk} - 2a_{ij}\eta_k) = 0 \ ; \ d\eta = 0. \tag{33}$$

By a quite long computation we obtain that the condition (ii) from Proposition 7 becomes:

$$3(\nabla_j b_{ik} - \nabla_k b_{ij}) + a_{ij}\eta_k - a_{ik}\eta_j - 2a_{jk}\eta_i - \nabla_k a_{ij} + \tag{34}$$
$$+\nabla_j a_{ik} + 2\nabla_i a_{jk} - 3b_{ij}\eta_k + 3b_{ik}\eta_j = 0$$

$$R^l_{ijk} + \lambda(b_{ij}\delta^l_k - b_{ik}\delta^l_j) + 3\lambda(a_{ij}\delta^l_k - a_{ik}\delta^l_j - 2a_{jk}\delta^l_i) + 3\eta_i\eta_k\delta^l_j - \tag{35}$$
$$-3\eta_i\eta_j\delta^l_k - d\eta_{jk}\delta^l_i - (\nabla_k\eta_i)\delta^l_j + (\nabla_j\eta_i)\delta^l_k + 2(\nabla_i\eta_j)\delta^l_k - 2(\nabla_i\eta_k)\delta^l_j = 0$$

$$\lambda \cdot \eta = 0 \tag{36}$$

and the corresponding of the condition (vi) from Proposition 7 are obtained from (34)-(36) by changing a_{ij} with $-a_{ij}$ and η_i with $-\eta_i$. Hence we may state:

Theorem 8. *The almost parahermitian manifold (T^*M, G, J) with G and J obtained from the above expressions of N_{ij} and h_{ij} is in the class $\underset{j\neq 1}{\oplus}\mathcal{W}_j$ (respectively in the class $\underset{j\neq 5}{\oplus}\mathcal{W}_j$) if the $1-$form η is closed and the almost symplectic structure defined by the tensor field a_{ij} is locally conformal symplectic.*

Consider now a locally conformal almost Kaehler manifold M, i.e. an almost Hermitian manifold on which it is defined a closed $1-$form ω (called the Lee form) such that the fundamental $2-$form Φ of M satisfies

$$d\Phi = \omega\wedge\Phi.$$

Denote by F the almost complex structure on M, by g the Hermitian metric on M and by ∇ the Levi Civita connection of g with the coefficients Γ^k_{ij}. Suppose that $\omega\neq 0$ at every point of M and denote by θ the unit $1-$form corresponding to ω on M. Consider the torsion-free linear connection ∇^1 on M, called the Weyl connection, defined by the coefficients:

$$\overset{1}{\Gamma}{}^k_{ij} = \Gamma^k_{ij} - c(\theta_i\delta^k_j + \theta_j\delta^k_i - g_{ij}g^{kl}\theta_l)$$

where $c = \frac{|\omega|}{2}$. Then M is a locally conformal Kaehler manifold if and only if $\nabla^1 F = 0$ (see [7]). A locally conformal Kaehler manifold is called a $\mathcal{PK}-$manifold if its Lee form θ is parallel with respect to the connection ∇. Then, if M is a $\mathcal{PK}-$manifold, we have that c is constant and

$$\nabla_i\theta_j = 0 \ ; \ \overset{1}{\nabla}_i\theta_j = c(2\theta_i\theta_j - g_{ij}) \ ; \ \overset{1}{\nabla}_i\Phi_{jk} = 2c\Phi_{jk}\theta_i.$$

Consider the nondegenerate $M-$tensor field of type $(0,2)$ on T^*M, defined by

$$h_{ij} = \Phi_{ij} + c(\theta_j p_i - \theta_i p_j)$$

and denote by J the almost product structure on T^*M defined by (24) with h_{ij} given as above. Consider also on T^*M the (nonlinear) connection defined by the coefficients:

$$N_{ij} = -\overset{1}{\Gamma}{}^k_{ij}\, p_k - c(\theta_i p_j + \theta_j p_i)$$

and denote by G the pseudo-Riemannian metric on T^*M defined by (11) with N_{ij} given as above. Then, according to Proposition 6 we have that (T^*M, G, J) is an almost parahermitian manifold and by a straightforward computation we obtain:

$$\overline{\nabla}_i h_{jk} = c(\Phi_{ki}\theta_j - \Phi_{ji}\theta_k) + c^2(g_{ij}p_k - g_{ik}p_j) \tag{37}$$

$$h_{hi}\partial^h h_{kj} = c(\Phi_{ki}\theta_j - \Phi_{ji}\theta_k) + c^2(\theta_i\theta_j p_k - \theta_i\theta_k p_j) \tag{38}$$

$$R_{ijk} = -\overset{1}{R}{}^h_{ijk}\, p_h + c^2(\theta_i\theta_k p_j - \theta_i\theta_j p_k + g_{ij}p_k - g_{ik}p_j) \tag{39}$$

where $\overset{1}{R}{}^h_{ijk}$ denotes the components of the curvature tensor of the connection $\overset{1}{\nabla}$ on M.

A $\mathcal{PK}-$manifold M is said to be a $\mathcal{P}_0\mathcal{K}-$manifold if $\overset{1}{R}{}^h_{ijk} = 0$. Suppose that M is a $\mathcal{P}_0\mathcal{K}-$manifold. Then, by using (37), (38) and (39) we get that the conditions (iii),(v) and (vi) from Proposition 7 are satisfied. Hence, we have:

Theorem 9. *Let (M, g, F) be a $\mathcal{P}_0\mathcal{K}-$manifold. Then (T^*M, G, J) is a $\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_4\oplus\mathcal{W}_7\oplus\mathcal{W}_8-$manifold.*

In a similar way, if (M, g, F) is a $\mathcal{P}_0\mathcal{K}-$manifold and consider on T^*M the almost product structure $\overline{J}$ defined by (24) with h_{ij} given by

$$h_{ij} = \Phi_{ij} - c(\theta_j p_i - \theta_i p_j)$$

and the same pseudo-Riemannian metric G as above, we get that the conditions (i), (ii) and (vii) from Proposition 7 are satisfied. Hence we have:

Theorem 10. *Let (M, g, F) be a $\mathcal{P}_0\mathcal{K}-$manifold. Then $(T^*M, G, \overline{J})$ is a $\mathcal{W}_3\oplus\mathcal{W}_4\oplus\mathcal{W}_5\oplus\mathcal{W}_6\oplus\mathcal{W}_8-$manifold.*

Consider now the particular case where (M, g, F) is a Kaehler manifold and denote by ∇ the Levi Civita connection on M with its coefficients Γ^k_{ij}. Denote by h_{ij} the components of the fundamental $2-$form on M, i.e.

$$h_{ij} = g_{ik}F^k_j$$

and let J be the almost product structure on T^*M defined by (24) with h_{ij} given above. On T^*M consider the (nonlinear) connection defined by the coefficients

$$N_{ij} = -\Gamma^k_{ij}p_k + a(F^k_i p_k p_j + F^k_j p_k p_i)$$

where a is a real constant and denote by G the pseudo-Riemannian metric on T^*M defined by (11) with N_{ij} given above. By a straightforward computation we get:

$$\overline{\nabla}_i h_{jk} = -\overset{0}{R}{}^h_{ijk}\, p_h \tag{40}$$

where $\overset{0}{R}{}^h_{ijk}$ denotes the components of the curvature tensor of a Kaehler manifold of constant holomorphic sectional curvature $-4a$. On the other hand, by direct computation we obtain:

$$R_{ijk} = -R^h_{ijk}p_h \tag{41}$$

where R^h_{ijk} are the components of the curvature tensor of the Levi Civita connection ∇ on M. Then, by using (40), (41) and Proposition 7 we have:

Theorem 11. *Let (M, g, F) be a Kaehler manifold of constant holomorphic sectional curvature k and let (T^*M, G, J) be the almost parahermitian manifold with G and J defined above. Then we have: (i) if $k = -4a\neq 0$, then (T^*M, G, J) is a $\mathcal{W}_2\oplus\mathcal{W}_7\oplus\mathcal{W}_8-$manifold. (ii) if $k = 4a\neq 0$,*

then (T^*M, G, J) *is a* $\mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_6-$*manifold.* (iii) *if* $k = a = 0$*, then* (T^*M, G, J) *is a parakaehlerian manifold.*

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